Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 619525, 14 pages
doi:10.1155/2008/619525

Research Article

Weighted Composition Operators from Generalized Weighted Bergman Spaces to Weighted-Type Spaces

Dinggui Gu

Department of Mathematics, JiaYing University, Meizhou, GuangDong 514015, China

Correspondence should be addressed to Dinggui Gu, gudinggui@163.com

Received 3 November 2008; Revised 22 November 2008; Accepted 24 November 2008

Recommended by Kunquan Lan

Let φ be a holomorphic self-map and let ψ be a holomorphic function on the unit ball B. The boundedness and compactness of the weighted composition operator ψC_φ from the generalized weighted Bergman space into a class of weighted-type spaces are studied in this paper.

1. Introduction

Let B be the unit ball of $\mathbb{C}^n$ and let $H(B)$ be the space of all holomorphic functions on B. For $f \in H(B)$, let

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) \tag{1.1}$$

represent the radial derivative of $f \in H(B)$. We write $\Re^m f = \Re(\Re^{m-1} f)$.

For any $p > 0$ and $\alpha \in \mathbb{R}$, let N be the smallest nonnegative integer such that $pN + \alpha > -1$. The generalized weighted Bergman space A_α^p is defined as follows:

$$A_\alpha^p = \left\{ f \in H(B) \mid \|f\|_{A_\alpha^p} = |f(0)| + \left[\int_B |\Re^N f(z)|^p (1 - |z|^2)^{pN+\alpha} dv(z) \right]^{1/p} < \infty \right\}. \tag{1.2}$$

Here dv is the normalized Lebesgue measure of B (i.e., $v(B) = 1$). The generalized weighted Bergman space A_α^p is introduced by Zhao and Zhu (see, e.g., [1]). This space covers the (classical) weighted Bergman space ($\alpha > -1$), the Besov space $A^p_{-(n+1)}$, and the Hardy space H^2. See [1, 2] for some basic facts on the weighted Bergman space.

Let μ be a positive continuous function on $[0,1)$. We say that μ is normal if there exist positive numbers α and β, $0 < \alpha < \beta$, and $\delta \in [0,1)$ such that (see [3])

$$\frac{\mu(r)}{(1-r)^{\alpha}} \text{ is decreasing on } [\delta,1), \qquad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^{\alpha}} = 0;$$
$$\frac{\mu(r)}{(1-r)^{\beta}} \text{ is increasing on } [\delta,1), \qquad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^{\beta}} = \infty. \tag{1.3}$$

An $f \in H(B)$ is said to belong to the weighted-type space, denoted by $H^\infty_\mu = H^\infty_\mu(B)$, if

$$\|f\|_{H^\infty_\mu} = \sup_{z\in B}\mu(|z|)\left|f(z)\right| < \infty, \tag{1.4}$$

where μ is normal on $[0,1)$. The little weighted-type space, denoted by $H^\infty_{\mu,0}$, is the subspace of H^∞_μ consisting of those $f \in H^\infty_\mu$ such that

$$\lim_{|z|\to 1}\mu(|z|)\left|f(z)\right| = 0. \tag{1.5}$$

See [4, 5] for more information on H^∞_μ.

Let φ be a holomorphic self-map of B. The composition operator C_φ is defined as follows:

$$\left(C_\varphi f\right)(z) = (f\circ\varphi)(z), \quad f \in H(B). \tag{1.6}$$

Let $\psi \in H(B)$. For $f \in H(B)$, the weighted composition operator ψC_φ is defined by

$$\left(\psi C_\varphi f\right)(z) = \psi(z) f\left(\varphi(z)\right), \quad z \in B. \tag{1.7}$$

The book [6] contains a plenty of information on the composition operator and the weighted composition operator.

In the setting of the unit ball, Zhu studied the boundedness and compactness of the weighted composition operator between Bergman-type spaces and H^∞ in [7]. Some extensions of these results can be found in [8]. Some necessary and sufficient conditions for the weighted composition operator to be bounded or compact between the Bloch space and H^∞ are given in [9]. In the setting of the unit polydisk, some necessary and sufficient conditions for a weighted composition operator to be bounded and compact between the Bloch space and H^∞ are given in [10, 11] (see also [12] for the case of composition operators). In [13], Zhu studied the boundedness and compactness of the Volterra composition operators from generalized weighted Bergman space to μ-Bloch-type space. Other related results can be found, for example, in [4, 5, 14–22].

