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Research Article

Brézis-Wainger Inequality on Riemannian Manifolds

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The Brézis-Wainger inequality on a compact Riemannian manifold without boundary is shown. For this purpose, the Moser-Trudinger inequality and the Sobolev embedding theorem are applied.

1. Introduction

There is no doubt that the Brézis-Wainger inequality (see [1]) is a very useful tool in the examination of partial differential equations. Namely, a lot of estimates to a solution of PDE are obtained with the help of the Brézis-Wainger inequality. Especially, the inequality is often applied in the theory of wave maps.

In this paper, we extend the Brézis-Wainger result onto a compact Riemannian manifold. We show the following theorem.

Theorem 1.1. *Let $(\mathcal{M}, g)$ be an n-dimensional compact Riemannian manifold, and $u \in H^{k,p}(\mathcal{M})$, $\int_{\mathcal{M}} u\,dV_g = 0$, for $n > k > n/p$, where k is a positive integer and $p \geq 1$ is a real number. Then, $u \in L^{\infty}(\mathcal{M})$ and*

$$\|u\|_{L^{\infty}(\mathcal{M})} \leq \|u\|_{H^{k,n/k}(\mathcal{M})}\left(C + \log\frac{\|u\|_{H^{k,p}(\mathcal{M})}}{\|u\|_{H^{k,n/k}(\mathcal{M})}}\right), \tag{1.1}$$

where $C = C(k, \mathcal{M})$ is a positive constant.

The proof relies on the application of a Moser-Trudinger inequality (see Theorem 2.2) and the Sobolev embedding theorem (see Theorem 2.1). Moreover, we will use the integral representation of a smooth function via the Green function (see [2]).

2. Preliminaries

In order to make this paper more readable, we recall some definitions and facts from the theory of Sobolev spaces on Riemannian manifolds. In particular, we present useful inequalities and embeddings.

Let $(\mathcal{M}, g)$ be a smooth, compact Riemannian manifold without boundary. We will denote by $C^\infty(\mathcal{M})$ a space of smooth real functions. For $\phi \in C^\infty(\mathcal{M})$ and integer m, we denote by $\nabla^m \phi$ the mth covariant derivative of ϕ. Next, for $\phi \in C^\infty(\mathcal{M})$ and for a fixed integer m and a real $p \geq 1$, we set

$$\|\phi\|_{H^{m,p}(\mathcal{M})} = \sum_{i=0}^{m} \left(\int_{\mathcal{M}} |\nabla^i \phi|^p dV_g \right)^{1/p}, \tag{2.1}$$

where by V_g we have denoted the Riemannian measure on the manifold $(\mathcal{M}, g)$.

We define the Sobolev space $H^{m,p}(\mathcal{M})$ as a completion of $C^\infty(\mathcal{M})$ with respect to $\|\cdot\|_{H^{m,p}(\mathcal{M})}$.

We close this section stating the following results, which will be used in the proof of the main result.

Theorem 2.1 (Sobolev embedding theorem [3, 4]). *Let $(\mathcal{M}, g)$ be a smooth, compact Riemannian n-manifold. Then, for any real numbers $1 \leq p < q$ and any integers $0 \leq m < k$, if $1/q = 1/p - (k-m)/n$, then $H^{k,p}(\mathcal{M}) \hookrightarrow H^{m,q}(\mathcal{M})$. Moreover, there exists a constant C such that for all $u \in H^{k,p}(\mathcal{M})$, the following inequality holds:*

$$\|u\|_{H^{m,q}(\mathcal{M})} \leq C \|u\|_{H^{k,p}(\mathcal{M})}. \tag{2.2}$$

Theorem 2.2 (Moser-Trudinger inequality [5]). *Let $(\mathcal{M}, g)$ be a smooth, compact Riemannian n-manifold and k a positive integer, strictly smaller than n. There exist a constant $C = C(k, \mathcal{M})$ and $\lambda(k, n)$ such that for all $u \in C^n(\mathcal{M})$ with $\int_{\mathcal{M}} u dV_g = 0$ and $\int_{\mathcal{M}} |\nabla^k u|^{n/k} dV_g \leq 1$, the following inequality holds:*

$$\int_{\mathcal{M}} e^{\lambda(k,n)(u/\|u\|_{H^{k,n/k}(\mathcal{M})})^{n/(n-k)}} dV_g \leq C. \tag{2.3}$$

Let us stress that this inequality is a generalization of the Moser and Trudinger result (see [6–8]).

3. Proof of the main result

In this section, we will prove the main result, that is, Theorem 1.1.

Proof. Let us notice that by the assumptions we have an embedding $H^{k,p}(\mathcal{M}) \hookrightarrow H^{k,n/k}(\mathcal{M})$. First of all, we show the following lemma.

