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Research Article

Generic Well-Posedness for a Class of Equilibrium Problems

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We study a class of equilibrium problems which is identified with a complete metric space of functions. For most elements of this space of functions (in the sense of Baire category), we establish that the corresponding equilibrium problem possesses a unique solution and is well-posed.

1. Introduction

The study of equilibriumproblems has recently been a rapidly growing area of research. See, for example, [1–3] and the references mentioned therein.

Let (X,ρ) be a complete metric space. In this paper, we consider the following equilibrium problem:

$$\text{To find } x \in X \text{ such that } f(x,y) \geq 0 \quad \forall y \in X, \tag{P}$$

where f belongs to a complete metric space of functions $\mathcal{A}$ defined below. In this paper, we show that for most elements of this space of functions $\mathcal{A}$ (in the sense of Baire category) the equilibrium problem (P) possesses a unique solution. In other words, the problem (P) possesses a unique solution for a generic (typical) element of $\mathcal{A}$ [4–6].

Set

$$\rho_1((x_1,y_1),(x_2,y_2)) = \rho(x_1,x_2) + \rho(y_1,y_2), \quad x_1,x_2,y_1,y_2 \in X. \tag{1.1}$$

Clearly, $(X \times X, \rho_1)$ is a complete metric space.

Denote by $\mathcal{A}_0$ the set of all continuous functions $f : X \times X \to R^1$ such that

$$f(x,x) = 0 \quad \forall x \in X. \tag{1.2}$$

We equip the set $\mathcal{A}_0$ with the uniformity determined by the base

$$U(\epsilon) = \{(f,g) \in \mathcal{A}_0 \times \mathcal{A}_0 : |f(z) - g(z)| \leq \epsilon \ \forall z \in X \times X\}, \tag{1.3}$$

where $\epsilon > 0$. It is clear that the space $\mathcal{A}_0$ with this uniformity is metrizable (by a metric d) and complete.

Denote by $\mathcal{A}$ the set of all $f \in \mathcal{A}_0$ for which the following properties hold.

(P1) For each $\epsilon > 0$, there exists $x_\epsilon \in X$ such that $f(x_\epsilon, y) \geq -\epsilon$ for all $x \in X$.

(P2) For each $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y)| \leq \epsilon$ for all $x, y \in X$ satisfying $\rho(x,y) \leq \delta$.

Clearly, $\mathcal{A}$ is a closed subset of X. We equip the space $\mathcal{A}$ with the metric d and consider the topological subspace $\mathcal{A} \subset \mathcal{A}_0$ with the relative topology.

For each $x \in X$ and each subset $D \subset X$, put

$$\rho(x,D) = \inf\{\rho(x,y) : y \in D\}. \tag{1.4}$$

For each $x \in X$ and each $r > 0$, set

$$\begin{aligned} B(x,r) &= \{y \in X : \rho(x,y) \leq r\}, \\ B^o(x,r) &= \{y \in X : \rho(x,y) < r\}. \end{aligned} \tag{1.5}$$

Assume that the following property holds.

(P3) There exists a positive number Δ such that for each $y \in X$ and each pair of real numbers t_1, t_2 satisfying $0 < t_1 < t_2 < \Delta$, there is $z \in X$ such that $\rho(z,y) \in [t_1, t_2]$.

In this paper, we will establish the following result.

Theorem 1.1. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that for each $f \in \mathcal{F}$, the following properties hold:*

(i) *there exists a unique $x_f \in X$ such that*

$$f(x_f, y) \geq 0 \quad \forall x, y \in X; \tag{1.6}$$

(ii) *for each $\epsilon > 0$, there are $\delta > 0$ and a neighborhood V of f in $\mathcal{A}$ such that for each $h \in V$ and each $x \in X$ satisfying $\inf\{h(x,y) : y \in X\} > -\delta$, the inequality $\rho(x_f, x) < \epsilon$ holds.*

In other words, for a generic (typical) $f \in \mathcal{A}$, the problem (P) is well-posed [7–9].