In this paper, we study the weighted composition operator ψC_φ from the generalized weighted Bergman space to the spaces H_μ^∞ and $H_{\mu,0}^\infty$. Some necessary and sufficient conditions for the weighted composition operator ψC_φ to be bounded and compact are given.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Main results and proofs

Before we formulate our main results, we state several auxiliary results which will be used in the proofs. They are incorporated in the lemmas which follow.

Lemma 2.1 (see [1]). (i) *Suppose that $p > 0$ and $\alpha + n + 1 > 0$. Then there exists a constant $C > 0$ such that*

$$|f(z)| \leq \frac{C\|f\|_{A_\alpha^p}}{(1-|z|^2)^{(n+\alpha+1)/p}} \tag{2.1}$$

for all $f \in A_\alpha^p$ and $z \in B$.

(ii) *Suppose that $p > 0$ and $\alpha + n + 1 < 0$ or $0 < p \leq 1$ and $\alpha + n + 1 = 0$. Then every function in A_α^p is continuous on the closed unit ball. Moreover, there is a positive constant C such that*

$$\|f\|_\infty \leq C\|f\|_{A_{-(n+1)}^p}, \tag{2.2}$$

for every $f \in A_{-(n+1)}^p$.

(iii) *Suppose that $p > 1$, $1/p + 1/q = 1$, and $\alpha + n + 1 = 0$. Then there exists a constant $C > 0$ such that*

$$|f(z)| \leq C\left[\ln \frac{e}{1-|z|^2}\right]^{1/q} \tag{2.3}$$

for all $f \in A_\alpha^p$ and $z \in B$.

The following criterion for compactness of weighted composition operators follows from standard arguments similar to those outlined in [6, Proposition 3.11] (see also [12, proof of Lemma 2]). We omit the details of the proof.

Lemma 2.2. *Assume that $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0,1)$. Then $\psi C_\varphi : A_\alpha^p \to H_\mu^\infty$ is compact if and only if $\psi C_\varphi : A_\alpha^p \to H_\mu^\infty$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in A_α^p which converges to zero uniformly on compact subsets of B as $k \to \infty$, one has $\|\psi C_\varphi f_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$.*

Note that when $p > 0$ and $\alpha + n + 1 < 0$, the functions in A_α^p are Lipschitz continuous (see [1, Theorem 66]). By Lemma 2.1 and Arzela-Ascoli theorem, similarly to [19, proof of Lemma 3.6], we have the following result.

Lemma 2.3. *Let $p > 0$ and $\alpha + n + 1 < 0$. Let (f_k) be a bounded sequence in A^p_α which converges to 0 uniformly on compact subsets of B, then*

$$\lim_{k \to \infty} \sup_{z \in B} |f_k(z)| = 0. \tag{2.4}$$

The following lemma is from [21] (one-dimensional case is [20, Lemma 2.1]).

Lemma 2.4. *Assume that μ is a normal function on $[0,1)$. A closed set K in $H^\infty_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z| \to 1} \sup_{f \in K} \mu(|z|) |f(z)| = 0. \tag{2.5}$$

We will consider three cases: $n + 1 + \alpha > 0$, $n + 1 + \alpha = 0$, and $n + 1 + \alpha < 0$.

2.1. Case $n + 1 + \alpha > 0$

Theorem 2.5. *Assume that $p > 0$, α is a real number such that $n + \alpha + 1 > 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0,1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded if and only if*

$$M := \sup_{z \in B} \frac{\mu(|z|) |\psi(z)|}{\left(1 - |\varphi(z)|^2\right)^{(n+1+\alpha)/p}} < \infty. \tag{2.6}$$

Proof. Assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Let

$$t > n \max\left(1, \frac{1}{p}\right) + \frac{\alpha + 1}{p}. \tag{2.7}$$

For $a \in B$, set

$$f_a(z) = \frac{\left(1 - |a|^2\right)^{t-(n+1+\alpha)/p}}{\left(1 - \langle z, a \rangle\right)^t}. \tag{2.8}$$

It follows from [1, Theorem 32] that $f_a \in A^p_\alpha$ and $\sup_{a \in B} \|f_a\|_{A^p_\alpha} < \infty$. Hence

$$\begin{aligned} C\|\psi C_\varphi\|_{A^p_\alpha \to H^\infty_\mu} &\geq \|\psi C_\varphi f_{\varphi(b)}\|_{H^\infty_\mu} \\ &= \sup_{z \in B} \mu(|z|) \left|\left(\psi C_\varphi f_{\varphi(b)}\right)(z)\right| \\ &\geq \frac{\mu(|b|) |\psi(b)|}{\left(1 - |\varphi(b)|^2\right)^{(n+1+\alpha)/p}}, \end{aligned} \tag{2.9}$$

from which we get (2.6).

