Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 720371, 15 pages
doi:10.1155/2008/720371

Research Article

Common Solutions of an Iterative Scheme for Variational Inclusions, Equilibrium Problems, and Fixed Point Problems

Jian-Wen Peng,[1] Yan Wang,[1] David S. Shyu,[2] and Jen-Chih Yao[3]

[1] *College of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China*
[2] *Department of Finance, National Sun Yat-Sen University, Kaohsiung 80424, Taiwan*
[3] *Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung 80424, Taiwan*

Correspondence should be addressed to David S. Shyu, dshyu@cm.nsysu.edu.tw

Received 24 October 2008; Accepted 6 December 2008

Recommended by Ram U. Verma

We introduce an iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem, and the set of fixed points of a nonexpansive mapping. We obtain a strong convergence theorem for the sequences generated by these processes in Hilbert spaces. The results in this paper unify, extend, and improve some well-known results in the literature.

Copyright © 2008 Jian-Wen Peng et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. 2^H denotes the family of all the nonempty subsets of H.

Let $A : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. We consider the following variational inclusion, which is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{1.1}$$

where θ is the zero vector in H. The set of solutions of problem (1.1) is denoted by $I(A, M)$. If $H = R^m$, then problem (1.1) becomes the generalized equation introduced by Robinson [1]. If $A = 0$, then problem (1.1) becomes the inclusion problem introduced by Rockafellar [2]. It is known that (1.1) provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, and so forth. Also various types of variational inclusions problems have been extended and generalized, for more details, please see [3–20] and the references therein.

There are many algorithms for solving variational inclusions (see [3, 5, 7, 8, 10–13, 16–20] and the references therein). We note that recently Zhang et al. [20] introduced the following new iterative scheme for finding a common element of the set of solutions to the problem (1.1) and the set of fixed points of nonexpansive mappings in Hilbert spaces. Starting with an arbitrary point $x_1 = x \in H$, define sequences $\{x_n\}$ by

$$\begin{aligned} x_{n+1} &= \alpha_n x + (1-\alpha_n) S y_n, \\ y_n &= J_{M,\lambda}(x_n - \lambda A x_n), \quad \forall n \geq 0, \end{aligned} \tag{1.2}$$

where $J_{M,\lambda} = (I + \lambda M)^{-1}$ is the resolvent operator associated with M and a positive number λ, $\{\alpha_n\}$ is a sequence in the interval $[0, 1]$.

Let F be a bifunction from $C \times C$ to R, where R is the set of real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $EP(F)$. The problem (1.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; for more details (see [21]).

Recall that a mapping S of a closed convex subset C into itself is nonexpansive if there holds that

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

A mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|fx - fy\| \leq \alpha \|x - y\|, \quad \forall x, y \in C. \tag{1.5}$$

We denote the set of fixed points of S by $\mathrm{Fix}(S)$. It is known that $\mathrm{Fix}(S)$ is closed convex, but possibly empty. There are some methods for approximation of fixed points of a nonexpansive mapping. In 2000, Moudafi [22] introduced the viscosity approximation method for nonexpansive mappings.

Some methods have been proposed to solve the equilibrium problem; see, for instance, [23–30] and the references therein. Recently, S. Takahashi and W. Takahashi [23] introduced the following Mann iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.3) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary point $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S u_n, \quad \forall n \in N. \end{aligned} \tag{1.6}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.6) converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{\mathrm{Fix}(S)\cap EP(F)} f(z)$.

The purpose of this paper is twofold. On one hand, it is easy to see that the authors in [28–32] introduced some algorithms for finding a common element of the set of solutions of an equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality. It is natural to raise and to give an answer to the following question: can one construct algorithms for finding a common element of the set of solutions of an equilibrium problem, the set of fixed points of a nonexpansive mapping, and the set of solutions of a variational inclusion? In other words, can we construct algorithms for finding a solution of a mathematical model which consists of an equilibrium problem and a variational inclusion if this mathematical model has a solution? In this paper, we will give a positive answer to this question. By combining the algorithm (1.2) for a variational inclusion problem and the algorithm (1.6) for an equilibrium problem, we introduce the following iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the variational inclusion with a set-valued maximal monotone mapping and an inverse strongly monotone mapping, the set of solutions of equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary point $x_1 \in H$, define sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n, \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \quad \forall n \geq 0, \end{gathered} \tag{1.7}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$.