2. An auxiliary density result

Lemma 2.1. *Let $f \in \mathcal{A}$ and $\epsilon \in (0,1)$. Then there exist $f_0 \in \mathcal{A}$ and $x_0 \in X$ such that $(f, f_0) \in U(\epsilon)$ and $f(x_0, y) \geq 0$ for all $y \in X$.*

Proof. By (P1) there is $x_0 \in X$ such that

$$f(x_0, y) \geq -\frac{\epsilon}{16} \quad \forall y \in X. \tag{2.1}$$

Set

$$\begin{aligned} E_1 &= \left\{(x, y) \in X \times X : f(x, y) \geq -\frac{\epsilon}{16}\right\}, \\ E_2 &= \left\{(x, y) \in (X \times X) \setminus E_1 : f(x, y) \geq -\frac{\epsilon}{8}\right\}, \\ E_3 &= (X \times X) \setminus (E_1 \cup E_2). \end{aligned} \tag{2.2}$$

For each $(y_1, y_2) \in E_1$, there is $r_1(y_1, y_2) \in (0, 1)$ such that

$$f(z_1, z_2) > -\frac{\epsilon}{14} \quad \forall z_1, z_2 \in X \text{ satisfying } \rho(z_i, y_i) \leq r_1(y_1, y_2), \quad i = 1, 2. \tag{2.3}$$

For each $(y_1, y_2) \in E_2$, there is $r_1(y_1, y_2) \in (0, 1)$ such that

$$f(z_1, z_2) > -\frac{\epsilon}{6} \quad \forall z_1, z_2 \in X \text{ satisfying } \rho(z_i, y_i) \leq r_1(y_1, y_2), \quad i = 1, 2. \tag{2.4}$$

For each $(y_1, y_2) \in E_3$, there is $r_1(y_1, y_2) \in (0, 1)$ such that

$$f(z_1, z_2) < -\frac{\epsilon}{8} \quad \forall z_1, z_2 \in X \text{ satisying } \rho(z_i, y_i) \leq r_1(y_1, y_2), \quad i = 1, 2. \tag{2.5}$$

For each $(y_1, y_2) \in X \times X$, set

$$U(y_1, y_2) = B^o(y_1, r_1(y_1, y_2)) \times B^o(y_2, r_1(y_1, y_2)). \tag{2.6}$$

For any $(y_1, y_2) \in E_1 \cup E_2$, put

$$g_{y_1, y_2}(z) = \max\{f(z), 0\}, \quad z \in X \times X \tag{2.7}$$

and for any $(y_1, y_2) \in E_3$, put

$$g_{y_1, y_2}(z) = f(z), \quad z \in X \times X. \tag{2.8}$$

Clearly, $\{U(y_1, y_2) : y_1, y_2 \in X\}$ is an open covering of $X \times X$. Since any metric space is paracompact, there is a continuous locally finite partition of unity $\{\phi_\beta : \beta \in \mathcal{B}\}$ subordinated to the covering $\{U(y_1, y_2) : y_1, y_2 \in X\}$. Namely, for any $\beta \in \mathcal{B}$, $\phi_\beta : X \times X \to [0, 1]$ is a continuous function and there exist $y_1(\beta), y_2(\beta) \in X$ such that $\mathrm{supp}(\phi_\beta) \subset U(y_1(\beta), y_2(\beta))$ and that

$$\sum_{\beta \in \mathcal{B}} \phi_\beta(z) = 1 \quad \forall z \in X \times X. \tag{2.9}$$

Define

$$f_0(z) = \sum_{\beta \in \mathcal{B}} \phi_\beta(z) g_{(y_1(\beta), y_2(\beta))}(z), \quad z \in X \times X. \tag{2.10}$$

Clearly, f_0 is well defined, continuous, and satisfies

$$f_0(z) \geq f(z) \quad \forall z \in X \times X. \tag{2.11}$$

Let $(z_1, z_2) \in E_1$. Then

$$f(z_1, z_2) \geq -\frac{\epsilon}{16}. \tag{2.12}$$

Assume that $\beta \in \mathcal{B}$ and that $\phi_\beta(z_1, z_2) > 0$. Then

$$(z_1, z_2) \in \operatorname{supp}(\phi_\beta) \subset U(y_1(\beta), y_2(\beta)). \tag{2.13}$$

If $(y_1(\beta), y_2(\beta)) \in E_3$, then in view of (2.5), (2.6), and (2.13), $f(z_1, z_2) < -\epsilon/8$, a contradiction (see (2.12)). Then $(y_1(\beta), y_2(\beta)) \in E_1 \cup E_2$, and by (2.7),

$$g_{y_1(\beta), y_2(\beta)}(z_1, z_2) = \max\{f(z_1, z_2), 0\}. \tag{2.14}$$

Since this equality holds for any $\beta \in \mathcal{B}$ satisfying $\phi_\beta(z_1, z_2) > 0$, it follows from (2.10) that

$$f_0(z_1, z_2) = \max\{f(z_1, z_2), 0\} \tag{2.15}$$

for all $(z_1, z_2) \in E_1$.

