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Research Article

Exponential Inequalities for Positively Associated Random Variables and Applications

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We establish some exponential inequalities for positively associated random variables without the boundedness assumption. These inequalities improve the corresponding results obtained by Oliveira (2005). By one of the inequalities, we obtain the convergence rate $n^{-1/2}(\log\log n)^{1/2}(\log n)^2$ for the case of geometrically decreasing covariances, which closes to the optimal achievable convergence rate for independent random variables under the Hartman-Wintner law of the iterated logarithm and improves the convergence rate $n^{-1/3}(\log n)^{5/3}$ derived by Oliveira (2005) for the above case.

1. Introduction

A finite family of random variables $\{X_i,\ 1 \le i \le n\}$ is said to be positively associated (PA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\{f_1(X_i,\ i \in A_1), f_2(X_j, j \in A_2)\} \ge 0 \tag{1.1}$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is positively associated if every finite subfamily is positively associated.

The exponential inequalities and moment inequalities for partial sum $\sum_{i=1}^{n}(X_i - EX_i)$ play a very important role in various proofs of limit theorems. For positively associated random variables, Birkel [1] seems the first to get some moment inequalities. Shao and Yu [2] generalized later the previous results. Recently, Ioannides and Roussas [3] established a Bernstein-Hoeffding-type inequality for stationary and positively associated random variables being bounded; and Oliveira [4] gave a similar inequality dropping the boundedness assumption by the existence of Laplace transforms. By the inequality, he obtained that the rate of $\sum_{i=1}^{n}(X_i - EX_i)/n \to 0$ *a.s.* is $n^{-1/3}(\log n)^{5/3}$ under the rate of covariances supposed to be geometrically decreasing, that is, ρ^n for some $0 < \rho < 1$. The convergence rate is partially improved by Yang and Chen [5] only for positively associated random variables being bounded. Furthermore, the rate of convergence in [4] is even lower than that obtained by [3]. These motivate us to establish some new exponential inequalities in order to improve the inequalities and the convergence rate which [4] obtained without the boundedness assumption. It is the main purpose of this paper. Our inequalities in Sections 3–5 improve the corresponding results in [4]. Moreover, by Corollary 5.4 (which can be seen in Section 5), we may get the rate $n^{-1/2}(\log\log n)^{1/2}(\log n)^2$ if the rate of covariances is geometrically decreasing. The result closes to the optimal achievable convergence rate for independent random variables under the Hartman-Wintner law of the iterated logarithm and improves the relevant result obtained by [4] without the boundedness assumption.

Throughout this paper, we always suppose that C denotes a positive constant which only depends on some given numbers, $[x]$ denotes the integral of x; and this paper is organized as follows. Section 2 contains some lemmas used later in the proof of theorems, and some notations. Section 3 studies the truncated part giving conditions on the truncating sequence to enable the proof of some exponential inequalities for these terms. Section 4 treats the tails left aside from the truncation. Section 5 summarizes the partial results into some theorems and gives some applications.

2. Some lemmas and notations

Firstly, we quote two lemmas as follows.

Lemma 2.1 (see [6]). *Let $\{X_i, 1 \le i \le n\}$ be positively associated random variables bounded by a constant M. Then for any $\lambda > 0$,*

$$\left|E\left(\exp\left(\lambda\sum_{i=1}^{n}X_i\right)\right) - \prod_{i=1}^{n}E(\exp(\lambda X_i))\right| \le \lambda^2\exp(n\lambda M)\sum_{1\le i<j\le n}\operatorname{Cov}(X_i, X_j). \tag{2.1}$$

Lemma 2.2 (see [7]). *Let $\{X_i,\ i \ge 1\}$ be a positively associated sequence with zero mean and*

$$\sum_{i=1}^{\infty} v^{1/2}(2^i) < \infty, \tag{2.2}$$

where $v(n) = \sup_{i\ge 1}\sum_{j:j-i\ge n}\operatorname{Cov}^{1/2}(X_i, X_j)$. Then there exists a positive constant C such that

$$E\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|^2 \le Cn\left\{\sup_{i\ge 1}EX_i^2 + \left(\sup_{i\ge 1}EX_i^2\right)^{1/2}\right\}. \tag{2.3}$$

Remark 2.3 (see condition (2.2) is quite weak). In fact, it is satisfied only if $v(n) \le C(\log n)^{-2}(\log\log n)^{-2-\xi}$ for some $\xi > 0$. So it is weaker than the corresponding condition in [1, 2].