On the other hand, we will show that the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ generated by (1.7) converge strongly to $z \in F(S) \cap EP(F) \cap I(A,M)$ if the parameter sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy appropriate conditions. The results in this paper unify, extend, and improve some corresponding results in [20, 23] and the references therein.

2. Preliminaries

In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C x \in C$ and

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.2) is equivalent to

$$\|x-y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$.

Recall that a mapping $A : H \to H$ is called α-inverse strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in H. \tag{2.4}$$

Let I be the identity mapping on H. It is well known that if $A : H \to H$ is an α-inverse strongly monotone, then A is $(1/\alpha)$-Lipschitz continuous and monotone mapping. In addition, if $0 < \lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping.

Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. It is known that H satisfies the Opial condition [33], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.5}$$

holds for every $y \in H$ with $x \neq y$. A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$, and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if its graph $G(M) := \{(f, x) \in H \times H \mid f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Let the set-valued mapping $M : H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{M,\lambda}$ associated with M and λ as follows:

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H, \tag{2.6}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive, and 1-inverse strongly monotone, (see, e.g., [34]), and that a solution of problem (1.1) is a fixed point of the operator $J_{M,\lambda}(I - \lambda A)$ for all $\lambda > 0$ (see, e.g., [35]).

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F\big(tz + (1 - t)x, y\big) \leq F(x, y), \tag{2.7}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.1 (see [21]). *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.8}$$

Lemma 2.2 (see [23, 24]). *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \tag{2.9}$$

for all $x \in H$. Then, the following statements hold

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle, \tag{2.10}$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Lemma 2.3 (see [34]). *Let $M : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous mapping. Then the mapping $S = M + A : H \to 2^H$ is a maximal monotone mapping.*

Remark 2.4. Lemma 2.3 implies that $I(A, M)$ is closed and convex if $M : H \to 2^H$ is a maximal monotone mapping and $A : H \to H$ be an inverse strongly monotone mapping.

Lemma 2.5 (see [36, 37]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \tag{2.11}$$

where γ_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$\begin{aligned} &\text{(i)}\ \sum_{n=1}^{\infty} \gamma_n = \infty, \\ &\text{(ii)}\ \limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty. \end{aligned} \tag{2.12}$$

Then, $\lim_{n\to\infty} \alpha_n = 0$.

3. Strong convergence theorem

In this section, we establish a strong convergence theorem which solves the problem of finding a common element of the set of solutions of variational inclusion, the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let S be a nonexpansive mapping of C into H. Let $A : H \to H$ be an α-inverse strongly monotone mapping, $M : H \to 2^H$ be a maximal monotone mapping such that $\Omega = \mathrm{Fix}(S) \cap EP(F) \cap I(A, M) \neq \emptyset$, and let $f : H \to H$ be a contraction mapping with a constant*

$\xi \in (0,1)$. *Let* x_1 *be an arbitrary point in* H, $\{x_n\}$, $\{u_n\}$, *and* $\{y_n\}$ *be sequences generated by algorithm* (1.7). *If* $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} \left|\alpha_{n+1} - \alpha_n\right| < \infty,$$
$$\liminf_{n\to\infty} r_n > 0, \qquad \sum_{n=1}^{\infty} \left|r_{n+1} - r_n\right| < \infty. \tag{3.1}$$

Then, $\{x_n\}$, $\{y_n\}$, *and* $\{u_n\}$ *converge strongly to* $z \in \Omega$, *where* $z = P_\Omega f(z)$.

Proof. We show that $P_\Omega f$ is a contraction mapping. In fact, we have

$$\left\|P_\Omega f(x) - P_\Omega f(y)\right\| \le \left\|f(x) - f(y)\right\| \le \xi\|x - y\| \tag{3.2}$$

for any $x, y \in H$. Since H is complete, there exists a unique element $z \in H$ such that $z = P_\Omega f(z)$. Let $v \in \Omega$. Then, from $u_n = T_{r_n} x_n$, we have

$$\left\|u_n - v\right\| = \left\|T_{r_n} x_n - T_{r_n} v\right\| \le \left\|x_n - v\right\|, \tag{3.3}$$

for all $n \in N$. It is easy to see that $v = J_{M,\lambda}(v - \lambda A v)$. As $I - \lambda A$ is nonexpansive, we have

$$\begin{aligned}
\left\|y_n - v\right\| &= \left\|J_{M,\lambda}(u_n - \lambda A u_n) - J_{M,\lambda}(v - \lambda A v)\right\| \\
&\le \left\|(u_n - \lambda A u_n) - (v - \lambda A v)\right\| \\
&\le \left\|u_n - v\right\| \\
&\le \left\|x_n - v\right\|,
\end{aligned} \tag{3.4}$$