Relations (2.1), (2.2), and (2.15) imply that

$$f_0(x_0, y) \geq 0, \quad y \in X. \tag{2.16}$$

By (1.2), (2.7), (2.8), and (2.10)

$$f_0(x, x) = 0, \quad x \in X. \tag{2.17}$$

Assume that

$$(z_1, z_2) \in E_2. \tag{2.18}$$

Then in view of (2.2) and (2.18), $f(z_1, z_2) \geq -\epsilon/8$. Together with (2.7) and (2.10), this implies that

$$f_0(z_1, z_2) \leq \sum_{\beta \in \mathcal{B}} \phi_\beta(z_1, z_2)\left(f(z_1, z_2) + \frac{\epsilon}{8}\right) = f(z_1, z_2) + \frac{\epsilon}{8}. \tag{2.19}$$

Combined with (2.11), this implies that

$$f(z_1, z_2) \leq f_0(z_1, z_2) \leq f(z_1, z_2) + \frac{\epsilon}{8} \tag{2.20}$$

for all $(z_1, z_2) \in E_2$.

Let

$$(z_1, z_2) \in E_3 \tag{2.21}$$

and assume that

$$\beta \in \mathcal{B}, \quad \phi_\beta(z_1, z_2) > 0. \tag{2.22}$$

Then in view of (2.22),

$$(z_1, z_2) \in \operatorname{supp}(\phi_\beta) \subset U(y_1(\beta), y_2(\beta)). \tag{2.23}$$

By (2.23) and the choice of $U(y_1(\beta), y_2(\beta))$ (see (2.3)–(2.6)), $(y_1(\beta), y_2(\beta)) \notin E_1$ and by (2.4), (2.6), (2.7), and (2.8),

$$g_{y_1(\beta), y_2(\beta)}(z_1, z_2) \le f(z_1, z_2) + \frac{\epsilon}{6}. \tag{2.24}$$

Since the inequality above holds for any $\beta \in \mathcal{B}$ satisfying (2.22), the relation (2.10) implies that

$$f_0(z_1, z_2) \le f(z_1, z_2) + \frac{\epsilon}{6}. \tag{2.25}$$

Together with (2.11), (2.12), and (2.15), this implies that for all $(z_1, z_2) \in X \times X$

$$f(z_1, z_2) \le f_0(z_1, z_2) \le f(z_1, z_2) + \frac{\epsilon}{6}. \tag{2.26}$$

By (2.17), $f_0 \in \mathcal{A}_0$. In view of (2.16), f_0 possesses (P1). Since f possesses (P2), it follows from (2.7), (2.8), and (2.10) that f_0 possesses (P2). Therefore $f_0 \in \mathcal{A}$ and Lemma 2.1 now follows from (2.16) and (2.26). □

3. A perturbation lemma

Lemma 3.1. *Let $\epsilon \in (0,1)$, $f \in \mathcal{A}$, and let $x_0 \in X$ satisfy*

$$f(x_0, y) \ge 0 \quad \forall y \in X. \tag{3.1}$$

Then there exist $g \in \mathcal{A}$ and $\delta > 0$ such that

$$g(x_0, y) \ge 0 \quad \forall y \in X, \qquad |(g - f)(x, y)| \le \frac{\epsilon}{4} \quad \forall x, y \in X \tag{3.2}$$

and if $x \in X$ satisfies $\inf\{g(x, y) : y \in X\} > -\delta$, *then $\rho(x_0, x) < \epsilon/8$.*

Proof. By (P2) there is a positive number

$$\delta_0 < \min\{16^{-1}\epsilon, 16^{-1}\Delta\} \tag{3.3}$$

such that

$$|f(y, z)| \le \frac{\epsilon}{16} \quad \forall y, z \in X \text{ satisfying } \rho(y, z) \le 4\delta_0. \tag{3.4}$$