For the formulation of the assumptions to be made in this paper, some notations are required. Thus let $c_n,\ n \geq 1$ be a sequence of nonnegative real numbers such that $c_n \to \infty$ and $u(n) = \sup_{i\geq 1}\sum_{j:j-i\geq n}\operatorname{Cov}(X_i, X_j)$. Also, for convenience, we define X_{ni} by $X_{ni} = X_i$ for $1 \leq i \leq n$ and $X_{ni} = 0$ for $i > n$, and let

$$X_{1,i,n} = -\frac{c_n}{2}I_{(-\infty,-c_n/2)}(X_{ni}) + X_{ni}I_{(-c_n/2,c_n/2)}(X_{ni}) + \frac{c_n}{2}I_{(c_n/2,+\infty)}(X_{ni}), \tag{2.4}$$

$$X_{2,i,n} = \left(X_{ni} - \frac{c_n}{2}\right)I_{(c_n/2,+\infty)}(X_{ni}), \qquad X_{3,i,n} = \left(X_{ni} + \frac{c_n}{2}\right)I_{(-\infty,-c_n/2)}(X_{ni}), \tag{2.5}$$

for each $n, i \geq 1$, where I_A represents the characteristic function of the set A. Consider now a sequence of natural numbers p_n such that for each $n \geq 1,\ p_n < n/2$, and set $r_n = [n/(2p_n)] + 1$. Define, then,

$$Y_{q,j,n} = \sum_{i=2(j-1)p_n+1}^{2(j-1)p_n+p_n}(X_{q,i,n} - E(X_{q,i,n})), \qquad Z_{q,j,n} = \sum_{i=2(j-1)p_n+p_n+1}^{2jp_n}(X_{q,i,n} - E(X_{q,i,n})), \tag{2.6}$$

for $q = 1,2,3,\ j = 1,2,\dots,r_n$, and

$$S_{q,n,od} = \sum_{j=1}^{r_n} Y_{q,j,n}, \qquad S_{q,n,ev} = \sum_{j=1}^{r_n} Z_{q,j,n}. \tag{2.7}$$

Clearly, $n \leq 2r_np_n < 2n$.

The proofs given later will be divided into the control of the bounded terms that correspond to the index $q = 1$ and the control of the unbounded terms, corresponding to the indices $q = 2, 3$.

3. Control of the bounded terms

In this section, we will work hard to control the bounded terms. For this purpose, we give some lemmas as follows.

Lemma 3.1. *Let $\{X_i,\ i \geq 1\}$ be a positively associated sequence. Then on account of definitions* (2.5), (2.6), (2.7), *and for every $\lambda > 0$,*

$$\begin{gathered}\left|E(\exp(\lambda S_{1,n,od})) - \prod_{j=1}^{r_n}E(\exp(\lambda Y_{1,j,n}))\right| \leq \lambda^2 nu(p_n)\exp(\lambda nc_n),\\ \left|E(\exp(\lambda S_{1,n,ev})) - \prod_{j=1}^{r_n}E(\exp(\lambda Z_{1,j,n}))\right| \leq \lambda^2 nu(p_n)\exp(\lambda nc_n).\end{gathered} \tag{3.1}$$

Proof. Similarly to the proof of Lemma 3.2 in [4], it is omitted here. □

Lemma 3.2. *Let $\{X_i,\ i \geq 1\}$ be a positively associated sequence and let* (2.2) *hold. If $0 < \lambda p_n c_n \leq 1$ for $\lambda > 0$, then*

$$\prod_{j=1}^{r_n} E(\exp(\lambda Y_{1,j,n})) \leq \exp(C_1 \lambda^2 n c_n^2), \tag{3.2}$$

$$\prod_{j=1}^{r_n} E(\exp(\lambda Z_{1,j,n})) \leq \exp(C_1 \lambda^2 n c_n^2), \tag{3.3}$$

where C_1 is a constant, not depending on n.