for all $n \in N$. It follows from (1.7) and (3.4) that

$$\begin{aligned}
\left\|x_{n+1} - v\right\| &= \left\|\alpha_n f(x_n) + (1 - \alpha_n) S y_n - v\right\| \\
&= \left\|\alpha_n (f(x_n) - v) + (1 - \alpha_n)(S y_n - v)\right\| \\
&\le \alpha_n \left\|f(x_n) - v\right\| + (1 - \alpha_n)\left\|S y_n - v\right\| \\
&\le \alpha_n \left(\left\|f(x_n) - f(v)\right\| + \left\|f(v) - v\right\|\right) + (1 - \alpha_n)\left\|y_n - v\right\| \\
&\le \alpha_n \xi \left\|x_n - v\right\| + \alpha_n \left\|f(v) - v\right\| + (1 - \alpha_n)\left\|x_n - v\right\| \\
&= \left[1 - \alpha_n(1 - \xi)\right]\left\|x_n - v\right\| + \alpha_n(1 - \xi)\frac{1}{1 - \xi}\left\|f(v) - v\right\| \\
&\le \max\left\{\left\|x_n - v\right\|; \frac{1}{1 - \xi}\left\|f(v) - v\right\|\right\} \\
&\le \cdots \\
&\le \max\left\{\left\|x_1 - v\right\|; \frac{1}{1 - \xi}\left\|f(v) - v\right\|\right\}
\end{aligned} \tag{3.5}$$

for all $n \geq 1$. This implies that $\{x_n\}$ is bounded. From which it follows that $\{u_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{Sy_n\}$, and $\{Au_n\}$ are also bounded. Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. In fact, since $I - \lambda A$ is nonexpansive, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|J_{M,\lambda}(u_{n+1} - \lambda Au_{n+1}) - J_{M,\lambda}(u_n - \lambda Au_n)\| \\ &\leq \|u_{n+1} - \lambda Au_{n+1} - (u_n - \lambda Au_n)\| \\ &\leq \|u_{n+1} - u_n\|. \end{aligned} \tag{3.6}$$

It follows from (1.7) and (3.6) that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)Sy_n - \alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})Sy_{n-1}\| \\ &= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1}f(x_{n-1}) \\ &\quad + (1-\alpha_n)Sy_n - (1-\alpha_n)Sy_{n-1} + (1-\alpha_n)Sy_{n-1} - (1-\alpha_{n-1})Sy_{n-1}\| \\ &\leq \alpha_n\xi\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| + (1-\alpha_n)\|Sy_n - Sy_{n-1}\| \\ &\leq \alpha_n\xi\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| + (1-\alpha_n)\|y_n - y_{n-1}\| \\ &\leq \alpha_n\xi\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| + (1-\alpha_n)\|u_n - u_{n-1}\|, \end{aligned} \tag{3.7}$$

where $K = \sup\{\|f(x_n)\| + \|Sy_n\|, n \geq 1\}$.

On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C, \tag{3.8}$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \quad \forall y \in C. \tag{3.9}$$

Putting $y = u_{n+1}$ in (3.8) and $y = u_n$ in (3.9), we have

$$\begin{gathered} F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0, \\ F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \end{gathered} \tag{3.10}$$

It follows from the monotonicity of F that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0. \tag{3.11}$$

Thus,

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geq 0. \tag{3.12}$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in N$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\ &\leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\}. \end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{L}{b} |r_{n+1} - r_n|, \end{aligned} \tag{3.14}$$

where $L = \sup\{\|u_n - x_n\| : n \in N\}$. From (3.7) and (3.14), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n \xi \|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| + (1 - \alpha_n)\|x_n - x_{n-1}\| + (1 - \alpha_n)\frac{L}{b}|r_n - r_{n-1}| \\ &\leq (1 - \alpha_n(1 - \xi))\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| + \frac{L}{b}|r_n - r_{n-1}|. \end{aligned} \tag{3.15}$$

It follows from Lemma 2.5 that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.16}$$

From (3.14) and $|r_{n+1} - r_n| \to 0$, we have

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0. \tag{3.17}$$

It follows from (3.6) that

$$\lim_{n \to \infty} \|y_{n+1} - y_n\| = 0. \tag{3.18}$$

Since $x_n = \alpha_{n-1} f(x_{n-1}) + (1 - \alpha_{n-1}) S y_{n-1}$, we have

$$\begin{aligned} \|x_n - Sy_n\| &\leq \|x_n - Sy_{n-1}\| + \|Sy_{n-1} - Sy_n\| \\ &\leq \alpha_{n-1}\|f(x_{n-1}) - Sy_{n-1}\| + \|y_{n-1} - y_n\|. \end{aligned} \tag{3.19}$$

It follows from $\alpha_n \to 0$ that $\|x_n - Sy_n\| \to 0$.