Set

$$\delta = 2^{-1}\delta_0. \tag{3.5}$$

Define

$$\begin{aligned} \phi(t) &= 1, \quad t \in [0, \delta_0], \\ \phi(t) &= 0, \quad t \in [2\delta_0, \infty), \\ \phi(t) &= 2 - t\delta_0^{-1}, \quad t \in (\delta_0, 2\delta_0), \end{aligned} \tag{3.6}$$

$$f_1(x, y) = -\phi(\rho(x, y))\rho(x, y) + (1 - \phi(\rho(x, y)))f(x, y), \quad (x, y \in X). \tag{3.7}$$

Clearly, f_1 is continuous and

$$f_1(x, x) = 0 \quad \forall x \in X. \tag{3.8}$$

By (3.6) and (3.7),

$$f_1(x, y) = -\rho(x, y) \quad \forall\, x, y \in X \text{ satisfying } \rho(x, y) \leq \delta_0. \tag{3.9}$$

Let $x, y \in X$. We estimate $|f(x, y) - f_1(x, y)|$. If $\rho(x, y) \geq 2\delta_0$, then by (3.6) and (3.7),

$$|f_1(x, y) - f(x, y)| = 0. \tag{3.10}$$

Assume that

$$\rho(x, y) \leq 2\delta_0. \tag{3.11}$$

By (3.3) and (3.11),

$$|f(x, y)| \leq \frac{\epsilon}{16}. \tag{3.12}$$

By (3.5), (3.6), (3.7), (3.11), and (3.12),

$$|f_1(x, y) - f(x, y)| \leq \rho(x, y) + |f(x, y)| \leq 2\delta_0 + \frac{\epsilon}{16} < \frac{\epsilon}{4}. \tag{3.13}$$

Together with (3.10) this implies that

$$|f_1(x, y) - f(x, y)| < \frac{\epsilon}{4} \quad \forall x, y \in X. \tag{3.14}$$

Assume that $x \in X$. In view of (P3) and (3.3), there is $y \in X$ such that

$$\rho(y, x) \in [2^{-1}\delta_0, \delta_0]. \tag{3.15}$$

It follows from (3.15) and (3.9) that

$$f_1(x, y) = -\rho(y, x) \leq -2^{-1}\delta_0, \tag{3.16}$$

$$\inf\{f_1(x, z) : z \in X\} \leq -2^{-1}\delta_0 \tag{3.17}$$

for all $x \in X$. Set

$$g(x,y) = \phi(\rho(x,x_0)) f(x,y) + (1-\phi(\rho(x,x_0))) f_1(x,y), \quad x,y \in X. \tag{3.18}$$

Clearly, the function g is continuous and

$$g(x,x) = 0 \quad \forall x \in X. \tag{3.19}$$

In view of (3.1), (3.18), and (3.6),

$$g(x_0,y) = f(x_0,y) \geq 0 \quad \forall y \in X. \tag{3.20}$$

Since the function f possesses (P2), it follows from (3.9), (3.20), and (3.18) that g possesses the property (P2). Thus $g \in \mathcal{A}$.

By (3.6), (3.14), and (3.18) for all $x, y \in X$

$$|(f-g)(x,y)| \leq |f_1(x,y) - f(x,y)| \leq \frac{\epsilon}{4}. \tag{3.21}$$

Assume that

$$x \in X, \quad \inf\{g(x,y) : y \in X\} > -2^{-1}\delta_0 = -\delta. \tag{3.22}$$

If $\rho(x_0,x) \geq 2\delta_0$, then by (3.6) and (3.18),

$$g(x,y) = f_1(x,y) \quad \forall y \in Y \tag{3.23}$$

and together with (3.17), this implies that

$$\inf\{g(x,y) : y \in X\} \leq -2^{-1}\delta_0. \tag{3.24}$$

This inequality contradicts (3.22). The contradiction we have reached proves that

$$\rho(x_0,x) < 2\delta_0 < \frac{\epsilon}{8}. \tag{3.25}$$

This completes the proof of the lemma. □

4. Proof of Theorem 1.1

Denote by E the set of all $f \in \mathcal{A}$ for which there exists $x \in X$ such that $f(x,y) \geq 0$ for all $y \in X$. By Lemma 2.1, E is an everywhere dense subset of $\mathcal{A}$.

Let $f \in E$ and n be a natural number. There exists $x_f \in X$ such that

$$f(x_f,y) \geq 0 \quad \forall y \in X. \tag{4.1}$$

By Lemma 3.1, there exist $g_{f,n} \in \mathcal{A}$ and $\delta_{f,n} > 0$ such that

$$g_{f,n}(x_f,y) \geq 0 \quad \forall y \in X, \qquad |(g_{f,n} - f)(x,y)| \leq (4n)^{-1} \quad \forall x,y \in X, \tag{4.2}$$

and the following property holds.

