Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 546829, 14 pages
doi:10.1155/2008/546829

Research Article

On Some Extensions of Hardy-Hilbert's Inequality and Applications

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Received 23 October 2007; Accepted 2 January 2008

Recommended by Shusen Ding

By introducing some parameters we establish an extension of Hardy-Hilbert's integral inequality and the corresponding inequality for series. As an application, the reverses, some particular results and their equivalent forms are considered.

1. Introduction

If $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$, and $0 < \int_0^\infty g^2(x)dx < \infty$, then (see [1])

$$\iint_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \pi\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.1}$$

$$\iint_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dx\,dy < 4\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.2}$$

where the constant factors π and 4 are the best possible in (1.1) and (1.2), respectively. Inequality (1.1) is called Hilbert's integral inequality and (1.2) is called Hilbert's type which have been extended by Hardy (see [2]) as follows: if $p > 1$, $1/p + 1/q = 1$, $f(x), g(x) > 0$, $0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\iint_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(x)dx\right\}^{1/q}, \tag{1.3}$$

$$\iint_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dx\,dy < pq\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(x)dx\right\}^{1/q}, \tag{1.4}$$

where the constant factors $\pi/\sin(\pi/p)$ and pq are the best possible in (1.3) and (1.4), respectively. Hardy-Hilbert's inequality and its applications are important in analysis (see [3]). Recently, Yang [4] gave some generalizations and the reverse form of (1.3) as follows: if $p > 1$, $1/p + 1/q = 1$, $r > 1$, $1/r + 1/s = 1$, $\lambda > 0$, $f(x), g(x) \geq 0$, $0 < \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x)dx < \infty$, and $0 < \int_0^\infty x^{q(1-\lambda/s)-1} g^q(x)dx < \infty$, then

$$\iint_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx\, dy < \frac{\pi}{\lambda \sin(\pi/r)} \left\{ \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x)dx \right\}^{1/p} \left\{ \iint_0^\infty x^{q(1-\lambda/s)-1} g^q(x)dx \right\}^{1/q}, \tag{1.5}$$

where the constant factor $\pi/\lambda \sin(\pi/r)$ is the best possible.

The corresponding inequalities for series (1.3) and (1.4) are

$$\begin{gathered} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=0}^\infty a_n^p \right\}^{1/p} \left\{ \sum_{n=0}^\infty b_n^q \right\}^{1/q}, \\ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < pq \left\{ \sum_{n=0}^\infty a_n^p \right\}^{1/p} \left\{ \sum_{n=0}^\infty b_n^q \right\}^{1/q}, \end{gathered} \tag{1.6}$$

where the sequences $\{a_n\}$ and $\{b_n\}$ are such that $0 < \sum_{n=0}^\infty a_n^p < \infty$, $0 < \sum_{n=0}^\infty a_n^q < \infty$, and the constant factor $\pi/\sin(\pi/p)$ and pq are the best possible. By introducing a parameter $0 < \lambda \leq 2$, some extensions of (1.6) ($p = q = 2$) were given by Yang [5, 6] as

$$\begin{gathered} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda} \left\{ \sum_{n=0}^\infty n^{1-\lambda} a_n^2 \right\}^{1/2} \left\{ \sum_{n=0}^\infty n^{1-\lambda} b_n^2 \right\}^{1/2}, \\ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \frac{4}{\lambda} \left\{ \sum_{n=0}^\infty n^{1-\lambda} a_n^2 \right\}^{1/2} \left\{ \sum_{n=0}^\infty n^{1-\lambda} b_n^2 \right\}^{1/2}. \end{gathered} \tag{1.7}$$

Very recently, in [7] the following extensions were given:

$$\begin{gathered} \iint_0^\infty \frac{f(x)g(x)}{A\min\{x,y\} + B\max\{x,y\}} dx\, dy < D(A,B) \left\{ \int_0^\infty f^2(x)dx \right\}^{1/2} \left\{ \int_0^\infty g^2(x)dx \right\}^{1/2}, \\ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{A\min\{m,n\} + B\max\{m,n\}} < D(A,B) \left\{ \sum_{n=1}^\infty a_n^2 \right\}^{1/2} \left\{ \sum_{n=1}^\infty b_n^2 \right\}^{1/2}, \end{gathered} \tag{1.8}$$

where the constant factor $D(A,B)$ (see [7, Lemma 2.1]) is the best possible in both inequalities. For more information related to this subject see, for example, [8, 9].

In this paper by introducing some parameters, we generalize (1.8) and we obtain the reverse form for each of them. Some particular results and the equivalent form are also considered.