Conversely, suppose that (2.6) holds. Then for arbitrary $z \in B$ and $f \in A^p_\alpha$, by Lemma 2.1 we have

$$\mu(|z|)\left|(\psi C_\varphi f)(z)\right| = \mu(|z|)\left|f(\varphi(z))\right|\left|\psi(z)\right| \leq C\|f\|_{A^p_\alpha}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}}. \tag{2.10}$$

In light of condition (2.6), the boundedness of the operator $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ follows from (2.10) by taking the supremum over B. This proof is completed. □

Theorem 2.6. *Assume that $p > 0$, α is a real number such that $n + \alpha + 1 > 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is compact if and only if $\psi \in H^\infty_\mu$ and*

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} = 0. \tag{2.11}$$

Proof. Assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is compact, then $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Taking $f(z) \equiv 1$, we get that $\psi \in H^\infty_\mu$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist that condition (2.11) is vacuously satisfied). Set

$$f_k(z) = \frac{\left(1-\left|\varphi(z_k)\right|^2\right)^{t-(n+\alpha+1)/p}}{\left(1-\langle z,\varphi(z_k)\rangle\right)^t}, \quad k \in \mathbb{N}, \tag{2.12}$$

where t satisfies (2.7). From [1, Theorem 32], we see that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A^p_α. Moreover, it is easy to see that f_k converges to zero uniformly on compact subsets of B. By Lemma 2.2, $\limsup_{k\to\infty}\|\psi C_\varphi f_k\|_{H^\infty_\mu} = 0$. On the other hand, we have

$$\left\|\psi C_\varphi f_k\right\|_{H^\infty_\mu} = \sup_{z\in B}\mu(|z|)\left|(\psi C_\varphi f_k)(z)\right| \geq \frac{\mu(|z_k|)\left|\psi(z_k)\right|}{\left(1-\left|\varphi(z_k)\right|^2\right)^{(n+1+\alpha)/p}}. \tag{2.13}$$

Hence

$$\limsup_{k\to\infty}\frac{\mu(|z_k|)\left|\psi(z_k)\right|}{\left(1-\left|\varphi(z_k)\right|^2\right)^{(n+1+\alpha)/p}} = 0, \tag{2.14}$$

from which (2.11) follows.

Conversely, assume that $\psi \in H^\infty_\mu$ and (2.11) holds. Then, it is easy to check that (2.6) holds. Hence $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. According to (2.11), for given $\varepsilon > 0$, there is a constant $\delta \in (0, 1)$ such that

$$\sup_{\{z\in B:\delta<|\varphi(z)|<1\}}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} < \varepsilon. \tag{2.15}$$

Let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in A^p_α such that $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. Let $\delta\mathbb{D} = \{w \in B : |w| \le \delta\}$. From (2.15) and $\psi \in H^\infty_\mu$, we have

$$\begin{aligned}\|\psi C_\varphi f_k\|_{H^\infty_\mu} &= \sup_{z\in B}\mu(|z|)\left|f_k(\varphi(z))\psi(z)\right| \\ &= \left(\sup_{\{z\in B:|\varphi(z)|\le\delta\}} + \sup_{\{z\in B:\delta<|\varphi(z)|<1\}}\right)\mu(|z|)|\psi(z)||f_k(\varphi(z))| \\ &= \|\psi\|_{H^\infty_\mu}\sup_{w\in\delta\mathbb{D}}|f_k(w)| + C\|f_k\|_{A^p_\alpha}\sup_{\{z\in B:\delta<|\varphi(z)|<1\}}\frac{\mu(|z|)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+\alpha)/p}} \\ &\le \|\psi\|_{H^\infty_\mu}\sup_{w\in\delta\mathbb{D}}|f_k(w)| + C\varepsilon.\end{aligned} \tag{2.16}$$