(P4) For each $x \in X$ satisfying $\inf\{g_{f,n}(x,y) : y \in X\} > -\delta_{f,n}$, the inequality $\rho(x_f, x) < (4n)^{-1}$ holds.

Denote by $V(f,n)$ the open neighborhood of $g_{f,n}$ in $\mathcal{A}$ such that

$$V(f,n) \subset \{h \in \mathcal{A} : (h, g_{f,n}) \in U(4^{-1}\delta_{f,n})\}. \tag{4.3}$$

Assume that

$$x \in X, \quad h \in V(f,n), \quad \inf\{h(x,y) : y \in X\} > -2^{-1}\delta_{f,n}. \tag{4.4}$$

By (1.3), (4.3), and (4.4),

$$\inf\{g_{f,n}(x,y) : y \in X\} \geq \inf\{h(x,y) : y \in X\} - 4^{-1}\delta_{f,n} > -\delta_{f,n}. \tag{4.5}$$

In view of (4.5) and (P4),

$$\rho(x_f, x) < (4n)^{-1}. \tag{4.6}$$

Thus we have shown that the following property holds.

(P5) For each $x \in X$ and each $h \in V(f,n)$ satisfying (4.4), the inequality $\rho(x_f, x) < (4n)^{-1}$ holds.

Set

$$\mathcal{F} = \bigcap_{k=1}^{\infty} \cup \{V(f,n) : f \in E \text{ and an integer } n \geq k\}. \tag{4.7}$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subset of $\mathcal{A}$. Let

$$\xi \in \mathcal{F}, \quad \epsilon > 0. \tag{4.8}$$

Choose a natural number $k > 8(\epsilon^{-1} + 1)$. There exist $f \in E$ and an integer $n \geq k$ such that

$$\xi \in V(f,n). \tag{4.9}$$

The property (P4), (4.3), and (4.9) imply that for each $x \in X$ satisfying

$$\inf\{\xi(x,y) : y \in X\} > -2^{-1}\delta_{f,n}, \tag{4.10}$$

we have

$$\begin{gathered}\inf\{g_{f,n}(x,y) : y \in X\} > -2^{-1}\delta_{f,n} - 4^{-1}\delta_{f,n} > -\delta_{f,n}, \\ \rho(x_f, x) < (4n)^{-1} < \frac{\epsilon}{8}.\end{gathered} \tag{4.11}$$

Thus we have shown that the following property holds.

(P6) For each $x \in X$ satisfying (4.10), the inequality $\rho(x_f, x) < \epsilon/8$ holds.

By (P1) there is a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ such that

$$\liminf_{i\to\infty} \left(\inf\left\{\xi(x_i, y) : y \in X\right\}\right) \geq 0. \tag{4.12}$$

In view of (4.12) and (P6) for all large enough natural numbers $i,\ j$, we have

$$\rho(x_i, x_j) \leq \rho(x_i, x_f) + \rho(x_f, x_j) < \frac{\epsilon}{4}. \tag{4.13}$$

Since ϵ is any positive number, we conclude that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence and there exists

$$x_\xi = \lim_{i\to\infty} x_i. \tag{4.14}$$

Relations (4.12) and (4.14) imply that for all $y \in X$

$$\xi(x_\xi, y) = \lim_{i\to\infty} \xi(x_i, y) \geq 0. \tag{4.15}$$

We have also shown that any sequence $\{x_i\}_{i=1}^{\infty} \subset X$ satisfying (4.12) converges. This implies that if $x \in X$ satisfies $\xi(x, y) \geq 0$ for all $y \in X$, then $x = x_\xi$. By (P6) and (4.15),

$$\rho(x_\xi, x_f) \leq \frac{\epsilon}{8}. \tag{4.16}$$

Let $x \in X$ and $h \in V(f, n)$ satisfy (4.4). By (P5), $\rho(x_f, x) < (4n)^{-1}$. Together with (4.16), this implies that

$$\rho(x, x_\xi) \leq \rho(x, x_f) + \rho(x_f, x_\xi) < (4n)^{-1} + \frac{\epsilon}{8} < \epsilon. \tag{4.17}$$

Theorem 1.1 is proved.

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