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Research Article

Strong Convergence of an Iterative Method for Inverse Strongly Accretive Operators

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We study the strong convergence of an iterative method for inverse strongly accretive operators in the framework of Banach spaces. Our results improve and extend the corresponding results announced by many others.

1. Introduction and preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, C a nonempty closed convex subset of H, and A a monotone operator of C into H. The classical variational inequality problem is formulated as finding a point $x \in C$ such that

$$\langle y - x, Ax \rangle \geq 0 \tag{1.1}$$

for all $y \in C$. Such a point $x \in C$ is called a solution of the variational inequality (1.1). Next, the set of solutions of the variational inequality (1.1) is denoted by $\mathrm{VI}(C, A)$. In the case when $C = H$, $\mathrm{VI}(H, A) = A^{-1}0$ holds, where

$$A^{-1}0 = \{x \in H : Ax = 0\}. \tag{1.2}$$

Recall that an operator A of C into H is said to be inverse strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \tag{1.3}$$

for all $x, y \in C$ (see [1–4]). For such a case, A is said to be α-inverse strongly monotone.

Recall that $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1.4}$$

for all $x, y \in C$. It is known that if T is a nonexpansive mapping of C into itself, then $A = (I - T)$ is 1/2-inverse strongly monotone and $F(T) = \mathrm{VI}(C, A)$, where $F(T)$ denotes the set of fixed points of T.

Let P_C be the projection of H onto the convex subset C. It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \tag{1.5}$$

for $x, y \in H$. Moreover, $P_C x$ is characterized by the properties $P_C x \in C$ and $\langle x - P_C x, P_C x - y \rangle \geq 0$ for all $y \in C$.

One can see that the variational inequality problem (1.1) is equivalent to some fixed-point problem. The element $x \in C$ is a solution of the variational inequality (1.1) if and only if $x \in C$ satisfies the relation $x = P_C(x - \lambda A x)$, where $\lambda > 0$ is a constant.

To find a solution of the variational inequality for an inverse strongly monotone operator, Iiduka et al. [2] proved the following weak convergence theorem.

Theorem ITT. *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be an α-inverse strongly monotone operator of C into H with $VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P_C\left[\alpha_n x_n + (1 - \alpha_n) P_C(x_n - \lambda_n A x_n)\right] \end{aligned} \tag{1.6}$$

for all $n = 1, 2, \ldots$, where P_C is the metric projection from H onto C, $\{\alpha_n\}$ is a sequence in $[-1, 1]$, and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\alpha_n \in [a, b]$ for some a, b with $-1 < a < b < 1$ and $\lambda_n \in [c, d]$ for some c, d with $0 < c < d < 2(1 + a)\alpha$, then the sequence $\{x_n\}$ defined by (1.6) *converges weakly to some element of $VI(C, A)$.*

Next, we assume that C is a nonempty closed and convex subset of a Banach space E. Let E^* be the dual space of E and let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\} \tag{1.7}$$

for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|q\|^{q-2} J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$. Further, we have the following properties of the generalized duality mapping J_q:

(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in E$ with $x \neq 0$;

(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$;

(3) $J_q(-x) = -J_q(x)$ for all $x \in E$.

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \tag{1.8}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{1.9}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit (1.9) is attained uniformly for $x, y \in U$. The norm of E is said to be Fréchet differentiable if, for any $x \in U$, the limit (1.9) is attained uniformly for all $y \in U$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1 : x, y \in X,\ \|x\| = 1,\ \|y\| = \tau\right\}, \tag{1.10}$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0}(\rho(\tau)/\tau) = 0$. Let q be a fixed real number with $1 < q \le 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \le c\tau^q$ for all $\tau > 0$.

Note that

(1) E is a uniformly smooth Banach space if and only if J_q is single-valued and uniformly continuous on any bounded subset of E;

(2) all Hilbert spaces, L_p (or l_p) spaces ($p \ge 2$), and the Sobolev spaces, W_m^p ($p \ge 2$), are 2-uniformly smooth, while L_p (or l_p) and W_m^p spaces ($1 < p \le 2$) are p-uniformly smooth.

