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Research Article

Remarks on Sum of Products of (h, q)-Twisted Euler Polynomials and Numbers

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The main purpose of this paper is to construct generating functions of higher-order twisted (h, q)-extension of Euler polynomials and numbers, by using p-adic, q-deformed fermionic integral on $\mathbb{Z}_p$. By applying these generating functions, we prove complete sums of products of the twisted (h, q)-extension of Euler polynomials and numbers. We also define some identities involving twisted (h, q)-extension of Euler polynomials and numbers.

1. Introduction, definitions, and notations

Higher-order twisted Bernoulli and Euler numbers and polynomials were studied by many authors (see for details [1–10]). In [1, 3], Kim constructed p-adic, q-Volkenborn integral identities. He proved p-adic, q-integral representation of q-Euler and Bernoulli numbers and polynomials. In [11], the second author constructed a new approach to the complete sums of products of (h, q)-extension of higher-order Euler polynomials and numbers. Kim and Rim [12], by using q-deformed fermionic integral on $\mathbb{Z}_p$, defined twisted generating functions of the q-Euler numbers and polynomials, respectively. By using these functions, they also constructed interpolation functions of these numbers and polynomials.

By the same motivation of the above studies, in this paper, we construct a new approach to the complete sums of products of twisted (h, q)-extension of Euler polynomials and numbers.

Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of rational integers, the set of positive integers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. Here, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. If $q \in \mathbb{C}$, then we assume that $|q| < 1$ (cf. [1, 3, 4, 9]).

We use the following notations:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Note that $\lim_{q\to 1}[x]_q = x$.

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. Let $f \in \mathrm{UD}(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function$\}$. For $f \in \mathrm{UD}(\mathbb{Z}_p, \mathbb{C}_p)$, let

$$\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x = \sum_{x=0}^{p^N-1} f(x)\mu_q(a+dp^N\mathbb{Z}_p) \tag{1.2}$$

representing the q-analogue of the Riemann sums for f. The integral of f on $\mathbb{Z}_p$ is defined as the limit ($N \to \infty$) of the above sums when it exists. Thus, Kim [1, 3] defined the p-adic invariant q-integral on $\mathbb{Z}_p$ as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x, \tag{1.3}$$

where

$$\mu_q(a+dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}, \quad N \in \mathbb{Z}^+. \tag{1.4}$$

Note that if $f \in \mathrm{UD}(\mathbb{Z}_p, \mathbb{C}_p)$, then

$$\left|\int_{\mathbb{Z}_p} f(x)d\mu_q(x)\right|_p \leq p\|f\|_1, \tag{1.5}$$

where

$$\|f\|_1 = \sup\left\{\left|f(0)\right|_p, \sup_{x\neq y}\left|\frac{f(x)-f(y)}{x-y}\right|_p\right\} \quad \text{(cf. [3])}. \tag{1.6}$$

The bosonic integral was considered from a physical point of view to the bosonic limit $q \to 1$, $I_1(f) = \lim_{q\to 1} I_q(f)$ (cf. [1, 3, 4, 12]). By using the q-bosonic integral on $\mathbb{Z}_p$, not only generating functions of the Bernoulli numbers and polynomials are constructed but also Witt-type formula of these numbers and polynomials are defined (cf. for detail [1, 9, 10, 13, 14]).

The fermionic integral, which is called the q-deformed fermionic integral on $\mathbb{Z}_p$, is defined by

$$I_{-q}(f) = \lim_{q\to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x), \tag{1.7}$$

where

$$\mu_{-q}(a+dp^N\mathbb{Z}_p)=\frac{(-q)^a}{[dp^N]_{-q}},\quad N\in\mathbb{Z}^+\quad(\text{cf. }[3,4,6,12]).\tag{1.8}$$

In view of the notation I_{-1} is written symbolically by

$$I_{-1}(f)=\lim_{q\to -1}I_q(f).\tag{1.9}$$

By using q-deformed fermionic integral on $\mathbb{Z}_p$, generating functions of the Euler numbers and polynomials, Genocchi numbers and polynomials, and Frobenius-Euler numbers and polynomials are constructed (cf. for detail [1, 3, 6–8, 10–12, 15]).