2. Main results

Lemma 2.1. *Suppose that* $\lambda > 0$, $A \geq 0$, $B > 0$. *Define the weight coefficients* $\omega_\lambda(A,B,x)$ *and* $\omega_\lambda(A,B,y)$ *by*

$$\omega_\lambda(A,B,x) := \int_0^\infty \frac{x^{\lambda/2} y^{-1+\lambda/2}}{A\min\{x^\lambda, y^\lambda\} + B\max\{x^\lambda, y^\lambda\}} dy, \tag{2.1}$$

$$\omega_\lambda(A,B,y) := \int_0^\infty \frac{x^{-1+\lambda/2} y^{\lambda/2}}{A\min\{x^\lambda, y^\lambda\} + B\max\{x^\lambda, y^\lambda\}} dx, \tag{2.2}$$

then $\omega_\lambda(A,B,x) = \omega_\lambda(A,B,y) = C_\lambda(A,B)$ *is a constant defined by*

$$C_\lambda(A,B) = \begin{cases} \dfrac{4}{\lambda\sqrt{AB}} \arctan\sqrt{\dfrac{A}{B}} & \text{for } A > 0,\ B > 0, \\ \dfrac{4}{\lambda B} & \text{for } A = 0, B > 0. \end{cases} \tag{2.3}$$

Proof. Setting $t = (y/x)^\lambda$, we get

$$\begin{aligned} \omega_\lambda(A,B,x) &= \int_0^\infty \frac{x^{\lambda/2} y^{-1+\lambda/2}}{A\min\{x^\lambda, y^\lambda\} + B\max\{x^\lambda, y^\lambda\}} dy, \\ &= \frac{1}{\lambda}\int_0^\infty \frac{t^{-1/2}}{A\min\{1,t\} + B\max\{1,t\}} dt := I, \end{aligned} \tag{2.4}$$

(i) for $A, B > 0$, we obtain

$$\begin{aligned} I &= \frac{1}{\lambda}\left\{ \int_0^1 \frac{t^{-1/2}}{At+B} dt + \int_1^\infty \frac{t^{-1/2}}{A+Bt} dt \right\} \\ &= \frac{1}{\lambda}\left\{ \frac{2}{\sqrt{AB}} \int_0^{\sqrt{A/B}} \frac{dt}{t^2+1} + \frac{2}{\sqrt{AB}} \int_0^{\sqrt{A/B}} \frac{dt}{t^2+1} \right\} \\ &= \frac{4}{\lambda\sqrt{AB}} \arctan\sqrt{\frac{A}{B}}; \end{aligned} \tag{2.5}$$

(ii) for $A = 0$, $B > 0$, we find

$$I = \frac{1}{\lambda}\left\{ \int_0^1 \frac{t^{-1/2}}{B} dt + \int_1^\infty \frac{t^{-1/2}}{Bt} dt \right\} = \frac{4}{\lambda B}. \tag{2.6}$$

Hence, $\omega_\lambda(A,B,x) = C_\lambda(A,B)$. By the symmetry we still have $\omega_\lambda(A,B,y) = C_\lambda(A,B)$. The lemma is proved. □

Lemma 2.2. *For* $p > 1$ (*or* $0 < p < 1$), $1/p + 1/q = 1$, $\lambda > 0$, $A \geq 0$, $B > 0$ *and* $0 < \varepsilon < p\lambda/2$, *setting*

$$J(\varepsilon) = \iint_1^\infty \frac{x^{\lambda/2-1-\varepsilon/p} y^{\lambda/2-1-\varepsilon/q}}{A \min\{x^\lambda, y^\lambda\} + B \max\{x^\lambda, y^\lambda\}} dx\, dy, \tag{2.7}$$

then for $\varepsilon \to 0^+$,

$$\frac{1}{\varepsilon}\left[C_\lambda(A,B) + o(1)\right] - O(1) < J(\varepsilon) < \frac{1}{\varepsilon}\left[C_\lambda(A,B) + o(1)\right]. \tag{2.8}$$

Proof. Setting $t = (x/y)^\lambda$, we find

$$\begin{aligned} J(\varepsilon) &= \iint_1^\infty \frac{x^{\lambda/2-1-\varepsilon/p} y^{\lambda/2-1-\varepsilon/q}}{A \min\{x^\lambda, y^\lambda\} + B \max\{x^\lambda, y^\lambda\}} dx\, dy \\ &= \frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon} \int_{y^{-\lambda}}^\infty \frac{t^{-1/2-\varepsilon/\lambda p}}{A\min\{t,1\} + B\max\{t,1\}} dt\, dy \\ &= \frac{1}{\lambda\varepsilon}\int_0^\infty \frac{t^{-1/2-\varepsilon/\lambda p}}{A\min\{t,1\} + B\max\{t,1\}} dt\, dy \\ &\quad - \frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon} \int_0^{y^{-\lambda}} \frac{t^{-1/2-\varepsilon/\lambda p}}{A\min\{t,1\} + B\max\{t,1\}} dt\, dy \\ &= \frac{1}{\varepsilon}\left[C_\lambda(A,B) + o(1)\right] - \frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon}\int_0^{y^{-\lambda}} \frac{t^{-1/2-\varepsilon/\lambda p}}{At + B} dt\, dy \\ &\geq \frac{1}{\varepsilon}\left[C_\lambda(A,B) + o(1)\right] - \frac{1}{\lambda}\int_1^\infty y^{-1}\int_0^{y^{-\lambda}} \frac{t^{-1/2-\varepsilon/\lambda p}}{B} dt\, dy \\ &= \frac{1}{\varepsilon}\left[C_\lambda(A,B) + o(1)\right] - O(1). \end{aligned} \tag{2.9}$$