Recall that an operator A of C into E is said to be accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \ge 0 \tag{1.11}$$

for all $x, y \in C$.

For $\alpha > 0$, recall that an operator A of C into E is said to be α-inverse strongly accretive if

$$\langle Ax - Ay, J(x-y)\rangle \ge \alpha\|Ax - Ay\|^2 \tag{1.12}$$

for all $x, y \in C$. Evidently, the definition of the inverse strongly accretive operator is based on that of the inverse strongly monotone operator.

Let D be a subset of C and let Q be a mapping of C into D. Then Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx, \tag{1.13}$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \ge 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. We know the following lemma concerning sunny nonexpansive retraction.

Lemma 1.1 (see [5]). *Let C be a closed convex subset of a smooth Banach space E, let D be a nonempty subset of C, and let Q be a retraction from C onto D. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Pu, J(y - Pu)\rangle \le 0 \tag{1.14}$$

for all $u \in C$ and $y \in D$.

Recently, Aoyama et al. [6] first considered the following generalized variational inequality problem in a smooth Banach space. Let A be an accretive operator of C into E. Find a point $x \in C$ such that

$$\langle Ax, J(y-x)\rangle \geq 0 \tag{1.15}$$

for all $y \in C$. In order to find a solution of the variational inequality (1.15), the authors proved the following theorem in the framework of Banach spaces.

Theorem AIT. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$, and A an α-inverse strongly accretive operator of C into E with $S(C, A) \neq \emptyset$, where*

$$S(C,A) = \{x^* \in C : \langle Ax^*, J(x-x^*)\rangle \geq 0, \ x \in C\}. \tag{1.16}$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \alpha/K^2]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners:

$$\begin{gathered} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) Q_C(x_n - \lambda_n A x_n), \end{gathered} \tag{1.17}$$

converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E.

In this paper, motivated by Aoyama et al. [6], Iiduka et al. [2], Takahahsi and Toyoda [4], we introduce an iterative method to approximate a solution of variational inequality (1.15) for an α-inverse strongly accretive operators. Strong convergence theorems are obtained in the framework of Banach spaces under appropriate conditions on parameters.

We also need the following lemmas for proof of our main results.

Lemma 1.2 (see [7]). *Let q be a given real number with $1 < q \leq 2$ and let E be a q-uniformly smooth Banach space. Then*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q \tag{1.18}$$

for all $x, y \in X$, where K is the q-uniformly smoothness constant of E.

The following lemma is characterized by the set of solutions of variational inequality (1.15) by using sunny nonexpansive retractions.

Lemma 1.3 (see [6]). *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$S(C,A) = F(Q(I-\lambda A)). \tag{1.19}$$

Lemma 1.4 (see [8]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$, then x is a fixed point of T.*

Lemma 1.5 (see [9]). *Let $\{x_n\}$, $\{l_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1. \tag{1.20}$$

Suppose that

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) l_n \tag{1.21}$$

for all $n = 0,1,3,\dots$ and

$$\limsup_{n\to\infty} \left(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \right) \le 0. \tag{1.22}$$

Then $\lim_{n\to\infty} \|l_n - x_n\| = 0$.

Lemma 1.6 (see[10]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n) a_n + \delta_n \tag{1.23}$$

for all $n = 0,1,3,\dots$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} (\delta_n/\gamma_n) \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

2. Main results

Theorem 2.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $u \in C$ an arbitrarily fixed point, and A an α-inverse strongly accretive operator of C into E such that $S(C,A) \neq \varnothing$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ and let $\{\lambda_n\}$ a real number sequence in $[a, \alpha/K^2]$ for some $a > 0$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ defined by

$$\begin{gathered} x_0 \in C, \\ y_n = \beta_n x_n + (1-\beta_n) Q_C (I - \lambda_n A) x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad n \ge 0, \end{gathered} \tag{2.1}$$

converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of C onto $S(C,A)$.