Proof. Since $EY_{1,j,n} = 0$ and $0 < \lambda p_n c_n \leq 1$, we may have

$$\begin{aligned} E(\exp(\lambda Y_{1,j,n})) &= \sum_{k=0}^{\infty} \frac{E(\lambda Y_{1,j,n})^k}{k!} = 1 + \sum_{k=2}^{\infty} \frac{E(\lambda Y_{1,j,n})^k}{k!} \\ &\leq 1 + E(\lambda Y_{1,j,n})^2 \sum_{k=2}^{\infty} \frac{1}{k!} \leq 1 + \lambda^2 E Y_{1,j,n}^2 \leq \exp(\lambda^2 E Y_{1,j,n}^2). \end{aligned} \tag{3.4}$$

By this, Lemma 2.2 and $|X_{1,i,n}| \leq c_n/2$,

$$\begin{aligned} \prod_{j=1}^{r_n} E(\exp(\lambda Y_{1,j,n})) &\leq \exp\left(\lambda^2 \sum_{j=1}^{r_n} E Y_{1,j,n}^2 \right) \\ &\leq \exp\left(C \lambda^2 p_n \sum_{j=1}^{r_n} \left\{ \sup_{i \geq 1} \operatorname{Var}(X_{1,i,n}) + \left(\sup_{i \geq 1} \operatorname{Var}(X_{1,i,n}) \right)^{1/2} \right\} \right) \\ &\leq \exp\left(C \lambda^2 p_n \sum_{j=1}^{r_n} \left\{ \sup_{i \geq 1} E X_{1,i,n}^2 + \left(\sup_{i \geq 1} E X_{1,i,n}^2 \right)^{1/2} \right\} \right) \\ &\leq \exp(C \lambda^2 r_n p_n (c_n/2)^2) \leq \exp(C_1 \lambda^2 n c_n^2) \end{aligned} \tag{3.5}$$

as desired. The proof is completed. □

Remark 3.3. The upper bound of [4, Lemma 3.1] is $\exp(\lambda^2 n p_n c_n^2)$, and so the upper bound of Lemma 3.1 is much sharper than that of [4] when $p_n \to \infty$, this is the reason why we choose the condition $0 < \lambda p_n c_n \leq 1$, which is equivalent to $0 < \lambda \leq 1/(p_n c_n)$ and enables us to get the desired upper bound by Lemma 2.2.

Combining Lemmas 3.1 and 3.2 yields easily the following result.

Lemma 3.4. *Let $\{X_i,\ i \geq 1\}$ be a positively associated sequence and let* (2.2) *hold. If $0 < \lambda p_n c_n \leq 1$ for $\lambda > 0$, then for any $\varepsilon > 0$,*

$$P\left(\left| \sum_{i=1}^{n} (X_{1,i,n} - EX_{1,i,n}) \right| > n\varepsilon \right) \leq 4\{\lambda^2 n u(p_n) e^{\lambda n c_n} + e^{C_1 \lambda^2 n c_n^2}\} e^{-n\lambda\varepsilon/2}, \tag{3.6}$$

where $X_{1,i,n}$ and C_1 are just as in (2.5) *and* (3.2).

By Lemma 3.4, one can show a result as follows.