Since $\delta\mathbb{D}$ is a compact subset of B, we have $\lim_{k\to\infty}\sup_{w\in\delta\mathbb{D}}|f_k(w)| = 0$. Using this fact and letting $k \to \infty$ in (2.16), we obtain

$$\limsup_{k\to\infty}\|\psi C_\varphi f_k\|_{H^\infty_\mu} \le C\varepsilon. \tag{2.17}$$

Since ε is an arbitrary positive number, we obtain $\limsup_{k\to\infty}\|\psi C_\varphi f_k\|_{H^\infty_\mu} = 0$. By Lemma 2.2, the implication follows. □

Theorem 2.7. *Assume that $p > 0$, α is a real number such that $n + \alpha + 1 > 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded if and only if $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded and $\psi \in H^\infty_{\mu,0}$.*

Proof. Assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded. Then it is clear that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Taking $f(z) = 1$ and employing the boundedness of $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$, we see that $\psi \in H^\infty_{\mu,0}$.

Conversely, assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded and $\psi \in H^\infty_{\mu,0}$. Suppose that $f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} \le L$, using polynomial approximations we obtain (see, e.g., [1])

$$\lim_{|z|\to1}\left(1-|z|^2\right)^{(n+1+\alpha)/p}|f(z)| = 0. \tag{2.18}$$

From the above equality and $\psi \in H^\infty_{\mu,0}$, we have that for every $\varepsilon > 0$, there exists a $\delta \in (0, 1)$ such that when $\delta < |z| < 1$,

$$\left(1-|z|^2\right)^{(n+1+\alpha)/p}|f(z)| < \frac{\varepsilon}{M}, \tag{2.19}$$

$$\mu(|z|)|\psi(z)| < \frac{\varepsilon\left(1-\delta^2\right)^{(n+1+\alpha)/p}}{L}, \tag{2.20}$$

where M is defined in (2.6). Therefore, if $\delta < |z| < 1$ and $\delta < |\varphi(z)| < 1$, from (2.6) and (2.19) we have

$$\begin{aligned}\mu(|z|)\left|(\psi C_\varphi f)(z)\right| &= \frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}}\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}\left|f(\varphi(z))\right| \\ &\le M\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}\left|f(\varphi(z))\right| < \varepsilon.\end{aligned} \tag{2.21}$$

If $\delta < |z| < 1$ and $|\varphi(z)| \le \delta$, using Lemma 2.1 and (2.20) we have

$$\begin{aligned}\mu(|z|)\left|(\psi C_\varphi f)(z)\right| &= \frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}}\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}\left|f(\varphi(z))\right| \\ &\le C\|f\|_{A^p_\alpha}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} \\ &\le \frac{C\|f\|_{A^p_\alpha}}{(1-\delta^2)((n+1+\alpha)/p)}\mu(|z|)\left|\psi(z)\right| < \varepsilon.\end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22), we obtain that $\psi C_\varphi f \in H^\infty_{\mu,0}$. Since f is an arbitrary element of A^p_α we see that

$$\psi C_\varphi\left(A^p_\alpha\right) \subset H^\infty_{\mu,0}, \tag{2.23}$$

which, along with the boundedness of $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$, implies the result. □

Theorem 2.8. *Assume that $p > 0$, α is a real number such that $n + \alpha + 1 > 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} = 0. \tag{2.24}$$

Proof. Assume that (2.24) holds. For any $f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} \le 1$, by (2.10) we have

$$\mu(|z|)\left|(\psi C_\varphi f)(z)\right| \le C\|f\|_{A^p_\alpha}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}}. \tag{2.25}$$

Using (2.24), we get

$$\lim_{|z|\to 1}\sup_{\|f\|_{A^p_\alpha}\le 1}\mu(|z|)\left|(\psi C_\varphi f)(z)\right| \le C\lim_{|z|\to 1}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}}=0. \tag{2.26}$$

From this and Lemma 2.4, we see that $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is compact.

Conversely, assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is compact. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded and $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu}$ is compact. By Theorems 2.6 and 2.7, we obtain

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}}=0, \tag{2.27}$$

$$\lim_{|z|\to 1}\mu(|z|)\left|\psi(z)\right|=0. \tag{2.28}$$

If $\|\varphi\|_\infty < 1$, it holds that

$$\lim_{|z|\to 1}\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} \le \frac{1}{\left(1-\|\varphi\|_\infty^2\right)^{(n+1+\alpha)/p}}\lim_{|z|\to 1}\mu(|z|)\left|\psi(z)\right|=0, \tag{2.29}$$

from which the result follows in this case.