Now we prove that for any given $v \in \Omega$,

$$\|Au_n - Av\| \longrightarrow 0. \tag{3.20}$$

In fact, for $v \in \Omega$, we have

$$\begin{aligned} \|u_n - v\|^2 &= \|T_{r_n} x_n - T_{r_n} v\|^2 \\ &\leq \langle T_{r_n} x_n - T_{r_n} v, x_n - v \rangle \\ &= \langle u_n - v, x_n - v \rangle \\ &= \frac{1}{2}\left(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2\right), \end{aligned} \tag{3.21}$$

and hence,

$$\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - u_n\|^2. \tag{3.22}$$

It follows from (1.7), (3.4), and (3.22) that

$$\begin{aligned}
\|x_{n+1}-v\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)Sy_n-v\|^2\\
&\le \alpha_n\|f(x_n)-v\|^2+(1-\alpha_n)\|Sy_n-v\|^2\\
&\le \alpha_n\|f(x_n)-v\|^2+(1-\alpha_n)\|y_n-v\|^2\\
&\le \alpha_n\|f(x_n)-v\|^2+(1-\alpha_n)\left[\|x_n-v\|^2-\|x_n-u_n\|^2\right]\\
&\le \alpha_n\|f(x_n)-v\|^2+\|x_n-v\|^2-(1-\alpha_n)\|x_n-u_n\|^2.
\end{aligned}\tag{3.23}$$

Thus, we have

$$\begin{aligned}
(1-\alpha_n)\|x_n-u_n\|^2 &\le \alpha_n\|f(x_n)-v\|^2+\|x_n-v\|^2-\|x_{n+1}-v\|^2\\
&\le \alpha_n\|f(x_n)-v\|^2+\|x_n-x_{n+1}\|\left(\|x_n-v\|+\|x_{n+1}-v\|\right).
\end{aligned}\tag{3.24}$$

Since $\alpha_n \to 0$, $\|x_n-x_{n+1}\| \to 0$, and $\{x_n\}$ is bounded, we have $\|x_n-u_n\| \to 0$. From $\|u_n-x_{n+1}\| \le \|u_n-x_n\|+\|x_n-x_{n+1}\|$, we get $\|u_n-x_{n+1}\| \to 0$.

Equation (3.23), the nonexpansiveness of $J_{M,\lambda}$, and the inverse strongly monotonicity of A imply that

$$\begin{aligned}
\|x_{n+1}-v\|^2 &\le \alpha_n\|f(x_n)-v\|^2+(1-\alpha_n)\|y_n-v\|^2\\
&\le \alpha_n\|f(x_n)-v\|^2+(1-\alpha_n)\|u_n-\lambda Au_n-(v-\lambda Av)\|^2\\
&\le \alpha_n\|f(x_n)-v\|^2+(1-\alpha_n)\left\{\|u_n-v\|^2+\lambda(\lambda-2\alpha)\|Au_n-Av\|^2\right\}\\
&\le \alpha_n\|f(x_n)-v\|^2+\|u_n-v\|^2+(1-\alpha_n)\lambda(\lambda-2\alpha)\|Au_n-Av\|^2.
\end{aligned}\tag{3.25}$$

Thus, we have

$$\begin{aligned}
(1-\alpha_n)\lambda(\lambda-2\alpha)\|Au_n-Av\|^2 &\le \alpha_n\|f(x_n)-v\|^2+\left(\|u_n-v\|^2-\|x_{n+1}-v\|^2\right)\\
&\le \alpha_n\|f(x_n)-v\|^2+\left(\|u_n-x_{n+1}\|\right)\left(\|u_n-v\|+\|x_{n+1}-v\|\right).
\end{aligned}\tag{3.26}$$