Theorem 3.5. *Let $\{X_i,\ i \geq 1\}$ be a positively associated sequence and let* (2.2) *hold. Suppose that $p_n \leq n/\alpha \log n$ for some $\alpha > 0$, $p_n \to \infty$, and*

$$\frac{\log n}{n^{2\alpha/3} p_n c_n^2} \exp\left\{\left(\frac{\alpha n \log n}{p_n}\right)^{1/2}\right\} u(p_n) \leq C_0 < \infty, \tag{3.7}$$

where C_0 is a constant which does not depend on n. Set $\varepsilon_n = (10/3)(\alpha p_n c_n^2 \log n/n)^{1/2}$. Then there exists a positive constant C_2, which only depends on $\alpha > 0$, such that

$$P\left(\left|\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right| > n\varepsilon_n\right) \leq C_2 \exp(-\alpha \log n). \tag{3.8}$$

Proof. Let $\lambda = 10\alpha \log n/3n\varepsilon_n = (\alpha \log n/np_n c_n^2)^{1/2}$ and $\varepsilon = \varepsilon_n$ in Lemma 3.4. Then it is obvious that $\lambda p_n c_n \leq 1$ from $p_n \leq n/\alpha \log n$ and that

$$e^{-n\lambda\varepsilon_n/2} = e^{-(5/3)\alpha \log n}. \tag{3.9}$$

Noting that $p_n \to \infty$, we may have

$$e^{C_1 \lambda^2 n c_n^2} = \exp\left(\frac{C_1 \alpha \log n}{p_n}\right) \leq \exp\left(\frac{2}{3}\alpha \log n\right), \tag{3.10}$$

$$\begin{aligned} \lambda^2 n u(p_n) e^{\lambda n c_n} &= \frac{\alpha \log n}{p_n c_n^2} \exp\left\{\left(\frac{\alpha n \log n}{p_n}\right)^{1/2}\right\} u(p_n) \\ &\leq C_2 n^{2\alpha/3} = C_2 \exp\left(\frac{2}{3}\alpha \log n\right) \end{aligned} \tag{3.11}$$

by (3.7). Combining (3.9)–(3.11), we can get (3.8) by Lemma 3.4. The proof is completed. □

Remark 3.6. (1) Let us compare Theorem 3.5 with [4, Theorem 3.6]. Our result drops the strict stationarity of the positively associated random variables; and to obtain (3.8), Oliveira [4] used the following condition:

$$\frac{\log n}{p_n c_n^2} \exp\left\{\left(\frac{\alpha n \log n}{p_n}\right)^{1/2}\right\} u(p_n) \leq C_0 < \infty. \tag{3.12}$$

Obviously, (3.7) is weaker than (3.12).

(2) Although Theorem 3.5 holds under weaker conditions, it cannot make us get a much faster convergence rate for the almost sure convergence to zero of $\sum_{i=1}^{n}(X_i - EX_i)/n$ than the one of convergence in [4]. This is because $\varepsilon_n = (10/3)(\alpha p_n c_n^2 \log n/n)^{1/2}$, preventing us from getting the convergence rate $n^{-1/2}(\log\log n)^{1/2}(\log n)^2$ for the case of geometrically decreasing covariances. So to obtain the above rate, we show another exponential inequality (3.20) in which $\varepsilon_n = p_n c_n \sqrt{\log\log n}\log n/2n$, permitting us to get the desired rate when we use condition (3.19) instead of condition (3.7), which is weaker than condition (3.19) for the case $\alpha > 2/3$, $0 < \delta < 1/2$, and $p_n \leq (4+3\delta)^2 n/\alpha\epsilon^2 \log n \log\log n$.

Now, let us consider (3.8) again. By Borel-Cantelli lemma, we need $\sum_{n=1}^{\infty} e^{-\alpha \log n} < \infty$ for some $\alpha > 0$ in order to get strong law of large numbers. However, it is not true for $0 < \alpha \leq 1$. To avoid this case, we show another exponential inequality.