Lemma 3.1. *If $u \in C^\infty(\mathcal{M})$, $\int_{\mathcal{M}} u dV_g = 0$, $\int_{\mathcal{M}} |\nabla^k u|^{n/k} dV_g \leq 1$, and $d \in [1, \infty)$, then $e^u \in L^d(\mathcal{M})$. Moreover, the following estimate holds:*

$$\|e^u\|_{L^d(\mathcal{M})} \leq C(\|u\|_{H^{k,n/k}(\mathcal{M})}, d, \mathcal{M}). \tag{3.1}$$

Proof. Let us put

$$a = \left(\lambda(k,n)\frac{n}{n-k}\right)^{(n-k)/n}\frac{u}{\|u\|_{H^{k,n/k}(\mathcal{M})}},$$
$$b = d\left(\lambda(k,n)\frac{n}{n-k}\right)^{n/(n-k)}\|u\|_{H^{k,n/k}(\mathcal{M})} \tag{3.2}$$

in Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{3.3}$$

where $1/p + 1/q = 1$. Then, we obtain an estimate

$$ud \leq \lambda(k,n)\left(\frac{u}{\|u\|_{H^{k,n/k}(\mathcal{M})}}\right)^{n/(n-k)} + \frac{k}{n}d^{n/k}\left(\lambda(k,n)\frac{n}{n-k}\right)^{(n-k)/n}\|u\|^{n/k}_{H^{k,n/k}(\mathcal{M})}, \tag{3.4}$$

where $\lambda(k,n)$ is a constant from the Moser-Trudinger inequality (see Theorem 2.2).

Next, we can estimate

$$\int_{\mathcal{M}} e^{ud}dV_g \leq C\left(d, \|u\|_{H^{k,n/k}(\mathcal{M})}\right)\int_{\mathcal{M}} e^{\lambda(k,n)(u/\|u\|_{H^{k,n/k}(\mathcal{M})})^{n/(n-k)}}dV_g. \tag{3.5}$$

Subsequently, we can apply the Moser-Trudinger inequality to the right-hand side of the above inequality:

$$\int_{\mathcal{M}} e^{\lambda(k,n)(u/\|u\|_{H^{k,n/k}(\mathcal{M})})^{n/(n-k)}}\,dV_g \leq c. \tag{3.6}$$

From this, the proof of Lemma 3.1 follows. □

Now, we can go back to the proof of Theorem 1.1. First, we prove the theorem for $u \in C^{\infty}(\mathcal{M})$ such that $\|u\|_{H^{k,n/k}(\mathcal{M})} = 1$ and $\int_{\mathcal{M}} u dV_g = 0$. Replacing u by $-u$ if necessary, we may suppose that

$$\|u\|_{L^{\infty}(\mathcal{M})} = u(x_0), \tag{3.7}$$

for $x_0 \in \mathcal{M}$.

Let us recall (see [2]) that for $(\mathcal{M}, g)$, a compact Riemannian n-manifold, there exists a Green function $G : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ such that

(1) for any $\phi \in C^{\infty}(\mathcal{M})$ and any $z \in \mathcal{M}$,

$$\phi(z) = \frac{1}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}} \phi\, dV_g + \int_{\mathcal{M}} G(z,x)\Delta_g\phi(x)\, dV_g, \tag{3.8}$$

where $\mathrm{Vol}(\mathcal{M})$ is a Riemannian volume of the manifold $(\mathcal{M}, g)$, and Δ_g is the Laplace-Beltrami operator on a manifold;

(2) there exists a constant C such that for any $(x,y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta$,

$$\left|\nabla_x G(x,y)\right| \le \frac{C}{\left(d_g(x,y)\right)^{n-1}}, \tag{3.9}$$

where Δ is a diagonal:

$$\Delta = \left\{(x,y) \in \mathcal{M} \times \mathcal{M} : x = y\right\}, \tag{3.10}$$

and $d_g(x,y)$ is a Riemannian distance from x to y.

Let us define the map $\psi = e^u - 1$. Next, we apply the first property of the Green function to the map ψ and to the point x_0. Namely,

$$\begin{aligned} 0 &\le \psi(x_0) \\ &= \frac{1}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}} (e^u - 1)\,dV_g + \int_{\mathcal{M}} G(x_0,x)\,\Delta_g \psi(x)\,dV_g \\ &\le 1 + \frac{1}{\mathrm{Vol}(\mathcal{M})}\left\|e^u\right\|_{L^1(\mathcal{M})} + \int_{\mathcal{M}} \left|\nabla_x G(x_0,x)\right|\left|\nabla_x \psi(x)\right| dV_g. \end{aligned} \tag{3.11}$$

Subsequently, by the second property of the Green function we can estimate $\psi(x_0)$ as follows:

$$\psi(x_0) \le 1 + \frac{1}{\mathrm{Vol}(\mathcal{M})}\left\|e^u\right\|_{L^1(\mathcal{M})} + \int_{\mathcal{M}} \frac{C}{\left(d_g(x_0,x)\right)^{n-1}} |\nabla u| e^u\,dV_g. \tag{3.12}$$

Now, we will try to estimate the last term in the inequality (3.12). Let us notice that if $u \in H^{k,p}(\mathcal{M})$, then $\nabla u \in H^{k-1,p}(\mathcal{M})$. Next, by the Sobolev theorem (see Theorem 2.1),

$$H^{k-1,p}(\mathcal{M}) \hookrightarrow L^q(\mathcal{M}), \quad \text{for } \frac{1}{q} = \frac{1}{p} - \frac{k-1}{n}, \tag{3.13}$$

and we have that $\nabla u \in L^q(\mathcal{M})$.