The main motivation of this paper is to construct generating functions of higher-order twisted (h,q)-extension of Euler polynomials and numbers by using q-deformed fernionic integral on $\mathbb{Z}_p$. Moreover, by this integral, we also define Witt-type formula of the higher-order twisted (h,q)-extension of Euler polynomials and numbers. By applying these generating functions and q-deformed fernionic integral, we obtain complete sums of products of the twisted (h,q)-extension of Euler polynomials and numbers as well.

The twisted (h,q)-Bernoulli and Euler numbers and polynomials have been studied by several authors (cf. [5, 8, 9, 11, 15–17]).

In [3, 6], Kim defined the following integral equation: for $f_1(x)=f(x+1)$,

$$I_{-1}(f_1)+I_{-1}(f)=2f(0).\tag{1.10}$$

Let

$$T_p=\bigcup_{n\geq 1}C_{p^n}=\lim_{n\to\infty}C_{p^n},\tag{1.11}$$

where $C_{p^n}=\{w\mid w^{p^n}=1\}$ is the cyclic group of order p^n. For $w\in T_p$, $\phi_w:\mathbb{Z}_p\to\mathbb{C}_p$ is the locally constant function $x\to w^x$ (cf. [9, 14, 16]).

Ozden and Simsek [7] defined new (h,q)-extension of Euler numbers and polynomials. In [15], Ozden et al. also defined twisted (h,q)-extension of Euler polynomials, $E_{n,w}^{(h)}(x,q)$, as follows:

$$F_{w,q}^{(h)}(t,x)=F_{w,q}^{(h)}(t)e^{tx}=\frac{2e^{tx}}{wq^he^t+1}=\sum_{n=0}^{\infty}E_{n,w}^{(h)}(x,q)\frac{t^n}{n!}.\tag{1.12}$$

Note that if $w\to 1$, then $E_{n,w}^{(h)}(q)\to E_n^{(h)}(q)$ and

$$F_{w,q}^{(h)}(t)\longrightarrow F_q^{(h)}(t)=\frac{2}{q^he^t+1}\tag{1.13}$$

(cf. [7]). If $q\to 1$, then

$$F_q^{(h)}(t)\longrightarrow F(t)=\frac{2}{e^t+1}=\sum_{n=1}^{\infty}E_n\frac{t^n}{n!},\tag{1.14}$$

where E_n is usual Euler numbers (cf. [3, 8, 10]).

For $x=0$, we have

$$F_q^{(h)}(t)=\frac{2}{wq^he^t+1}=\sum_{n=0}^{\infty}E_{n,w}^{(h)}(q)\frac{t^n}{n!}\quad(\text{cf. }[7]).\tag{1.15}$$

Theorem 1.1 ([15] Witt formula). *For $h \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$,*

$$\int_{\mathbb{Z}_p} q^{hx} w^n x^n d\mu_{-1}(x) = E_{n,w}^{(h)}(q), \tag{1.16}$$

$$\int_{\mathbb{Z}_p} q^{hy} (x+y)^n d\mu_{-1}(y) = E_{n,w}^{(h)}(x,q). \tag{1.17}$$

2. Higher-order twisted (h, q)-Euler polynomials and numbers

Here, we study on higher-order twisted (h, q)-Euler polynomials and numbers and complete sums of products of these polynomials and numbers, our method is similar to that of [11]. For constructions of them, we use multiple the q-deformed fermionic integral on $\mathbb{Z}_p$:

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{v\text{-times}} (wq^h)^{\sum_{j=1}^{v} x_j} \exp\left(t \sum_{j=1}^{v} x_j\right) \prod_{j=1}^{v} d\mu_{-1}(x_j) = \sum_{n=0}^{\infty} E_{n,w}^{(h,v)}(q) \frac{t^n}{n!}, \tag{2.1}$$

where $\prod_{j=1}^{v} d\mu_{-1}(x_j) = d\mu_{-1}(x_1) d\mu_{-1}(x_2) \cdots d\mu_{-1}(x_v)$. By using the above equation, we easily have

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (wq^h)^{\sum_{j=1}^{v} x_j} \left(\sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_{-1}(x_j) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,w}^{(h,v)}(q) \frac{t^n}{n!}. \tag{2.2}$$

By comparing coefficients of $t^n/n!$ in the above equation, we have the following theorem.