Theorem 3.7. *Let $\{X_i,\ i \geq 1\}$ be a positively associated sequence and let (2.2) hold. Assume that $\{\varepsilon_n : n \geq 1\}$ is a positive real sequence which satisfies*

$$\frac{p_n c_n \log n}{n\varepsilon_n} \longrightarrow 0, \qquad \frac{c_n^2 \log n}{n\varepsilon_n^2} \longrightarrow 0, \tag{3.13}$$

and for some $\epsilon > 0$ and $\delta > 0$,

$$n^{-(1+2\delta)} \left(\frac{\log n}{\varepsilon_n}\right)^2 \exp\left(\frac{2(1+3\delta)c_n \log n}{\varepsilon_n \epsilon}\right) u(p_n) \leq C_0 < \infty. \tag{3.14}$$

Then there exists a positive constant C, which depends on $\epsilon > 0$ and $\delta > 0$, such that

$$P\left(\left|\sum_{i=1}^{n} (X_{1,i,n} - EX_{1,i,n})\right| > n\varepsilon_n \epsilon\right) \leq C \exp(-(1+\delta)\log n). \tag{3.15}$$

Proof. Let $\lambda = 2(1+3\delta)\log n/n\varepsilon_n\epsilon$ and let $\varepsilon = \varepsilon_n\epsilon$ in Lemma 3.4. Then it is obvious that $\lambda p_n c_n \leq 1$ from (3.13) and that

$$e^{-n\lambda\varepsilon/2} = e^{-n\lambda\varepsilon_n\epsilon/2} = e^{-(1+3\delta)\log n}. \tag{3.16}$$

Also, we can get that

$$e^{C_1\lambda^2 n c_n^2} = \exp\left(\frac{C_1 4(1+3\delta)^2 c_n^2 \log n}{\epsilon^2 n\varepsilon_n^2}\right) \leq \exp(2\delta \log n) \tag{3.17}$$

by (3.13), and that

$$\begin{aligned} \lambda^2 n u(p_n) e^{\lambda n c_n} &= \left(\frac{2(1+3\delta)}{\epsilon}\right)^2 \left(\frac{\log n}{\varepsilon_n}\right)^2 n^{-1} \exp\left(\frac{2(1+3\delta)c_n \log n}{\varepsilon_n \epsilon}\right) u(p_n) \\ &\leq C n^{2\delta} = C \exp(2\delta \log n) \end{aligned} \tag{3.18}$$

by (3.14). Combining (3.16)–(3.18), we can obtain (3.15) by Lemma 3.4.

Taking $\varepsilon_n = p_n c_n \sqrt{\log\log n \log n/2n}$ in Theorem 3.7, we can get easily the following result. □

Corollary 3.8. *Let $\{X_i,\ i \geq 1\}$ be a positively associated sequence and let (2.2) hold. Suppose that p_n satisfies $\sqrt{n/\log n} \leq p_n < n/2$ and for some $\epsilon > 0$ and $\delta > 0$,*

$$\frac{n^{1-2\delta}}{p_n^2 c_n^2 \log\log n} \exp\left(\frac{4(1+3\delta)n}{\epsilon p_n \sqrt{\log\log n}}\right) u(p_n) \leq C_0 < \infty. \tag{3.19}$$

Then there exists a positive constant C_3, which depends on $\epsilon > 0$ and $\delta > 0$, such that

$$P\left(\left|\sum_{i=1}^{n}(X_{1,i,n}-EX_{1,i,n})\right| > \epsilon p_n c_n \sqrt{\log\log n}\log n\right) \le C_3 \exp\left(-(1+\delta)\log n\right). \tag{3.20}$$

4. Control of the unbounded terms

In this section, we will try ourselves to control the unbounded terms. Firstly, it is obvious that the variables $X_{2,i,n}$ and $X_{3,i,n}$ are positively associated but not bounded, even for fixed n. This means that Lemma 3.1 cannot be applied to the sum of such terms. While we may note that these variables depend only on the tails of distribution of the original variables. Hence by controlling the decrease rate of these tails, we may give some exponential inequalities for the sums of $X_{2,i,n}$ or $X_{3,i,n}$. The results we get are listed below.