Hence, assume that $\|\varphi\|_\infty = 1$. In terms of (2.27), for every $\varepsilon > 0$, there exists a $\delta \in (0,1)$, such that when $\delta < |\varphi(z)| < 1$,

$$\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} < \varepsilon. \tag{2.30}$$

According to (2.28), for the above ε, there exists an $r \in (0,1)$, such that when $r < |z| < 1$,

$$\mu(|z|)\left|\psi(z)\right| < \varepsilon\left(1-\delta^2\right)^{(n+1+\alpha)/p}. \tag{2.31}$$

Therefore, when $r < |z| < 1$ and $\delta < |\varphi(z)| < 1$, we have that

$$\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} < \varepsilon. \tag{2.32}$$

If $r < |z| < 1$ and $|\varphi(z)| \le \delta$, we obtain

$$\frac{\mu(|z|)\left|\psi(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} \le \frac{1}{\left(1-\delta^2\right)^{(n+1+\alpha)/p}}\mu(|z|)\left|\psi(z)\right| < \varepsilon. \tag{2.33}$$

Combining (2.32) with (2.33) we get (2.24), as desired. □

2.2. Case $n + 1 + \alpha = 0$

Theorem 2.9. *Assume that $p > 1$, α is a real number such that $n + \alpha + 1 = 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded if and only if*

$$M_1 := \sup_{z\in B} \mu(|z|)\left|\psi(z)\right|\left(\ln \frac{e}{1 - \left|\varphi(z)\right|^2}\right)^{1-1/p} < \infty. \tag{2.34}$$

Proof. Assume that (2.34) holds. Then for arbitrary $z \in B$ and $f \in A^p_\alpha$, by Lemma 2.1 we have

$$\begin{aligned}\mu(|z|)\left|(\psi C_\varphi f)(z)\right| &= \mu(|z|)\left|f(\varphi(z))\right|\left|\psi(z)\right| \\ &\le C\|f\|_{A^p_\alpha}\mu(|z|)\left|\psi(z)\right|\left(\ln \frac{e}{1 - \left|\varphi(z)\right|^2}\right)^{1-1/p}.\end{aligned} \tag{2.35}$$

From (2.34) and (2.35), the boundedness of $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ follows.

Now assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. For $a \in B$, set

$$f_a(z) = \left(\ln \frac{e}{1 - |a|^2}\right)^{-1/p}\left(\ln \frac{e}{1 - \langle z, a\rangle}\right). \tag{2.36}$$

By using [2, Theorem 1.12], we easily check that $f_a \in A^p_{-(n+1)}$. Therefore,

$$\begin{aligned}C\left\|\psi C_\varphi\right\|_{A^p_\alpha \to H^\infty_\mu} &\ge \left\|\psi C_\varphi f_{\varphi(b)}\right\|_{H^\infty_\mu} \\ &= \sup_{z\in B} \mu(|z|)\left|(\psi C_\varphi f_{\varphi(b)})(z)\right| \\ &\ge \mu(|b|)\left|\psi(b)\right|\left(\ln \frac{e}{1 - \left|\varphi(b)\right|^2}\right)^{1-1/p}.\end{aligned} \tag{2.37}$$

From the last inequality, we get the desired result. □

Theorem 2.10. *Assume that $p > 1$, α is a real number such that $n + \alpha + 1 = 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is compact if and only if $\psi \in H^\infty_\mu$ and*

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)\left|\psi(z)\right|\left(\ln \frac{e}{1 - \left|\varphi(z)\right|^2}\right)^{1-1/p} = 0. \tag{2.38}$$

Proof. First assume that (2.38) holds and $\psi \in H^\infty_\mu$. In this case, the proof of Theorem 2.6 still works with minor changes, hence we omit the details.