On the other hand,

$$\begin{aligned} J(\varepsilon) &= \iint_1^\infty \frac{x^{\lambda/2-1-\varepsilon/p} y^{\lambda/2-1-\varepsilon/q}}{A \min\{x^\lambda, y^\lambda\} + B \max\{x^\lambda, y^\lambda\}} dx\, dy \\ &< \int_1^\infty \left[\int_0^\infty \frac{x^{\lambda/2-1-\varepsilon/p}}{A \min\{x^\lambda, y^\lambda\} + B \max\{x^\lambda, y^\lambda\}} dx\right] y^{\lambda/2-1-\varepsilon/q} dy \\ &= \frac{1}{\varepsilon}\left[C_\lambda(A,B) + o(1)\right]. \end{aligned} \tag{2.10}$$

Hence, (2.8) is valid. The lemma is proved. □

Theorem 2.3. *If $p > 1$, $1/p + 1/q = 1$, $\lambda > 0$, $A \geq 0$, $B > 0$, $f(x)$, $g(x) \geq 0$ such that $0 < \int_0^\infty x^{p(1-\lambda/2)-1} f^p(x)dx < \infty$, $0 < \int_0^\infty x^{q(1-\lambda/2)-1} g^q(x)dx < \infty$, then*

$$\begin{aligned} S &:= \iint_0^\infty \frac{f(x)g(x)}{A\min\{x^\lambda, y^\lambda\} + B\max\{x^\lambda, y^\lambda\}} dx\,dy \\ &< C_\lambda(A,B)\left\{\int_0^\infty x^{p(1-\lambda/2)-1} f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{q(1-\lambda/2)-1} g^q(x)dx\right\}^{1/q}, \end{aligned} \tag{2.11}$$

where the constant factor $C_\lambda(A,B)$ defined in (2.3) *is the best possible. In particular,*

(i) *for $\lambda = A = B = 1$, $C_1(1,1) = \pi$, and inequality* (2.11) *reduces to Hardy-Hilbert's inequality*

$$\iint_0^\infty \frac{f(x)g(x)}{x+y} dx\,dy < \pi\left\{\int_0^\infty x^{p/2-1} f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{q/2-1} g^q(x)dx\right\}^{1/q}; \tag{2.12}$$

(ii) *for $A = 0$, $\lambda = B = 1$, $C_1(0,1) = 4$ and* (2.11) *reduces to Hardy-Hilbert's-type inequality*

$$\iint_0^\infty \frac{f(x)g(x)}{\max\{x,y\}} dx\,dy < 4\left\{\int_0^\infty x^{p/2-1} f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{q/2-1} g^q(x)dx\right\}^{1/q}. \tag{2.13}$$

Proof. By the Holder inequality, taking into account (2.1), we get

$$\begin{aligned} S &= \iint_0^\infty \left[\frac{1}{A\min\{x^\lambda, y^\lambda\} + B\max\{x^\lambda, y^\lambda\}}\right]^{1/p} \frac{x^{(1-\lambda/2)/q}}{y^{(1-\lambda/2)/p}} f(x) \\ &\quad \times \left[\frac{1}{A\min\{x^\lambda, y^\lambda\} + B\max\{x^\lambda, y^\lambda\}}\right]^{1/q}\left[\frac{y^{(1-\lambda/2)/p}}{x^{(1-\lambda/2)/q}} g(y)\right] dx\,dy \\ &\leq \left\{\iint_0^\infty \frac{x^{(1-\lambda/2)(p-1)} y^{\lambda/2-1}}{A\min\{x^\lambda, y^\lambda\} + B\max\{x^\lambda, y^\lambda\}} f^p(x)dx\right\}^{1/p} \\ &\quad \times \left\{\iint_0^\infty \frac{y^{(1-\lambda/2)(q-1)} x^{\lambda/2-1}}{A\min\{x^\lambda, y^\lambda\} + B\max\{x^\lambda, y^\lambda\}} g^q(y)dy\right\}^{1/q} \\ &= \left\{\int_0^\infty \omega_\lambda(A,B,x) x^{p(1-\lambda/2)-1} f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty \omega_\lambda(A,B,y) y^{q(1-\lambda/2)-1} g^q(y)dy\right\}^{1/q} \\ &\leq C_\lambda(A,B)\left\{\int_0^\infty x^{p(1-\lambda/2)-1} f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty y^{q(1-\lambda/2)-1} g^q(y)dy\right\}^{1/q}. \end{aligned} \tag{2.14}$$

If (2.14) takes the form of equality, then there exist constants M and N which are not all zero such that

$$M\frac{x^{(1-\lambda/2)(p-1)}y^{\lambda/2-1}}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}f^{p}(x)=N\frac{y^{(1-\lambda/2)(q-1)}x^{\lambda/2-1}}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}g^{q}(y),$$
$$Mx^{p(1-\lambda/2)}f^{p}(x)=Ny^{q(1-\lambda/2)}g^{q}(y),\quad \text{a.e. in } (0,\infty)\times(0,\infty) \tag{2.15}$$

Hence, there exists a constant c such that

$$Mx^{p(1-\lambda/2)}f^{p}(x)=Ny^{q(1-\lambda/2)}g^{q}(y)=c\quad \text{a.e. in } (0,\infty). \tag{2.16}$$

We claim that $M=0$. In fact, if $M\neq 0$, then

$$x^{p(1-\lambda/2)-1}f^{p}(x)=\frac{c}{Mx}\quad \text{a.e. in } (0,\infty) \tag{2.17}$$

which contradicts the fact that $0<\int_0^{\infty}x^{p(1-\lambda/2)-1}f^{p}(x)dx<\infty$. Hence, by (2.14) we get (2.11).