Lemma 4.1. *Let $\{X_i,\ i \ge 1\}$ be a positively associated sequence that satisfies*

$$\sup_{i\ge 1,|t|\le\omega} E\left(e^{tX_i}\right) \le M_\omega < \infty \tag{4.1}$$

for some $\omega > 0$ and let (2.2) hold. Then for $0 < t \le \omega$,

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(X_{q,i,n}-EX_{q,i,n})\right| > n\varepsilon\right) \le \frac{C\sqrt{2M_\omega e^{-tc_n/2}}}{nt\varepsilon^2}, \quad q=2,3. \tag{4.2}$$

Proof. Firstly, let us estimate $EX_{q,i,n}^2$. Without loss of generality, set $q = 2$. We will assume $F(x) = P(X_i > x)$. Then by Markov's inequality and $\sup_{i\ge 1,|t|\le\omega} E(e^{tX_i}) \le M_\omega < \infty$ for some $\omega > 0$, it follows that, for $0 < t \le \omega$,

$$F(x) \le e^{-tx}E\left(e^{tX_i}\right) \le M_\omega e^{-tx}. \tag{4.3}$$

Writing the mathematical expectation as a Stieltjes integral and integrating by parts, we have

$$\begin{aligned}
EX_{2,i,n}^2 &= -\int_{(c_n/2,+\infty)}\left(x-\frac{c_n}{2}\right)^2 dF(x)\\
&= -\left(x-\frac{c_n}{2}\right)^2 F(x)\Big|_{c_n/2}^{+\infty} + \int_{(c_n/2,+\infty)} 2\left(x-\frac{c_n}{2}\right)F(x)dx\\
&= -\lim_{x\to+\infty}\left(x-\frac{c_n}{2}\right)^2 F(x) + \int_{(c_n/2,+\infty)} 2\left(x-\frac{c_n}{2}\right)F(x)dx\\
&= \int_{(c_n/2,+\infty)} 2\left(x-\frac{c_n}{2}\right)F(x)dx\\
&\le 2M_\omega\int_{(c_n/2,+\infty)}\left(x-\frac{c_n}{2}\right)e^{-tx}dx\\
&= 2M_\omega\frac{e^{-tc_n/2}}{t^2}
\end{aligned} \tag{4.4}$$

by the inequality stated earlier. Hence using (4.4) and Lemma 2.2, we have, for n large enough,

$$\begin{aligned} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_{2,i,n}-EX_{2,i,n}\right)\right|>n\varepsilon\right) &\le \frac{E\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_{2,i,n}-EX_{2,i,n}\right)\right|^2}{n^2\varepsilon^2} \\ &\le \frac{Cn\left\{\sup_{i\ge 1}\operatorname{Var}\left(X_{2,i,n}\right)+\left(\sup_{i\ge 1}\operatorname{Var}\left(X_{2,i,n}\right)\right)^{1/2}\right\}}{n^2\varepsilon^2} \\ &\le \frac{C\left\{\sup_{i\ge 1}EX_{2,i,n}^2+\left(\sup_{i\ge 1}EX_{2,i,n}^2\right)^{1/2}\right\}}{n\varepsilon^2} \\ &\le \frac{C\sqrt{2M_\omega e^{-tc_n/2}}}{nt\varepsilon^2}. \end{aligned} \tag{4.5}$$

This completes the proof of the lemma. □

Remark 4.2. Let $\{X_i,\ i\ge 1\}$ be a positively associated sequence and let (2.2) hold (as mentioned above, it is a quite weak condition). Then Lemma 4.1 improves the corresponding result in [4] from the following aspects.

(i) The assumption of the stationarity of $\{X_i,\ i\ge 1\}$ is dropped.

(ii) The sum in (4.2) is

$$\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_{q,i,n}-EX_{q,i,n}\right)\right|,\ \text{not}\ \left|\sum_{i=1}^{n}\left(X_{q,i,n}-EX_{q,i,n}\right)\right|\ \text{in [4]}. \tag{4.6}$$

(iii) The upper bound of the exponential inequality in [4, Lemma 4.1] is $2M_\omega ne^{-t\widetilde{c_n}}/t^2\varepsilon^2$, where $\widetilde{c_n}\to\infty$. So, assuming $c_n=4\widetilde{c_n}$ in the inequality (4.2), we can obtain that the upper bound of our inequality is $C\sqrt{2M_\omega}e^{-t\widetilde{c_n}}/nt^2\varepsilon^2$. Obviously, $C\sqrt{2M_\omega}e^{-t\widetilde{c_n}}/nt^2\varepsilon^2\le 2M_\omega ne^{-t\widetilde{c_n}}/t^2\varepsilon^2$ for sufficiently large n. That is, the upper bound in Lemma 4.1 is much lower than that of [4, Lemma 4.1].