Proof. First, we show that $I - \lambda_n A$ is nonexpansive for all $n \geq 0$. Indeed, for all $x, y \in C$ and $\lambda_n \in [a, \alpha/K^2]$, from Lemma 1.2, one has

$$\begin{aligned}
\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 &= \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_n\langle Ax - Ay, J(x-y)\rangle \\
&\quad + 2K^2\lambda_n^2\|Ax - Ay\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_n\alpha\|Ax - Ay\|^2 \\
&\quad + 2K^2\lambda_n^2\|Ax - Ay\|^2 \\
&= \|x-y\|^2 + 2\lambda_n(K^2\lambda_n - \alpha)\|Ax - Ay\|^2 \\
&\leq \|x-y\|^2.
\end{aligned} \tag{2.2}$$

Therefore, one obtains that $I-\lambda_n A$ is a nonexpansive mapping for all $n \geq 0$. For all $p \in S(C, A)$, it follows from Lemma 1.3 that $p = Q_C(I - \lambda_n A)p$. Put $\rho_n = Q_C(I - \lambda_n A)x_n$. Noticing that

$$\begin{aligned}
\|\rho_n - p\| &= \|Q_C(I-\lambda_n A)x_n - Q_C(I-\lambda_n A)p\| \\
&\leq \|(I-\lambda_n A)x_n - (I-\lambda_n A)p\| \\
&\leq \|x_n - p\|,
\end{aligned} \tag{2.3}$$

one has

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(\rho_n - p)\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\|\rho_n - p\| \\
&\leq \beta_n\|x - p\| + (1-\beta_n)\|x_n - p\| \\
&= \|x_n - p\|,
\end{aligned} \tag{2.4}$$

from which it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(u-p) + (1-\alpha_n)(y_n - p)\| \\
&\leq \alpha_n\|u-p\| + (1-\alpha_n)\|y_n - p\| \\
&\leq \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \max\{\|u-p\|, \|x_n - p\|\}.
\end{aligned} \tag{2.5}$$

Now, an induction yields

$$\|x_n - p\| \leq \max\{\|u-p\|, \|x_0 - p\|\}, \quad n \geq 0. \tag{2.6}$$

Hence, $\{x_n\}$ is bounded, and so is $\{y_n\}$. On the other hand, one has

$$\begin{aligned}
\|\rho_{n+1} - \rho_n\| &= \|Q_C(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - Q_C(x_n - \lambda_n Ax_n)\| \\
&\leq \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\
&= \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n) + (\lambda_n - \lambda_{n+1})Ax_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\|.
\end{aligned} \tag{2.7}$$

Put $l_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$, that is,

$$x_{n+1} = (1-\beta_n)l_n + \beta_n x_n, \quad n \geq 0. \tag{2.8}$$

Next, we compute $l_{n+1} - l_n$. Observing that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1}u + (1-\alpha_{n+1})y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\alpha_n)y_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}(u - y_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(u-y_n)}{1-\beta_n} + \rho_{n+1} - \rho_n, \end{aligned} \tag{2.9}$$

we have

$$\|l_{n+1} - l_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - u\| + \|\rho_{n+1} - \rho_n\|. \tag{2.10}$$

Combining (2.7) with (2.10), one obtains

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - u\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\|. \tag{2.11}$$

It follows that

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.12}$$

Hence, from Lemma 1.5, we obtain $\lim_{n\to\infty}\|l_n - x_n\| = 0$. From (2.7) and the condition (ii), one arrives at

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.13}$$

On the other hand, from (2.1), one has

$$x_{n+1} - x_n = \alpha_n(u - x_n) + (1-\alpha_n)(1-\beta_n)(\rho_n - x_n), \tag{2.14}$$

which combines with (2.13), and from the conditions (i), (ii), one sees that

$$\lim_{n\to\infty}\|\rho_n - x_n\| = 0. \tag{2.15}$$

Next, we show that

$$\limsup_{n\to\infty}\langle u - Q'u, J(x_n - Q'u)\rangle \leq 0. \tag{2.16}$$

To show (2.16), we choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to x such that

$$\limsup_{n\to\infty}\langle u - Q'u, J(x_n - Q'u)\rangle = \lim_{i\to\infty}\langle u - Q'u, J(x_{n,i} - Q'u)\rangle. \tag{2.17}$$