Since $\alpha_n \to 0$, $\|u_n-x_{n+1}\| \to 0$, $\{u_n\}$ and $\{x_n\}$ are bounded, we have

$$\|Au_n-Av\| \longrightarrow 0. \tag{3.27}$$

Next, we show that $\|Sy_n-y_n\| \to 0$. Indeed, for any $v\in\Omega$,

$$\begin{aligned}
\|y_n-v\|^2 &= \|J_{M,\lambda}(u_n-\lambda Au_n)-J_{M,\lambda}(v-\lambda Av)\|^2\\
&\le \langle u_n-\lambda Au_n-(v-\lambda Av),\, y_n-v\rangle\\
&= \frac{1}{2}\left\{\|u_n-\lambda Au_n-(v-\lambda Av)\|^2+\|y_n-v\|^2-\|u_n-\lambda Au_n-(v-\lambda Av)-(y_n-v)\|^2\right\}\\
&\le \frac{1}{2}\left\{\|u_n-v\|^2+\|y_n-v\|^2-\|u_n-y_n-\lambda(Au_n-Av)\|^2\right\}\\
&= \frac{1}{2}\left\{\|u_n-v\|^2+\|y_n-v\|^2-\|u_n-y_n\|^2+2\lambda\langle u_n-y_n,\, Au_n-Av\rangle-\lambda^2\|Au_n-Av\|^2\right\}.
\end{aligned}\tag{3.28}$$

Thus, we have

$$\|y_n - v\|^2 \le \|u_n - v\|^2 - \|u_n - y_n\|^2 + 2\lambda\langle u_n - y_n, Au_n - Av\rangle - \lambda^2\|Au_n - Av\|^2. \tag{3.29}$$

As the function $\|\cdot\|^2$ is convex, by (3.29) and (3.23), we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &\le \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\|y_n - v\|^2\\ &\le \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\\ &\quad \times\{\|u_n - v\|^2 - \|u_n - y_n\|^2 + 2\lambda\langle u_n - y_n, Au_n - Av\rangle - \lambda^2\|Au_n - Av\|^2\}.\end{aligned} \tag{3.30}$$

Thus, we get

$$\begin{aligned}(1-\alpha_n)\|u_n - y_n\|^2 &\le \alpha_n\|f(x_n) - v\|^2 + (\|u_n - v\|^2 - \|x_{n+1} - v\|^2)\\ &\quad + 2(1-\alpha_n)\lambda\langle u_n - y_n, Au_n - Av\rangle - (1-\alpha_n)\lambda^2\|Au_n - Av\|^2\\ &\le \alpha_n\|f(x_n) - v\|^2 + (\|u_n - x_{n+1}\|)(\|u_n - v\| - \|x_{n+1} - v\|)\\ &\quad + 2(1-\alpha_n)\lambda\langle u_n - y_n, Au_n - Av\rangle - (1-\alpha_n)\lambda^2\|Au_n - Av\|^2.\end{aligned} \tag{3.31}$$

Since $\alpha_n \to 0$, $\|Au_n - Av\| \to 0$, and $\|u_n - x_{n+1}\| \to 0$, we have $\|u_n - y_n\| \to 0$. From the triangle inequality $\|x_n - y_n\| = \|x_n - u_n\| + \|u_n - y_n\|$, we deduce that $\|x_n - y_n\| \to 0$. It then follows from the inequality $\|Sy_n - y_n\| \le \|Sy_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\|$ that $\|Sy_n - y_n\| \to 0$.

Next, we show that

$$\limsup_{n\to\infty}\langle f(z) - z,\ x_n - z\rangle \le 0, \tag{3.32}$$

where $z = P_\Omega f(z)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{n_i\to\infty}\langle f(z) - z,\ x_{n_i} - z\rangle = \limsup_{n\to\infty}\langle f(z) - z,\ x_n - z\rangle. \tag{3.33}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $\{u_{n_i}\} \rightharpoonup w$. From $\|u_n - y_n\| \to 0$, we also obtain that $y_{n_i} \rightharpoonup w$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

Let us show $w \in EP(F)$. By $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C. \tag{3.34}$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \tag{3.35}$$

and hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F(y, u_{n_i}). \tag{3.36}$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup w$, from (A4), we have

$$F(y, w) \le 0, \quad \forall y \in C. \tag{3.37}$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since, $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence, $F(y_t, w) \le 0$. So we have

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, w) \le tF(y_t, y). \tag{3.38}$$

Dividing by t, we get

$$F(y_t, y) \ge 0. \tag{3.39}$$

Letting $t \to 0$ and from (A3), we get

$$F(w, y) \ge 0 \tag{3.40}$$

for all $y \in C$ and hence $w \in EP(F)$.