Applying Lemma 4.1, one can get immediately the following result by taking values for t and c_n.

Corollary 4.3. *Let $\{X_i,\ i\ge 1\}$ be a positively associated sequence that satisfies* $\sup_{i\ge 1,|t|\le\omega}E(e^{tX_i})\le M_\omega<\infty$ *for some $\omega>0$ and let* (2.2) *hold. Then*

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_{q,i,n}-EX_{q,i,n}\right)\right|>n\varepsilon\right)\le\frac{C\sqrt{2M_\omega}}{2\alpha n\varepsilon^2}\exp(-\alpha\log n),\qquad q=2,3, \tag{4.7}$$

provided $t=2\alpha$ and $c_n=2\log n$, and

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_{q,i,n}-EX_{q,i,n}\right)\right|>n\varepsilon\right)\le\frac{C\sqrt{2M_\omega}}{2\alpha n\varepsilon^2}\exp\left(-(1+\delta)\log n\right),\qquad q=2,3, \tag{4.8}$$

provided $t=2\alpha$ and $c_n=(2(1+\delta)/\alpha)\log n$, where α and δ are as in (3.8) *and* (3.13).

5. Strong convergences and rates

This section summarizes the results stated earlier. In addition, we give a convergence rate for geometrically decreasing covariances, which improves the relevant one obtained by [4].

Theorem 5.1. *Let* $\{X_i,\ i \geq 1\}$ *be a positively associated sequence satisfying*

$$\frac{1}{n^{2\alpha/3} p_n \log n} \exp\left\{ \left(\frac{\alpha n \log n}{p_n} \right)^{1/2} \right\} u(p_n) \leq C_0 < \infty \tag{5.1}$$

for some $\alpha > 0$, $n/\alpha \log n \geq p_n \to \infty$ *and let* (2.2) *hold. Suppose that* ε_n *is as in Theorem 3.5 and there exists* $\omega > \alpha$ *that satisfies* $\sup_{i\geq 1, |t|\leq\omega} E(e^{tX_i}) \leq M_\omega < \infty$. *Then for sufficiently large* n,

$$P\left(\left| \sum_{i=1}^{n} (X_i - EX_i) \right| > 3n\varepsilon_n \right) \leq \left(C_2 + \frac{9C\sqrt{2M_\omega}}{200\alpha^2 p_n \log^3 n} \right) \exp(-\alpha \log n). \tag{5.2}$$

Proof. Combining Theorem 3.5 and Corollary 4.3 yields the desired result (5.2). □

Remark 5.2. Theorem 5.1 improves [4, Theorem 5.1], because the latter uses the following more restrictive conditions.

(i) $\{X_i,\ i \geq 1\}$ is a strictly stationary sequence.

(ii) $\{X_i,\ i \geq 1\}$ satisfies $(1/p_n \log n) \exp\{(\alpha n \log n/p_n)^{1/2}\} u(p_n) \leq C_0 < \infty$. Clearly, it implies (5.1).

(iii) The latter has a higher upper bound than our result, because $9C\sqrt{2M_\omega}/200\alpha^2 p_n \log^3 n \leq 2M_\omega n^2/9\alpha^3 p_n \log^3 n$ for sufficiently large n.

Combining Corollaries 3.8 and 4.3, we may get easily the following result.