Now we assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is compact, then it is clear that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Similarly to the proof of Theorem 2.6, we see that $\psi \in H^\infty_\mu$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist that condition (2.38) is vacuously satisfied). Set

$$f_k(z) = \left(\ln \frac{e}{1-|\varphi(z_k)|^2}\right)^{-1/p} \left(\ln \frac{e}{1-\langle z,\varphi(z_k)\rangle}\right), \quad k \in \mathbb{N}. \tag{2.39}$$

From [2, Theorem 1.12], we see that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A^p_α. Moreover, $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. It follows from Lemma 2.2 that $\|\psi C_\varphi f_k\|_{H^\infty_\mu} \to 0$ as $k \to \infty$. Because

$$\begin{aligned} \|\psi C_\varphi f_k\|_{H^\infty_\mu} &= \sup_{z\in B} \mu(|z|)\left|(\psi C_\varphi f_k)(z)\right| \\ &\geq \mu(|z_k|)|\psi(z_k)|\left(\ln \frac{e}{1-|\varphi(z_k)|^2}\right)^{1-1/p}, \end{aligned} \tag{2.40}$$

we obtain

$$\lim_{k\to\infty} \mu(|z_k|)|\psi(z_k)|\left(\ln \frac{e}{1-|\varphi(z_k)|^2}\right)^{1-1/p} = 0, \tag{2.41}$$

from which we get the desired result. The proof is completed. □

Theorem 2.11. *Assume that $p > 1$, α is a real number such that $n + \alpha + 1 = 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0,1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded if and only if $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded and $\psi \in H^\infty_{\mu,0}$.*

Proof. First assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded. Then clearly $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Taking $f(z) = 1$, then employing the boundedness of $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$, we have that $\psi \in H^\infty_{\mu,0}$, as desired.

Conversely, assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded and $\psi \in H^\infty_{\mu,0}$. For each polynomial p, we have

$$\mu(|z|)\left|(\psi C_\varphi p)(z)\right| = \mu(|z|)\left|p(\varphi(z))\right|\left|\psi(z)\right| \leq \|p\|_\infty \mu(|z|)\left|\psi(z)\right|, \tag{2.42}$$

from which we have that $\psi C_\varphi(p) \in H^\infty_{\mu,0}$.

Since the set of all polynomials is dense in A^p_α (see [2]), for every $f \in A^p_\alpha$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that

$$\|p_k - f\|_{A^p_\alpha} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.43}$$

From the boundedness of $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$, we have that

$$\|\psi C_\varphi p_k - \psi C_\varphi f\|_{H^\infty_\mu} \le \|\psi C_\varphi\| \|p_k - f\|_{A^p_\alpha} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.44}$$

Since $H^\infty_{\mu,0}$ is a closed subset of H^∞_μ, we obtain

$$\psi C_\varphi f = \lim_{k\to\infty} \psi C_\varphi p_k \in H^\infty_{\mu,0}. \tag{2.45}$$

Therefore, $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded. □

Using Theorems 2.10 and 2.11, similarly to the proof of Theorem 2.8 we obtain the following result. We omit the proof.

Theorem 2.12. *Assume that $p > 1$, α is a real number such that $n + \alpha + 1 = 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \mu(|z|) |\psi(z)| \left(\ln \frac{e}{1 - |\varphi(z)|^2} \right)^{1-1/p} = 0. \tag{2.46}$$

Theorem 2.13. *Assume that $0 < p \le 1$, α is a real number such that $n + \alpha + 1 = 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded if and only if $\psi \in H^\infty_\mu$.*

Proof. Assume that $\psi \in H^\infty_\mu$. For any $f \in A^p_\alpha$, by Lemma 2.1 we have

$$\sup_{z\in B} \mu(|z|) |(\psi C_\varphi f)(z)| \le C\|f\|_{A^p_\alpha} \sup_{z\in B} \mu(|z|) |\psi(z)|. \tag{2.47}$$

From the above inequality, we obtain that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded.

Conversely, assume that $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Taking $f(z) = 1$ and using the boundedness of $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$, we get $\psi \in H^\infty_\mu$, as desired. □

Theorem 2.14. *Assume that $0 < p \le 1$, α is a real number such that $n + \alpha + 1 = 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then $\psi C_\varphi : A^p_\alpha \to H^\infty_\mu$ is compact if and only if $\psi \in H^\infty_\mu$ and*

$$\lim_{|\varphi(z)|\to 1} \mu(|z|) |\psi(z)| = 0. \tag{2.48}$$

Proof. First assume that $\psi \in H^\infty_\mu$ and (2.48) holds. In this case, the proof is similar to the corresponding part of the proof of Theorem 2.6 and hence will be omitted.