We next show that $w \in \operatorname{Fix}(S)$. Assume $w \notin \operatorname{Fix}(S)$. Since $y_{n_i} \rightharpoonup w$ and $w \ne Sw$, from the Opial theorem [33] we have

$$\begin{aligned}\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - Sw\| \\ &\le \liminf_{i\to\infty} \{ \|y_{n_i} - Sy_{n_i}\| + \|Sy_{n_i} - Sw\| \} \\ &\le \liminf_{i\to\infty} \|y_{n_i} - w\|.\end{aligned} \tag{3.41}$$

This is a contradiction. So, we get $w \in \operatorname{Fix}(S)$.

We now show that $w \in I(A, M)$. In fact, since A is an α-inverse strongly monotone, A is an $(1/\alpha)$-Lipschitz continuous monotone mapping and $D(A) = H$. It follows from Lemma 2.3 that $M + A$ is maximal monotone. Let $(p, g) \in G(M + A)$, that is, $g - Ap \in M(p)$. Again since $y_{n_i} = J_{M,\lambda}(u_{n_i} - \lambda A u_{n_i})$, we have $u_{n_i} - \lambda A u_{n_i} \in (I + \lambda M)(y_{n_i})$, that is,

$$\frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda A u_{n_i}) \in M(y_{n_i}). \tag{3.42}$$

By virtue of the maximal monotonicity of $M + A$, we have

$$\left\langle p - y_{n_i},\, g - Ap - \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda A u_{n_i}) \right\rangle \ge 0, \tag{3.43}$$

and so

$$\begin{aligned}\langle p - y_{n_i},\, g\rangle &\ge \left\langle p - y_{n_i},\, Ap + \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda A u_{n_i}) \right\rangle \\ &= \left\langle p - y_{n_i},\, Ap - Ay_{n_i} + Ay_{n_i} - Au_{n_i} + \frac{1}{\lambda}(u_{n_i} - y_{n_i}) \right\rangle \\ &\ge 0 + \langle p - y_{n_i},\, Ay_{n_i} - Au_{n_i}\rangle + \left\langle p - y_{n_i},\, \frac{1}{\lambda}(u_{n_i} - y_{n_i}) \right\rangle.\end{aligned} \tag{3.44}$$

It follows from $\|u_n - y_n\| \to 0$, $\|Au_n - Ay_n\| \to 0$ and $y_{n_i} \rightharpoonup w$ that

$$\lim_{n_i \to \infty} \langle p - y_{n_i}, g \rangle = \langle p - w, g \rangle \geq 0. \tag{3.45}$$

It follows from the maximal monotonicity of $A+M$ that $\theta \in (M+A)(w)$, that is, $w \in I(A, M)$. This implies that $w \in \Omega$.

Since $z = P_\Omega f(z)$, we have

$$\begin{aligned}\limsup_{n \to \infty} \langle f(z) - z, x_n - z \rangle &= \lim_{i \to \infty} \langle f(z) - z, x_{n_i} - z \rangle \\ &= \lim_{i \to \infty} \langle f(z) - z, u_{n_i} - z \rangle \\ &= \langle f(z) - z, w - z \rangle \leq 0.\end{aligned} \tag{3.46}$$

From $x_{n+1} - z = \alpha_n (f(x_n) - z) + (1 - \alpha_n)(Sy_n - z)$ and (3.4), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \langle \alpha_n (f(x_n) - z) + (1 - \alpha_n)(Sy_n - z), x_{n+1} - z \rangle \\ &\leq \alpha_n \langle f(x_n) - f(z), x_{n+1} - z \rangle + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle + (1 - \alpha_n) \|Sy_n - z\| \|x_{n+1} - z\| \\ &\leq \alpha_n \xi \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle + (1 - \alpha_n) \|y_n - z\| \|x_{n+1} - z\| \\ &\leq \frac{1}{2} \alpha_n \xi \{ \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \} + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\quad + \frac{1}{2}(1 - \alpha_n) \{ \|y_n - z\|^2 + \|x_{n+1} - z\|^2 \} \\ &\leq \frac{1}{2} \alpha_n \xi \{ \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \} + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\ &\quad + \frac{1}{2}(1 - \alpha_n) \{ \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \} \\ &\leq \frac{1}{2}(1 - \alpha_n(1 - \xi)) \|x_n - z\|^2 + \frac{1}{2} \|x_{n+1} - z\|^2 + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle.\end{aligned} \tag{3.47}$$

It follows that

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(1 - \xi)) \|x_n - z\|^2 + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle. \tag{3.48}$$

Lemma 2.5 implies that $x_n \to z = P_\Omega f(z)$. Since $\|x_n - u_n\| \to 0$ and $\|y_n - u_n\| \to 0$, we also obtain that $u_n \to z$ and $y_n \to z$. The proof is now complete. □

By Theorem 3.1, we can obtain some new and interesting results as follows: let $VI(C, A)$ denote the solution set of the following variational inequality: finding $u \in C$ such that

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in C, \tag{3.49}$$

where C is a closed convex subset of H.