Theorem 5.3. *Let* $\{X_i,\ i \geq 1\}$ *be a positively associated sequence satisfying* (3.19) *for* $\sqrt{n/\log n} \leq p_n < n/2$, *some* $\epsilon > 0$, *and* $\delta > 0$ *and let* (2.2) *hold. Suppose that* $\sup_{i\geq 1, |t|\leq\omega} E(e^{tX_i}) \leq M_\omega < \infty$ *for some* $\omega > \alpha$. *Then for* n *large enough,*

$$P\left(\left| \sum_{i=1}^{n} (X_i - EX_i) \right| > 3n\epsilon_n \right) \leq \left(C_3 + \frac{C\sqrt{2M_\omega}}{2\alpha n \epsilon_n^2} \right) \exp\big(-(1+\delta) \log n \big), \tag{5.3}$$

where $\epsilon_n = \epsilon p_n c_n \sqrt{\log\log n} \log n/n$ *and* $c_n = (2(1+\delta)/\alpha) \log n$.

Applying Theorem 5.3, one may have immediately some strong laws of large numbers by taking $p_n = [\sqrt{n}]$ *and* $p_n = [n/4]$, *respectively.*

Corollary 5.4. *Let* $\{X_i,\ i \geq 1\}$ *be a positively associated sequence which satisfies* $\sup_{i\geq 1, |t|\leq\omega} E(e^{tX_i}) \leq M_\omega < \infty$ *for some* $\omega > \alpha$. *Then*

$$\frac{\sum_{i=1}^{n} (X_i - EX_i)}{\sqrt{n \log\log n} \log^2 n} \longrightarrow 0, \ a.s., \tag{5.4}$$

provided that

$$\frac{\exp(\alpha\sqrt{n})u([\sqrt{n}])}{n^{2\delta}\log^2 n\log\log n} \le C < \infty \quad \text{for some } \alpha > 0,\ \delta > 0, \tag{5.5}$$

and (2.2) *holds; and*

$$\frac{\sum_{i=1}^{n}(X_i - EX_i)}{n\sqrt{\log\log n}\log^2 n} \longrightarrow 0, \ a.s., \tag{5.6}$$

provided that

$$\frac{u([n/4])}{n^{1+2\delta}\log^2 n\log\log n} \le C < \infty \quad \text{for some } \delta > 0, \tag{5.7}$$

and (2.2) *holds.*

Finally, one gives some applications of Corollary 5.4.

(1) Suppose now $\mathrm{Cov}(X_i, X_j) = C\rho^{|i-j|}$ *for some* $0 < \rho < 1$. *Then* $v([\sqrt{n}]) \sim C\rho^{\sqrt{n}/2}$ *and* $u([\sqrt{n}]) \sim C\rho^{\sqrt{n}}$, *so* (2.2) *is satisfied and*

$$\exp(\alpha\sqrt{n})u([\sqrt{n}]) \sim C(\rho e^{\alpha})^{\sqrt{n}} \longrightarrow 0 \tag{5.8}$$

by choosing $\alpha > 0$ *with* $0 < \rho e^{\alpha} < 1$. *This means that one requires only* $0 < \alpha < -\log\rho$, *not* $\alpha > 8/3$ *in [4]. It is due to* Lemma 4.1. *By* (5.8), *one knows that* (5.5) *holds. Hence one gets finally that* $\sum_{i=1}^{n}(X_i - EX_i)/n \to 0$, *a.s., converges at the rate* $n^{-1/2}(\log\log n)^{1/2}\log^2 n$ *which closes to the optimal achievable convergence rate for independent random variables under the Hartman-Wintner law of the iterated logarithm. However, Oliveira [4] only got* $n^{-1/3}\log^{5/3} n$ *for the case mentioned above. Clearly, the convergence rate is much lower than ours.*

(2) If $\mathrm{Cov}(X_i, X_j) = C|j-i|^{-\tau}$ *for some* $\tau > 2$, *or* $\mathrm{Cov}(X_i, X_j) = C|j-i|^{-2}\log^{-\eta}|j-i|$ *for some* $\eta > 8$, *then it is clear that* (5.7) *and* (2.2) *can be satisfied. Therefore By* (5.6), *one does have almost sure convergence but without rates. The explicit reason could be seen in [4].*

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