Now we suppose that $\psi C_\varphi : A_\alpha^p \to H_\mu^\infty$ is compact. It follows from Theorem 2.13 and the boundedness of $\psi C_\varphi : A_\alpha^p \to H_\mu^\infty$ that $\psi \in H_\mu^\infty$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Set

$$f_k(z) = \frac{1 - |\varphi(z_k)|^2}{1 - \langle z, \varphi(z_k)\rangle}, \quad k \in \mathbb{N}. \tag{2.49}$$

From [2, Theorem 6.6], we see that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A_α^p. Moreover, f_k converges to zero uniformly on compact subsets of B. Hence by Lemma 2.2 it follows that

$$\limsup_{k\to\infty} \|\psi C_\varphi f_k\|_{H_\mu^\infty} = 0. \tag{2.50}$$

On the other hand, we obtain

$$\|\psi C_\varphi f_k\|_{H_\mu^\infty} = \sup_{z\in B} \mu(|z|)|(\psi C_\varphi f_k)(z)| \geq \mu(|z_k|)|\psi(z_k)|. \tag{2.51}$$

Combining (2.50) with (2.51) we obtain that (2.48) holds. □

Theorem 2.15. *Assume that* $0 < p \leq 1$, α *is a real number such that* $n + \alpha + 1 = 0$, $\psi \in H(B)$, φ *is a holomorphic self-map of* B, *and* μ *is a normal function on* $[0, 1)$. *Then the following statements are equivalent:*

(i) $\psi C_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$ *is bounded;*

(ii) $\psi C_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$ *is compact;*

(iii) $\psi \in H_{\mu,0}^\infty$.

Proof. (ii)⇒(i). This implication is clear.

(i)⇒(iii). Taking $f(z) = 1$ and employing the boundedness of $\psi C_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$, we obtain that $\psi \in H_{\mu,0}^\infty$.

(iii)⇒(ii). For any $f \in A_\alpha^p$ with $\|f\|_{A_\alpha^p} \leq 1$, we have

$$\mu(|z|)|(\psi C_\varphi f)(z)| \leq C\|f\|_{A_\alpha^p}\mu(|z|)|\psi(z)| \leq C\mu(|z|)|\psi(z)|, \tag{2.52}$$

from which we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{A_\alpha^p}\leq 1} \mu(|z|)|(\psi C_\varphi f)(z)| \leq C \lim_{|z|\to 1} \mu(|z|)|\psi(z)| = 0. \tag{2.53}$$

Using Lemma 2.4, we obtain that $\psi C_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$ is compact. □

2.3. *Case* $n + 1 + \alpha < 0$

Theorem 2.16. *Assume that* $p > 0$, α *is a real number such that* $n + \alpha + 1 < 0$, $\psi \in H(B)$, φ *is a holomorphic self-map of* B, *and* μ *is a normal function on* $[0, 1)$. *Then the following statements are equivalent:*

(i) $\psi C_\varphi : A_\alpha^p \to H_\mu^\infty$ *is bounded;*

(ii) $\psi C_\varphi : A_\alpha^p \to H_\mu^\infty$ *is compact;*

(iii) $\psi \in H_\mu^\infty$.

Proof. (ii)⇒(i). This implication is obvious.

(i)⇒(iii). Taking $f(z) = 1$, then using the boundedness of $\psi C_\varphi : A_\alpha^p \to H_\mu^\infty$, we obtain that $\psi \in H_\mu^\infty$.

(iii)⇒(ii). If $f \in A_\alpha^p$, by Lemma 2.1 we obtain

$$\mu(|z|)\left|(\psi C_\varphi f)(z)\right| \le C\|f\|_{A_\alpha^p}\mu(|z|)\left|\psi(z)\right|, \tag{2.54}$$

from which it follows that $\psi C_\varphi : A_\alpha^p \to H_\mu^\infty$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in A_α^p and $f_k \to 0$ uniformly on B as $k \to \infty$. By Lemma 2.3, we have

$$\left\|\psi C_\varphi f_k\right\|_{H_\mu^\infty} = \sup_{z\in B}\mu(|z|)\left|f_k(\varphi(z))\psi(z)\right| \le \|\psi\|_{H_\mu^\infty}\sup_{z\in B}\left|f_k(\varphi(z))\right| \longrightarrow 0, \tag{2.55}$$

as $k \to \infty$. The result follows from Lemma 2.2. □

Similarly to the proof of Theorem 2.15, we have the following result. We omit the proof here.