Next, we prove that $x \in S(C, A)$. Since $\lambda_n \in [a, \alpha/K^2]$ for some $a > 0$, it follows that $\{\lambda_{n_i}\}$ is bounded and so there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ which converges to $\lambda_0 \in [a, \alpha/K^2]$. We may assume, without loss of generality, that $\lambda_{n_i} \to \lambda_0$. Since Q_C is nonexpansive, it follows from $y_{n_i} = Q_C(x_{n_i} - \lambda_{n_i} A x_{n_i})$ that

$$\begin{aligned}\left\|Q_C(x_{n_i} - \lambda_0 A x_{n_i}) - x_{n_i}\right\| &\le \left\|Q_C(x_{n_i} - \lambda_0 A x_{n_i}) - \rho_{n_i}\right\| + \left\|\rho_{n_i} - x_{n_i}\right\| \\ &\le \left\|(x_{n_i} - \lambda_0 A x_{n_i}) - (x_{n_i} - \lambda_{n_i} A x_{n_i})\right\| + \left\|\rho_{n_i} - x_{n_i}\right\| \\ &\le \left|\lambda_{n_i} - \lambda_0\right| \left\|A x_{n_i}\right\| + \left\|\rho_{n_i} - x_{n_i}\right\|.\end{aligned} \tag{2.18}$$

It follows from (2.15) that

$$\lim_{i\to\infty}\left\|Q_C(I - \lambda_0 A)x_{n_i} - x_{n_i}\right\| = 0. \tag{2.19}$$

From Lemma 1.4, we have $x \in F(Q_C(I - \lambda_0 A))$. It follows from Lemma 1.3 that $x \in S(C, A)$. Now, from (2.17) and Lemma 1.1, we have

$$\limsup_{n\to\infty}\langle u - Q'u, J(x_n - Q'u)\rangle = \lim_{i\to\infty}\langle u - Q'u, J(x_{n_i} - Q'u)\rangle = \langle u - Q'u, J(x - Q'u)\rangle \le 0. \tag{2.20}$$

From (2.1), we have

$$\begin{aligned}\left\|x_{n+1} - Q'u\right\|^2 &= \alpha_n\langle u - Q'u, J(x_{n+1} - Q'u)\rangle + (1 - \alpha_n)\langle y_n - Q'u, J(x_{n+1} - Q'u)\rangle \\ &\le \alpha_n\langle u - Q'u, J(x_{n+1} - Q'u)\rangle + \frac{1 - \alpha_n}{2}\left(\left\|y_n - Q'u\right\|^2 + \left\|x_{n+1} - Q'u\right\|^2\right) \\ &\le \alpha_n\langle u - Q'u, J(x_{n+1} - Q'u)\rangle + \frac{1 - \alpha_n}{2}\left(\left\|x_n - Q'u\right\|^2 + \left\|x_{n+1} - Q'u\right\|^2\right).\end{aligned} \tag{2.21}$$

It follows that

$$\left\|x_{n+1} - Q'u\right\|^2 \le (1 - \alpha_n)\left\|x_n - Q'u\right\|^2 + 2\alpha_n\langle u - Q'u, J(x_{n+1} - Q'u)\rangle. \tag{2.22}$$

Applying Lemma 1.6 to (2.22), we can conclude the desired conclusion. This completes the proof. □

As an application of Theorem 2.1, we have the following results in the framework of Hilbert spaces.

Corollary 2.2. *Let H be a Hilbert space and C a nonempty closed convex subset of H. Let P_C be a metric projection from H onto C, $u \in C$ an arbitrarily fixed point, and A an α-inverse strongly monotone operator of C into H such that $VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$be two sequences in $(0, 1)$*

and let $\{\lambda_n\}$ *be a real number sequence in* $[a, 2\alpha]$ *for some* $a > 0$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ *defined by*

$$\begin{gathered} x_0 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) P_C (I - \lambda_n A) x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \end{gathered} \tag{2.23}$$

converges strongly to Pu.

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