Theorem 2.1. *For positive integers n, v, and $h \in \mathbb{Z}$, then*

$$E_{n,w}^{(h,v)}(q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (wq^h)^{\sum_{j=1}^{v} x_j} \left(\sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_{-1}(x_j). \tag{2.3}$$

By (2.1), twisted (h, q)-Euler numbers of higher-order, $E_{n,w}^{(h,v)}(q)$, are defined by means of the following generating function:

$$\left(\frac{2}{wq^h e^t + 1} \right)^v = \sum_{n=0}^{\infty} E_{n,w}^{(h,v)}(q) \frac{t^n}{n!}. \tag{2.4}$$

Observe that for $v = 1$, the above equation reduces to (1.15):

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{v\text{-times}} (wq^h)^{\sum_{j=1}^{v} x_j} \exp\left(tz + \sum_{j=1}^{v} t x_j\right) \prod_{j=1}^{v} d\mu_{-1}(x_j) = \sum_{n=0}^{\infty} E_{n,w}^{(h,v)}(z,q) \frac{t^n}{n!}. \tag{2.5}$$

By using Taylor series of $\exp(tx)$ in the above equation, we have

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (wq^h)^{\sum_{j=1}^{v} x_j} \left(z + \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_{-1}(x_j) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,w}^{(h,v)}(z,q) \frac{t^n}{n!}. \tag{2.6}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following theorem.

Theorem 2.2 (Witt-type formula). *For $z \in \mathbb{C}_p$ and positive integers n, v, and $h \in \mathbb{Z}$, then*

$$E_{n,w}^{(h,v)}(z,q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (q^h w)^{\sum_{j=1}^{v} x_j} \left(z + \sum_{j=1}^{v} x_j \right)^n \prod_{j=1}^{v} d\mu_{-1}(x_j). \tag{2.7}$$

By (2.1), (h,q)-Euler polynomials of higher-order, $E_{n,q}^{(h,v)}(z)$, are defined by means of the following generating function:

$$F_{q,w}^{(h,v)}(z,t) = e^{tz}\left(\frac{2}{wq^h e^t + 1} \right)^v = \sum_{n=0}^{\infty} E_{n,w}^{(h,v)}(z,q)\frac{t^n}{n!}. \tag{2.8}$$

Note that when $v = 1$, then we have (1.12); when $q \to 1$ and $w \to 1$, then we have

$$F^{(v)}(z,t) = e^{tz}\left(\frac{2}{e^t + 1} \right)^v = \sum_{n=0}^{\infty} E_n^{(v)}(z)\frac{t^n}{n!}, \tag{2.9}$$

where $E_n^{(v)}(z)$ denote classical higher-order Euler polynomials (cf. [10]).

Theorem 2.3. *For $z \in \mathbb{C}_p$ and positive integers n, v, and $h \in \mathbb{Z}$, then*

$$E_{n,w}^{(h,v)}(z,q) = \sum_{l=0}^{n} \binom{n}{l} z^{n-l} E_{l,w}^{(h,v)}(q). \tag{2.10}$$

Proof. By using binomial expansion in (2.7), we have

$$E_{n,w}^{(h,v)}(z,q) = \sum_{l=0}^{n} \binom{n}{l} z^{n-l} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (q^h w)^{\sum_{j=1}^{v} x_j} \left(\sum_{j=1}^{v} x_j \right)^l \prod_{j=1}^{v} d\mu_{-1}(x_j). \tag{2.11}$$

By (2.3) in the above, we arrive at the desired result. □

Remark 2.4. If $w \to 1$, then $E_{n,w}^{(h,v)}(q) \to E_n^{(h,v)}(q)$ (cf. [11]). If $q \to 1$, $v = 1$, then $E_{n,w}^{(h,v)}(q) \to E_n$, where $E_{n,w}^{(v)}$ is usual twisted Euler numbers (cf. [10]).

3. The complete sums of products of (h,q)-extension of Euler polynomials and numbers

In this section, we prove main theorems related to the complete sums of products of (h,q)-extension of Euler polynomials and numbers. Firstly, we need the multinomial theorem, which is given as follows (cf. [18, 19]).

Theorem 3.1 (multinomial theorem). *Let*

$$\left(\sum_{j=1}^{v} x_j \right)^n = \sum_{\substack{l_1,l_2,\ldots,l_v \geq 0 \\ l_1+l_2+\cdots+l_v=n}} \binom{n}{l_1,l_2,\ldots,l_v} \prod_{a=1}^{v} x_a^{l_a}, \tag{3.1}$$

where $\binom{n}{l_1,l_2,\ldots,l_v}$ are the multinomial coefficients, which are defined by $\binom{n}{l_1,l_2,\ldots,l_v} = n!/l_1!l_2!\cdots l_v!$.