If the constant factor $C_{\lambda}(A,B)$ is not the best possible, then there exists a positive constant K (with $K<C_{\lambda}(A,B)$), thus (2.11) is still valid if we replace $C_{\lambda}(A,B)$ by K. For $0<\varepsilon<p\lambda/2$, setting $\widetilde{f}$ and $\widetilde{g}$ as $\widetilde{f}(x)=\widetilde{g}(x)=0$ for $x\in(0,1)$, $\widetilde{f}(x)=x^{\lambda/2-1-\varepsilon/p}$; $\widetilde{g}(x)=x^{\lambda/2-1-\varepsilon/q}$ for $x\in[1,\infty)$, then we have

$$K\left\{\int_0^{\infty}x^{p(1-\lambda/2)-1}\widetilde{f}^{p}(x)dx\right\}^{1/p}\left\{\int_0^{\infty}x^{q(1-\lambda/2)-1}\widetilde{g}^{q}(x)dx\right\}^{1/q}$$
$$=K\left\{\int_0^{\infty}x^{-1-\varepsilon}dx\right\}^{1/p}\left\{\int_0^{\infty}x^{-1-\varepsilon}dx\right\}^{1/q}=\frac{K}{\varepsilon}. \tag{2.18}$$

By using (2.8), we find

$$\iint_0^{\infty}\frac{\widetilde{f}(x)\widetilde{g}(x)dx\,dy}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}=\int_1^{\infty}\left[\int_1^{\infty}\frac{x^{\lambda/2-1-\varepsilon/p}dx}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}\right]y^{\lambda/2-1-\varepsilon/q}dy$$
$$>\frac{1}{\varepsilon}\left[C_{\lambda}(A,B)+o(1)\right]-O(1). \tag{2.19}$$

Therefore, we get

$$\frac{1}{\varepsilon}\left[C_{\lambda}(A,B)+o(1)\right]-O(1)<\frac{K}{\varepsilon} \tag{2.20}$$

or

$$\frac{1}{\lambda}\left[C_\lambda(A,B)+o(1)\right]-\varepsilon O(1)<K. \tag{2.21}$$

For $\varepsilon \to 0^+$, it follows that $C_\lambda(A,B) \le K$ which contradicts the fact that $K < C_\lambda(A,B)$. Hence, the constant factor $C_\lambda(A,B)$ in (2.11) is the best possible. The theorem is proved. □

Theorem 2.4. *If $0 < p < 1$, $1/p + 1/q = 1$, $\lambda > 0$, $A \ge 0$, $B > 0$, $f(x), g(x) \ge 0$ such that $0 < \int_0^\infty x^{p(1-\lambda/2)-1} f^p(x)dx < \infty$, $0 < 0 < \int_0^\infty x^{q(1-\lambda/2)-1} g^q(x)dx < \infty$, then*

$$\begin{aligned}&\iint_0^\infty \frac{f(x)g(x)}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}}dx\,dy\\ &\quad > C_\lambda(A,B)\left\{\int_0^\infty x^{p(1-\lambda/2)-1}f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{q(1-\lambda/2)-1}g^q(x)dx\right\}^{1/q},\end{aligned} \tag{2.22}$$

where the constant factor $C_\lambda(A,B)$ defined in (2.3) is the best possible. In particular,

(i) *for $\lambda = A = B = 1$, $C_1(1,1) = \pi$, and inequality (2.22) reduces to Hardy-Hilbert's inequality*

$$\iint_0^\infty \frac{f(x)g(x)}{x+y}dx\,dy > \pi\left\{\int_0^\infty x^{p/2-1}f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{q/2-1}g^q(x)dx\right\}^{1/q}, \tag{2.23}$$

(ii) *for $A = 0$, $\lambda = B = 1$, $C_1(0,1) = 4$ and (2.22) reduces to Hardy-Hilbert's-type inequality*

$$\iint_0^\infty \frac{f(x)g(x)}{\max\{x,y\}}dx\,dy > 4\left\{\int_0^\infty x^{p/2-1}f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{q/2-1}g^q(x)dx\right\}^{1/q}. \tag{2.24}$$