Theorem 3.2. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let S be a nonexpansive mapping of C into H. Let $A : C \to H$ be an α-inverse strongly monotone mapping such that $\Gamma = \mathrm{Fix}(S) \cap EF(F) \cap VI(C, A) \neq \emptyset$, where $VI(C, A)$ is the set of solutions of problem (3.49). Let $f : H \to H$ be a contraction mapping with a constant $\xi \in (0, 1)$ and x_1 be an arbitrary point in H. Let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sy_n, \\ y_n = P_C(u_n - \lambda Au_n) \end{gathered} \tag{3.50}$$

for all $n \in N$. If $\lambda \in (0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1]$, and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

$$\begin{gathered} \lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n \mathrm{ref} = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \liminf_{n\to\infty} r_n > 0, \qquad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{gathered} \tag{3.51}$$

Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $z \in \Gamma$ where $z = P_\Gamma f(z)$.

Proof. In Theorem 3.1 take $M = \partial\delta_C : H \to 2^H$, where $\delta_C : H \to [0, \infty]$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & v \notin C. \end{cases} \tag{3.52}$$

Then, the problem (1.1) is equivalent to problem (3.49). Again, since $M = \partial\delta_C$, then $J_{M,\lambda} = P_C$, so we have

$$y_n = P_C(u_n - \lambda Au_n). \tag{3.53}$$

The conclusion of Theorem 3.2 can be obtained from Theorem 3.1 immediately. □

Theorem 3.3. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let S be a nonexpansive mapping of C into H, Let $A : H \to H$ be an α-inverse strongly monotone mapping, $M : H \to 2^H$ be a maximal monotone mapping such that $\Omega = \mathrm{Fix}(S) \cap EP(F) \cap I(A, M) \neq \emptyset$. Let $x_1 = x$ be an arbitrary point in H and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)Sy_n, \\ y_n = J_{M,\lambda}(u_n - \lambda Au_n) \end{gathered} \tag{3.54}$$

for all $n \geq 1$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

$$\begin{gathered} \lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \liminf_{n\to\infty} r_n > 0, \qquad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{gathered} \tag{3.55}$$

Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $z \in \Omega$ where $z = P_\Omega x$.

Proof. Putting $f(y) = x$, for all $y \in H$ in Theorem 3.1. Then, $f(x_n) = x$ for all $n \geq 1$. By Theorem 3.1, we obtained the desired result. □

Remark 3.4. Putting $A = 0$ in Theorem 3.2, then $y_n = u_n$. By Theorem 3.2, we recover [23, Theorem 3.1]. Putting $C = H$ and $F(x, y) = 0$ for all $x, y \in H$ in Theorem 3.3, then $u_n = P_H x_n = x_n$. By Theorem 3.3, we recover [20, Theorem 2.1]. Hence, Theorem 3.1 unifies and extends the main results in [20, 23] and the references therein.

Acknowledgments

The authors would like to thank the referees for helpful suggestions. The first author was supported by the National Natural Science Foundation of China (Grant no. 10771228 and Grant no. 10831009), the Science and Technology Research Project of Chinese Ministry of Education (Grant no. 206123), the Education Committee project Research Foundation of Chongqing (Grant no. KJ070816). The Third author was partially supported by the grant NSC96-2628-E-110-014-MY3.

References

[1] S. M. Robinson, "Generalized equations and their solutions—I: basic theory," *Mathematical Programming Study*, no. 10, pp. 128–141, 1979.

[2] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[3] S. Adly, "Perturbed algorithms and sensitivity analysis for a general class of variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 201, no. 2, pp. 609–630, 1996.

[4] R. P. Agarwal, Y. J. Cho, and N. J. Huang, "Sensitivity analysis for strongly nonlinear quasi-variational inclusions," *Applied Mathematics Letters*, vol. 13, no. 6, pp. 19–24, 2000.