Theorem 2.17. *Assume that $p > 0$, α is a real number such that $n + \alpha + 1 < 0$, $\psi \in H(B)$, φ is a holomorphic self-map of B, and μ is a normal function on $[0, 1)$. Then the following statements are equivalent:*

(i) $\psi C_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$ *is bounded;*

(ii) $\psi C_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$ *is compact;*

(iii) $\psi \in H_{\mu,0}^\infty$.

Acknowledgment

The author is supported partly by the NSF of Guangdong Province of China (no. 07006700).

References

[1] R. Zhao and K. Zhu, "Theory of Bergman spaces on the unit ball," to appear *Mémoires de la Société Mathématique de France*.
[2] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, vol. 226 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2005.
[3] A. L. Shields and D. L. Williams, "Bonded projections, duality, and multipliers in spaces of analytic functions," *Transactions of the American Mathematical Society*, vol. 162, pp. 287–302, 1971.
[4] S. Stević, "Essential norms of weighted composition operators from the α-Bloch space to a weighted-type space on the unit ball," *Abstract and Applied Analysis*, vol. 2008, Article ID 279691, 10 pages, 2008.
[5] S. Stević, "Norm of weighted composition operators from Bloch space to H_μ^∞ on the unit ball," *Ars Combinatoria*, vol. 88, pp. 125–127, 2008.
[6] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.

[7] X. Zhu, “Weighted composition operators between H^∞ and Bergman type spaces,” *Communications of the Korean Mathematical Society*, vol. 21, no. 4, pp. 719–727, 2006.
[8] S. Stević, “Weighted composition operators between mixed norm spaces and H_α^∞ spaces in the unit ball,” *Journal of Inequalities and Applications*, vol. 2007, Article ID 28629, 9 pages, 2007.
[9] S. Li and S. Stević, “Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball,” *Taiwanese Journal of Mathematics*, vol. 12, pp. 1625–1639, 2008.
[10] S. Li and S. Stević, “Weighted composition operators from H^∞ to the Bloch space on the polydisc,” *Abstract and Applied Analysis*, vol. 2007, Article ID 48478, 13 pages, 2007.
[11] S. Li and S. Stević, “Weighted composition operators from α-Bloch space to H^∞ on the polydisc,” *Numerical Functional Analysis and Optimization*, vol. 28, no. 7-8, pp. 911–925, 2007.
[12] S. Stević, “Composition operators between H^∞ and α-Bloch spaces on the polydisc,” *Zeitschrift für Analysis und ihre Anwendungen*, vol. 25, no. 4, pp. 457–466, 2006.
[13] X. Zhu, “Volterra composition operators from generalized weighted Bergman spaces to μ-Bloch type spaces,” to appear in *Journal of Function Spaces and Applications*.
[14] D. D. Clahane and S. Stević, “Norm equivalence and composition operators between Bloch/Lipschitz spaces of the ball,” *Journal of Inequalities and Applications*, vol. 2006, Article ID 61018, 11 pages, 2006.
[15] X. Fu and X. Zhu, “Weighted composition operators on some weighted spaces in the unit ball,” *Abstract and Applied Analysis*, vol. 2008, Article ID 605807, 8 pages, 2008.
[16] S. Li and S. Stević, “Weighted composition operators from Bergman-type spaces into Bloch spaces,” *Proceedings of the Indian Academy of Sciences. Mathematical Sciences*, vol. 117, no. 3, pp. 371–385, 2007.
[17] S. Li and S. Stević, “Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces,” *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 40–52, 2008.
[18] K. Madigan and A. Matheson, “Compact composition operators on the Bloch space,” *Transactions of the American Mathematical Society*, vol. 347, no. 7, pp. 2679–2687, 1995.
[19] S. Ohno, K. Stroethoff, and R. Zhao, “Weighted composition operators between Bloch-type spaces,” *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 1, pp. 191–215, 2003.
[20] A. Montes-Rodríguez, “Weighted composition operators on weighted Banach spaces of analytic functions,” *Journal of the London Mathematical Society*, vol. 61, no. 3, pp. 872–884, 2000.
[21] S. Stević, “Essential norms of weighted composition operators from the Bergman space to weighted-type spaces on the unit ball,” to appear in *Ars Combinatoria*.
[22] X. Zhu, “Generalized weighted composition operators from Bloch type spaces to weighted Bergman spaces,” *Indian Journal of Mathematics*, vol. 49, no. 2, pp. 139–150, 2007.