Proof. By reverse Holder's inequality, and the same way, we have (2.22). If the constant factor $C_\lambda(A,B)$ in (2.22) is not the best possible, then there exists a positive constant H (with $H > C_\lambda(A,B)$) such that (2.22) is still valid if we replace $C_\lambda(A,B)$ by H. For $0 < \varepsilon < p\lambda/2$, setting $\tilde{f}$ and $\tilde{g}$ as in Theorem 2.3, then we have

$$\begin{aligned}&H\left\{\int_0^\infty x^{p(1-\lambda/2)-1}\tilde{f}^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{q(1-\lambda/2)-1}\tilde{g}^q(x)dx\right\}^{1/q}\\ &\quad = H\left\{\int_0^\infty x^{-1-\varepsilon}dx\right\}^{1/p}\left\{\int_0^\infty x^{-1-\varepsilon}dx\right\}^{1/q} = \frac{H}{\varepsilon}.\end{aligned} \tag{2.25}$$

By using (2.8), we find

$$\begin{aligned}\iint_0^\infty \frac{\tilde{f}(x)\tilde{g}(x)dx\,dy}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}} &= \int_1^\infty\left[\int_1^\infty \frac{x^{\lambda/2-1-\varepsilon/p}dx}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}}\right]y^{\lambda/2-1-\varepsilon/q}dy\\ &< \frac{1}{\varepsilon}\left[C_\lambda(A,B)+o(1)\right].\end{aligned} \tag{2.26}$$

Therefore, we get

$$\frac{1}{\varepsilon}\left[C_{\lambda}(A,B)+o(1)\right] > \frac{H}{\varepsilon}, \tag{2.27}$$

or

$$C_{\lambda}(A,B)+o(1) \geq H. \tag{2.28}$$

For $\varepsilon \to 0^{+}$, it follows that $C_{\lambda}(A,B) \geq H$ which contradicts the fact that $H > C_{\lambda}(A,B)$. Hence, the constant factor $C_{\lambda}(A,B)$ in (2.22) is the best possible. The theorem is proved. □

Theorem 2.5. *Under the assumption of Theorem 2.3,*

$$\int_0^{\infty} y^{\lambda p/2-1}\left[\int_0^{\infty}\frac{f(x)}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}dx\right]^{p}dy < \left[C_{\lambda}(A,B)\right]^{p}\int_0^{\infty} x^{p(1-\lambda/2)-1}f^{p}(x)dx, \tag{2.29}$$

where the constant factor $[C_{\lambda}(A,B)]^{p}$ *is the best possible. Inequalities* (2.11) *and* (2.29) *are equivalent.*

Proof. Setting

$$g(y)=y^{\lambda p/2-1}\left\{\int_0^{\infty}\frac{f(x)}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}dx\right\}^{p-1}, \tag{2.30}$$

then by (2.11) we have

$$\begin{aligned}
\int_0^{\infty} y^{q(1-\lambda/2)-1}g^{q}(y)dy &= \int_0^{\infty} y^{\lambda p/2-1}\left\{\int_0^{\infty}\frac{f(x)}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}dx\right\}^{p}dy \\
&= \int_0^{\infty}\left\{\int_0^{\infty}\frac{f(x)}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}dx\right\} \\
&\quad \times \left\{y^{\lambda p/2-1}\left\{\int_0^{\infty}\frac{f(x)}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}dx\right\}^{p-1}\right\}dy \\
&= \iint_0^{\infty}\frac{f(x)g(y)}{A\min\{x^{\lambda},y^{\lambda}\}+B\max\{x^{\lambda},y^{\lambda}\}}dx\,dy \\
&\leq C_{\lambda}(A,B)\left\{\int_0^{\infty} x^{p(1-\lambda/2)-1}f^{p}(x)dx\right\}^{1/p}\left\{\int_0^{\infty} y^{q(1-\lambda/2)-1}g^{q}(y)dy\right\}^{1/q}.
\end{aligned} \tag{2.31}$$

Hence, we obtain

$$\int_0^{\infty} y^{q(1-\lambda/2)-1}g^{q}(y)dy \leq \left[C_{\lambda}(A,B)\right]^{p}\int_0^{\infty} x^{p(1-\lambda/2)-1}f^{p}(x)dx. \tag{2.32}$$

Thus, by (2.11), both (2.31) and (2.32) keep the form of strict inequalities, then we have (2.29).

By Holder's inequality, we find

$$\begin{aligned}
&\iint_0^\infty \frac{f(x)g(y)}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}}dx\,dy\\
&=\int_0^\infty \left\{y^{\lambda/2-1/p}\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}}dx\right\}\left\{y^{1/p-\lambda/2}g(y)\right\}dy\\
&\le \left\{\int_0^\infty y^{p\lambda/2-1}\left\{\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}}dx\right\}^p\right\}^{1/p}\left\{\int_0^\infty y^{q(1-\lambda/2)-1}g^q(y)dy\right\}^{1/q}.
\end{aligned} \tag{2.33}$$

Therefore, by (2.29) we have (2.11), and inequalities (2.29) and (2.11) are equivalent. If the constant factor in (2.29) is not the best possible, then by (2.33) we can get a contradiction that the constant factor in (2.11) is not the best possible. The theorem is proved. □

Theorem 2.6. *Under the assumption of Theorem 2.4,*

$$\int_0^\infty y^{\lambda p/2-1}\left[\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+B\max\{x^\lambda,y^\lambda\}}dx\right]^p dy > [C_\lambda(A,B)]^p\int_0^\infty x^{p(1-\lambda/2)-1}f^p(x)dx, \tag{2.34}$$

where the constant factor $[C_\lambda(A,B)]^p$ *is the best possible. Inequalities* (2.22) *and* (2.34) *are equivalent.*

The proof of Theorem 2.6 is similar to that of Theorem 2.5, so we omit it.