Using elementary calculations, one can easily show the lemma.

Lemma 3.2. *There exist $r < n/(n-1)$ and a finite d such that the following equality holds:*

$$\frac{1}{r} + \frac{1}{q} + \frac{1}{d} = 1, \tag{3.14}$$

where q is the exponent from the Sobolev theorem.

By Hölder's inequality with exponents r, q, d, we can estimate the inequality (3.12) as follows:

$$\psi(x_0) \le 1 + \frac{1}{\mathrm{Vol}(\mathcal{M})}\left\|e^u\right\|_{L^1(\mathcal{M})} + \widetilde{C}\|\nabla u\|_{L^q(\mathcal{M})}\left\|e^u\right\|_{L^d(\mathcal{M})}, \tag{3.15}$$

where $\widetilde{C} = \sup_{x_0 \in \mathcal{M}} \|C/(d_g(x_0,\cdot))^{n-1}\|_{L^r(\mathcal{M})}$ and r is the exponent from Lemma 3.2. Finally, by Lemma 3.1 and the Sobolev theorem, we obtain

$$\varphi(x_0) \le C\left(1 + \|u\|_{H^{k,p}(\mathcal{M})}\right). \tag{3.16}$$

Let us stress that since $\|u\|_{H^{k,n/k}(\mathcal{M})} = 1$, the constant C does not depend on u. We can rewrite the inequality (3.16) as follows:

$$e^{u(x_0)} \le C\left(1 + \|u\|_{H^{k,p}(\mathcal{M})}\right). \tag{3.17}$$

Hence,

$$\|u\|_{L^\infty(\mathcal{M})} = u(x_0) \le \log\left(C\left(1 + \|u\|_{H^{k,p}(\mathcal{M})}\right)\right). \tag{3.18}$$

Taking into account $1 = \|u\|_{H^{k,n/k}(\mathcal{M})} \le c\|u\|_{H^{k,p}(\mathcal{M})}$, we obtain

$$\|u\|_{L^\infty(\mathcal{M})} \le C + \log \|u\|_{H^{k,p}(\mathcal{M})}. \tag{3.19}$$

This finishes the proof of the inequality in the case $u \in C^\infty(\mathcal{M})$ such that $\|u\|_{H^{k,n/k}(\mathcal{M})} = 1$ and $\int_{\mathcal{M}} u dV_g = 0$. Subsequently, one can easily obtain the inequality for $u \in C^\infty(\mathcal{M})$ such that $\int_{\mathcal{M}} u dV_g = 0$.

Now, we prove the theorem for an arbitrary $u \in H^{k,p}(\mathcal{M})$ such that $\int_{\mathcal{M}} u dV_g = 0$. We apply the density argument. Namely, for any $\varepsilon > 0$, there exists $\widetilde{u}_\varepsilon \in C^\infty(\mathcal{M})$ such that

$$\left\|\widetilde{u}_\varepsilon - u\right\|_{H^{k,p}(\mathcal{M})} \le \varepsilon. \tag{3.20}$$

Next, we define

$$u_\varepsilon = \widetilde{u}_\varepsilon - \frac{1}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}} \widetilde{u}_\varepsilon \, dV_g. \tag{3.21}$$

Such u_ε has zero-mean value. Moreover, the following inequality holds:

$$\left\|u_\varepsilon - u\right\|_{H^{k,p}(\mathcal{M})} \le 2\varepsilon. \tag{3.22}$$

Indeed,

$$\begin{aligned}
\left\|u_\varepsilon - u\right\|_{H^{k,p}(\mathcal{M})} &\le \left\|\widetilde{u}_\varepsilon - u\right\|_{H^{k,p}(\mathcal{M})} + \left\|\frac{1}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}} (\widetilde{u}_\varepsilon - u) dV_g\right\|_{H^{k,p}(\mathcal{M})} \\
&\le \varepsilon + \frac{1}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}} \left|\widetilde{u}_\varepsilon - u\right| dV_g \|1\|_{H^{k,p}(\mathcal{M})} \\
&\le \varepsilon + \left(\int_{\mathcal{M}} \left|\widetilde{u}_\varepsilon - u\right|^p dV_g\right)^{1/p} \\
&\le 2\varepsilon.
\end{aligned} \tag{3.23}$$

Hence,

$$\|u\|_{L^\infty(\mathcal{M})} \le \left\|u - u_\varepsilon\right\|_{L^\infty(\mathcal{M})} + \left\|u_\varepsilon\right\|_{H^{k,n/k}(\mathcal{M})}\left(C + \log\frac{\left\|u_\varepsilon\right\|_{H^{k,p}(\mathcal{M})}}{\left\|u_\varepsilon\right\|_{H^{k,n/k}(\mathcal{M})}}\right). \tag{3.24}$$

Finally, we take $\varepsilon \to 0$. This completes the proof. □

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