[5] R. P. Agarwal, N. J. Huang, and Y. J. Cho, "Generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings," *Journal of Inequalities and Applications*, vol. 7, no. 6, pp. 807–828, 2002.

[6] T. Q. Bao and B. S. Mordukhovich, "Existence of minimizers and necessary conditions in set-valued optimization with equilibrium constraints," *Applications of Mathematics*, vol. 52, no. 6, pp. 453–472, 2007.

[7] S. S. Chang, "Set-valued variational inclusions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 438–454, 2000.

[8] X. P. Ding, "Perturbed Ishikawa type iterative algorithm for generalized quasivariational inclusions," *Applied Mathematics and Computation*, vol. 141, no. 2-3, pp. 359–373, 2003.

[9] X. P. Ding, "Sensitivity analysis for generalized nonlinear implicit quasi-variational inclusions," *Applied Mathematics Letters*, vol. 17, no. 2, pp. 225–235, 2004.

[10] X. P. Ding, "Parametric completely generalized mixed implicit quasi-variational inclusions involving h-maximal monotone mappings," *Journal of Computational and Applied Mathematics*, vol. 182, no. 2, pp. 252–269, 2005.

[11] N.-J. Huang, "Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear implicit quasi-variational inclusions," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 1–7, 1998.

[12] Y.-P. Fang and N.-J. Huang, "H-monotone operator and resolvent operator technique for variational inclusions," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 795–803, 2003.

[13] Y.-P. Fang and N.-J. Huang, "H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 647–653, 2004.

[14] L.-J. Lin, "Variational inclusions problems with applications to Ekeland's variational principle, fixed point and optimization problems," *Journal of Global Optimization*, vol. 39, no. 4, pp. 509–527, 2007.

[15] L.-J. Lin, S. Y. Wang, and C.-S. Chuang, "Existence theorems of systems of variational inclusion problems with applications," *Journal of Global Optimization*, vol. 40, no. 4, pp. 751–764, 2008.

[16] J.-W. Peng, "A novel proof of the existence of solutions for a new system of generalized mixed quasi-variational-like inclusions involving (A, η, m)-accretive operators," *Journal of Global Optimization*, vol. 42, no. 4, pp. 559–575, 2008.

[17] R. U. Verma, "*A*-monotonicity and applications to nonlinear variational inclusion problems," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2004, no. 2, pp. 193–195, 2004.

[18] R. U. Verma, "General system of (A, η)-monotone variational inclusion problems based on generalized hybrid iterative algorithm," *Nonlinear Analysis: Hybrid Systems*, vol. 1, no. 3, pp. 326–335, 2007.

[19] R. U. Verma, "A-monotone nonlinear relaxed cocoercive variational inclusions," *Central European Journal of Mathematics*, vol. 5, no. 2, pp. 386–396, 2007.

[20] S.-S. Zhang, J. H. W. Lee, and C. K. Chan, "Algorithms of common solutions to quasi variational inclusion and fixed point problems," *Applied Mathematics and Mechanics*, vol. 29, no. 5, pp. 571–581, 2008.

[21] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[22] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[23] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[24] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[25] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.

[26] V. Colao, G. Marino, and H.-K. Xu, "An iterative method for finding common solutions of equilibrium and fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 340–352, 2008.

[27] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.

[28] J.-W. Peng and J.-C. Yao, "A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1401–1432, 2008.

[29] J.-W. Peng and J. C. Yao, "Weak convergence of an iterative scheme for generalized equilibrium problems," *Bulletin of the Australian Mathematical Society*. In press.

[30] J.-W. Peng and J. C. Yao, "A modified CQ method for equilibrium problems, fixed points and variational inequality," *Fixed Point Theory*, vol. 9, no. 2, pp. 515–531, 2008.

[31] Y. Su, M. Shang, and X. Qin, "An iterative method of solution for equilibrium and optimization problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2709–2719, 2008.

[32] S.-S. Chang, H. W. Joseph Lee, and C. K. Chan, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis: Theory, Methods & Applications*. In press.

[33] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[34] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies, no. 5. Notas de Matemática (50), North-Holland, Amsterdam, The Netherlands, 1973.

[35] B. Lemaire, "Which fixed point does the iteration method select?" in *Recent Advances in Optimization (Trier, 1996)*, vol. 452 of *Lecture Notes in Economics and Mathematical Systems*, pp. 154–167, Springer, Berlin, Germany, 1997.

[36] H.-K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[37] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.