3. Discrete analogous

Lemma 3.1. *Suppose that* $0<\lambda\le 2$, $A\ge 0$, $B>0$. *Then the weight coefficients* $\varpi_\lambda(A,B,m)$ *and* $\varpi_\lambda(A,B,n)$, *defined, respectively, by*

$$\varpi_\lambda(A,B,m) := \sum_{n=1}^\infty \frac{m^{\lambda/2}n^{-1+\lambda/2}}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}} \quad (m\in N), \tag{3.1}$$

$$\varpi_\lambda(A,B,n) := \sum_{m=1}^\infty \frac{n^{\lambda/2}m^{-1+\lambda/2}}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}} \quad (n\in N), \tag{3.2}$$

satisfy the following inequalities:

$$C_\lambda(A,B)\left[1-\theta_\lambda(A,B,m)\right] < \varpi_\lambda(A,B,m) < C_\lambda(A,B), \tag{3.3}$$

$$C_\lambda(A,B)\left[1-\theta_\lambda(A,B,n)\right] < \varpi_\lambda(A,B,n) < C_\lambda(A,B), \tag{3.4}$$

where $\theta_\lambda(A,B,r) := (1/C_1(A,B))\int_0^{r^{-\lambda}}(t^{-1/2}/(At+B))dt = O(1/r^{\lambda/2}) \in (0,1)$ $(r\in N)$ $(r\to\infty)$, *and* $C_\lambda(A,B)$ *is defined by* (2.3).

Proof. Since $0 < \lambda \le 2$, $A \ge 0$, $B > 0$, by Lemma 2.1 we get

$$\begin{aligned}\varpi_\lambda(A,B,m) &< \int_0^\infty \frac{m^{\lambda/2}y^{-1+\lambda/2}}{A\min\{m^\lambda,y^\lambda\}+B\max\{m^\lambda,y^\lambda\}}dy,\\ &= \omega_\lambda(A,B,m) = C_\lambda(A,B).\end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}\varpi_\lambda(A,B,m) &> \int_1^\infty \frac{m^{\lambda/2}y^{-1+\lambda/2}}{A\min\{m^\lambda,y^\lambda\}+B\max\{m^\lambda,y^\lambda\}}dy\\ &= \frac{1}{\lambda}\int_{m^{-\lambda}}^\infty \frac{t^{-1/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\\ &= I - \frac{1}{\lambda}\int_0^{m^{-\lambda}} \frac{t^{-1/2}}{At+B}dt\\ &= I\big(1-\theta_\lambda(A,B,m)\big),\end{aligned} \tag{3.6}$$

where $I = (1/\lambda)\,C_1(A,B)$ and

$$0 < \theta_\lambda(A,B,m) = \frac{1}{C_1(A,B)}\int_0^{m^{-\lambda}} \frac{t^{-1/2}}{At+B}dt < 1. \tag{3.7}$$

Since

$$\int_0^{m^{-\lambda}} \frac{t^{-1/2}}{At+B}dt \le \int_0^{m^{-\lambda}} \frac{t^{-1/2}}{B}dt = \frac{2}{Bm^{\lambda/2}}, \tag{3.8}$$

then $\theta_\lambda(A,B,m) = O(1/m^{\lambda/2})$. Therefore, (3.3) is valid. By the symmetry, (3.4) is still valid. The lemma is proved. □

Lemma 3.2. *If $p > 0$ $(p \ne 1)$, $1/p + 1/q = 1$, $0 < \lambda \le 2$, $A \ge 0$, $B > 0$, and $0 < \varepsilon < p\lambda/2$, setting*

$$L(\varepsilon) = \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{m^{\lambda/2-1-\varepsilon/p}n^{\lambda/2-1-\varepsilon/q}}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}, \tag{3.9}$$

then for $\varepsilon \to 0^+$,

$$\big[C_\lambda(A,B) - o(1)\big]\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} < L(\varepsilon) < \big[C_\lambda(A,B) + \tilde{o}(1)\big]\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}. \tag{3.10}$$

Proof. Setting $t = (x/n)^{\lambda}$ in the following, by (3.4), we have

$$
\begin{aligned}
L(\varepsilon) &< \sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{x^{\lambda/2-1-\varepsilon/p}n^{\lambda/2-1-\varepsilon/q}}{A\min\{x^{\lambda},n^{\lambda}\}+B\max\{x^{\lambda},n^{\lambda}\}}dx \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[\frac{1}{\lambda}\int_{0}^{\infty}\frac{t^{-1/2-\varepsilon/\lambda p}}{A\min\{t,1\}+B\max\{t,1\}}dt\right] \\
&= \left[C_{\lambda}(A,B)+\tilde{o}(1)\right]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} \quad (\varepsilon\longrightarrow 0^{+}), \\
L(\varepsilon) &> \sum_{n=1}^{\infty}\int_{1}^{\infty}\frac{x^{\lambda/2-1-\varepsilon/p}n^{\lambda/2-1-\varepsilon/q}}{A\min\{x^{\lambda},n^{\lambda}\}+B\max\{x^{\lambda},n^{\lambda}\}}dx \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[\frac{1}{\lambda}\int_{n^{-\lambda}}^{\infty}\frac{t^{-1/2-\varepsilon/\lambda p}}{A\min\{t,1\}+B\max\{t,1\}}dt\right] \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[C_{\lambda}(A,B)+\tilde{o}(1)-\frac{1}{\lambda}\int_{0}^{n^{-\lambda}}\frac{t^{-1/2-\varepsilon/\lambda p}}{At+B}dt\right] \\
&> \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[C_{\lambda}(A,B)+\tilde{o}(1)\right]-\frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{1}{n}\int_{0}^{n^{-\lambda}}\frac{t^{-1/2-\varepsilon/\lambda p}}{B}dt \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[C_{\lambda}(A,B)+\tilde{o}(1)\right]-\frac{1}{(\lambda B/2-\varepsilon B/p)}\sum_{n=1}^{\infty}\frac{1}{n^{1+\lambda/2-\varepsilon/p}} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[C_{\lambda}(A,B)+\tilde{o}(1)-\frac{1}{(\lambda B/2-\varepsilon B/p)}\sum_{n=1}^{\infty}\frac{1}{n^{1+\lambda/2-\varepsilon/p}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\right] \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[C_{\lambda}(A,B)-o(1)\right] \quad (\varepsilon\longrightarrow 0^{+}).
\end{aligned}
\tag{3.11}
$$

Thus, inequality (3.10) holds. The lemma is proved. □

Theorem 3.3. *If* $p > 1$, $1/p + 1/q = 1$, $0 < \lambda \le 2$, $A \ge 0$, $B > 0$, a_n, $b_n \ge 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\lambda/2)-1}a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{q(1-\lambda/2)-1}b_n^q < \infty$, *then*

$$
\begin{aligned}
D &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^{\lambda},n^{\lambda}\}+B\max\{m^{\lambda},n^{\lambda}\}} \\
&< C_{\lambda}(A,B)\left\{\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q\right\}^{1/q},
\end{aligned}
\tag{3.12}
$$

where the constant factor $C_\lambda(A,B)$ *defined in* (2.3) *is the best possible. In particular,*

(i) *for* $\lambda = A = B = 1$, $C_1(1,1) = \pi$, *and inequality* (3.12) *reduces to Hardy-Hilbert's inequality*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{n=1}^{\infty} n^{p/2-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q/2-1}b_n^q\right\}^{1/q}; \tag{3.13}$$

(ii) *for* $A = 0$, $\lambda = B = 1$, $C_1(0,1) = 4$ *and* (3.12) *reduces to Hardy-Hilbert's-type inequality*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < 4\left\{\sum_{n=1}^{\infty} n^{p/2-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q/2-1}b_n^q\right\}^{1/q}. \tag{3.14}$$

Proof. By the Holder inequality, taking into account (3.1), we get

$$\begin{aligned}
D &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\right\}^{1/p}\left[\frac{m^{(1-\lambda/2)/q}}{n^{(1-\lambda/2)/p}}a_m\right] \\
&\quad\times\left\{\frac{1}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\right\}^{1/q}\left[\frac{n^{(1-\lambda/2)/p}}{m^{(1-\lambda/2)/q}}b_n\right] \\
&\le \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\frac{m^{(1-\lambda/2)(p-1)}}{n^{1-\lambda/2}}a_m^p\right\}^{1/p} \\
&\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\frac{n^{(1-\lambda/2)(q-1)}}{m^{1-\lambda/2}}b_n^q\right\}^{1/q} \\
&= \left\{\sum_{m=1}^{\infty}\varpi_\lambda(A,B,m)m^{p(1-\lambda/2)-1}a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\varpi_\lambda(A,B,n)n^{q(1-\lambda/2)-1}b_n^q\right\}^{1/q}.
\end{aligned} \tag{3.15}$$

Then, by (3.3) and (3.4) we obtain (3.12).

It remains to show that the constant factor $C_\lambda(A,B)$ is the best possible, to do that we set for $0 < \varepsilon < p\lambda/2$, $\widetilde{a}_m = m^{\lambda/2-1-\varepsilon/p}$; $\widetilde{b}_n = n^{\lambda/2-1-\varepsilon/q}$, by (3.9) we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}} = L(\varepsilon). \tag{3.16}$$

If there exists a constant $0 < K \le C_\lambda(A,B)$ such that (3.12) is still valid if we replace $C_\lambda(A,B)$ by K, then in particular by (3.10) we find

$$\begin{aligned}
\left[C_\lambda(A,B)-o(1)\right]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} &< L(\varepsilon) < K\left\{\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}\widetilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}\widetilde{b}_n^q\right\}^{1/q} \\
&= K\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}},
\end{aligned} \tag{3.17}$$

it follows that $C_\lambda(A,B) - o(1) < K$ and then $C_\lambda(A,B) \le K$ $(\varepsilon \to 0^+)$. Therefor, $K = C_\lambda(A,B)$ is the best constant factor in (3.12). The theorem is proved. □

Theorem 3.4. *If* $0 < p < 1$, $1/p + 1/q = 1$, $0 < \lambda \le 2$, $A \ge 0$, $B > 0$, a_n, $b_n \ge 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\lambda/2)-1} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{q(1-\lambda/2)-1} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}} > C_\lambda(A,B)\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(A,B,n)\right]n^{p(1-\lambda/2)-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q\right\}^{1/q}, \tag{3.18}$$

where the constant factor $C_\lambda(A,B)$ *defined in* (2.4) *is the best possible. In particular,*

(i) *for* $\lambda = A = B = 1$, $C_1(1,1) = \pi$, *and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} > \pi\left\{\sum_{n=1}^{\infty}\left[1-\frac{2}{\pi}\arctan\frac{1}{n^{1/2}}\right]n^{p/2-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q/2-1}b_n^q\right\}^{1/q}; \tag{3.19}$$

(ii) *for* $A = 0$, $\lambda = B = 1$, $C_1(0,1) = 4$, *and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} > 4\left\{\sum_{n=1}^{\infty}\left[1-\frac{1}{2n^{1/2}}\right]n^{p/2-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q/2-1}b_n^q\right\}^{1/q}. \tag{3.20}$$

Proof. By reverse Holder's inequality, we get

$$\begin{aligned} D &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}} \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\right\}^{1/p}\left[\frac{m^{(1-\lambda/2)/q}}{n^{(1-\lambda/2)/p}}a_m\right] \\ &\quad\times\left\{\frac{1}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\right\}^{1/q}\left[\frac{n^{(1-\lambda/2)/p}}{m^{(1-\lambda/2)/q}}b_n\right] \\ &\ge \left\{\sum_{m=1}^{\infty}\varpi_\lambda(A,B,m)m^{p(1-\lambda/2)-1}a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\varpi_\lambda(A,B,n)n^{q(1-\lambda/2)-1}b_n^q\right\}^{1/q}. \end{aligned} \tag{3.21}$$

Then by (3.3) and (3.4), in view of $q < 0$, we have (3.18). For $0 < \varepsilon < p\lambda/2$, setting $\widetilde{a}_m = m^{\lambda/2-1-\varepsilon/p}$, $\widetilde{b}_n = n^{\lambda/2-1-\varepsilon/q}$ $(m,n \in N)$. If there exists a constant $K \ge C_\lambda(A,B)$ such that (3.18) is still valid if we replace $C_\lambda(A,B)$ by K, then in particular by (3.9) and (3.10) we find

$$\begin{aligned} \left[C_\lambda(A,B)+\widetilde{o}(1)\right]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} &> L(\varepsilon) \\ &> K\left\{\sum_{n=1}^{\infty}\left[1-\theta_\lambda(A,B,n)\right]n^{p(1-\lambda/2)-1}\widetilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}\widetilde{b}_n^q\right\}^{1/q} \\ &= K\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\lambda/2}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right\}^{1/q} \\ &= K\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{1-\left[\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right]^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\lambda/2}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}, \end{aligned} \tag{3.22}$$

it follows that

$$C_\lambda(A,B)+\tilde{o}(1)>K\left\{1-\left[\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right]^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\lambda/2}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}. \tag{3.23}$$

Hence, if $\varepsilon \to 0^+$, we get $C_\lambda(A,B) \geq K$. Thus, $K = C_\lambda(A,B)$ is the best constant factor in (3.18). □

Theorem 3.5. *Under the assumption of Theorem 3.3,*

$$\sum_{n=1}^{\infty}n^{\lambda p/2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\right]^p<[C_\lambda(A,B)]^p\sum_{m=1}^{\infty}m^{p(1-\lambda/2)-1}a_m^p, \tag{3.24}$$

where the constant factor $[C_\lambda(A,B)]^p$ *is the best possible. Inequalities* (3.12) *and* (3.24) *are equivalent.*

Proof. Setting

$$b_n=n^{\lambda p/2-1}\left\{\sum_{m=1}^{\infty}\frac{a_m}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\right\}^{p-1}, \tag{3.25}$$

we get

$$\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}. \tag{3.26}$$

By (3.12) and using the same method of Theorem 2.5, we obtain (3.24). We may show that the constant factor in (3.24) is the best possible and inequality (3.12) is equivalent to (3.24). □

Theorem 3.6. *Under the assumption of Theorem 3.4,*

$$\sum_{n=1}^{\infty}n^{\lambda p/2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{A\min\{m^\lambda,n^\lambda\}+B\max\{m^\lambda,n^\lambda\}}\right]^p>[C_\lambda(A,B)]^p\sum_{m=1}^{\infty}m^{p(1-\lambda/2)-1}a_m^p, \tag{3.27}$$

where the constant factor $[C_\lambda(A,B)]^p$ *is the best possible. Inequalities* (3.18) *and* (3.27) *are equivalent.*

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