Theorem 3.2. *For positive integers n, v, then*

$$E_{n,w}^{(h,v)}(q) = \sum_{\substack{l_1,l_2,\ldots,l_v\geq 0\\ l_1+l_2+\cdots+l_v=n}} \binom{n}{l_1,l_2,\ldots,l_v} \prod_{j=1}^{v} E_{l_j,w}^{(h)}(q), \tag{3.2}$$

where $\binom{n}{l_1,l_2,\ldots,l_v}$ *is the multinomial coefficient.*

Proof. By using Theorem 3.1 in (2.3), we have

$$E_{n,w}^{(h,v)}(q) = \sum_{\substack{l_1,l_2,\ldots,l_v\geq 0\\ l_1+l_2+\cdots+l_v=n}} \binom{n}{l_1,l_2,\ldots,l_v} \prod_{j=1}^{v} \int_{\mathbb{Z}_p} \left(wq^h\right)^{x_j} x_j^{l_j} d\mu_{-1}(x_j). \tag{3.3}$$

By (1.16) in the above, we obtain the desired result. □

By substituting (3.2) into (2.10), we have the following corollary.

Corollary 3.3. *For $z \in \mathbb{C}_p$ and positive integers n, v, then*

$$E_{n,w}^{(h,v)}(z,q) = \sum_{m=0}^{n} \sum_{\substack{l_1,l_2,\ldots,l_v\geq 0\\ l_1+l_2+\cdots+l_v=m}} \binom{n}{m} \binom{m}{l_1,l_2,\ldots,l_v} z^{n-m} \prod_{j=1}^{v} E_{l_j,w}^{(h)}(q). \tag{3.4}$$

Complete sum of products of the twisted (h,q)-Euler polynomials is given by the following theorem.

Theorem 3.4. *For $y_1, y_2, \ldots, y_v \in \mathbb{C}_p$ and positive integers n, v, then*

$$E_{n,w}^{(h,v)}(y_1+y_2+\cdots+y_v,q) = \sum_{\substack{l_1,l_2,\ldots,l_v\geq 0\\ l_1+l_2+\cdots+l_v=n}} \binom{n}{l_1,l_2,\ldots,l_v} \prod_{j=1}^{v} E_{l_j,w}^{(h)}(y_j,q). \tag{3.5}$$

Proof. By substituting $z = y_1 + y_2 + \cdots + y_v$ into (2.7), we have

$$E_{n,w}^{(h,v)}(y_1+y_2+\cdots+y_v,q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(wq^h\right)^{\sum_{j=1}^{v} x_j} \left(\sum_{j=1}^{v} (y_j+x_j)\right)^n \prod_{j=1}^{v} d\mu_{-1}(x_j). \tag{3.6}$$

By using Theorem 3.1 in the above, and after some elementary calculations, we get

$$\begin{aligned} &E_{n,w}^{(h,v)}(y_1+y_2+\cdots+y_v,q) \\ &\quad= \sum_{\substack{l_1,l_2,\ldots,l_v\geq 0\\ l_1+l_2+\cdots+l_v=n}} \binom{n}{l_1,l_2,\ldots,l_v} \prod_{j=1}^{v} \int_{\mathbb{Z}_p} \left(wq^h\right)^{x_j} (y_j+x_j)^{l_j} d\mu_{-1}(x_j). \end{aligned} \tag{3.7}$$

By substituting (1.17) into the above, we arrive at the desired result. □

Remark 3.5. If we take $y_1 = y_2 = \cdots = y_v = 0$ in Theorem 3.4, then Theorem 3.4 reduces to Theorem 3.2. Substituting $q \to 1$ and $w \to 1$ into (3.5), we obtain the following relation:

$$E_n^{(v)}(y_1 + y_2 + \cdots + y_v) = \sum_{\substack{l_1,l_2,\ldots,l_v \geq 0 \\ l_1+l_2+\cdots+l_v=m}} \binom{m}{l_1,l_2,\ldots,l_v} \prod_{j=1}^{v} E_{l_j}(y_j) \quad \text{(cf. [11])}. \tag{3.8}$$

I.-C. Huang and S.-Y. Huang [20] found complete sums of products of Bernoulli polynomials. Kim [13] defined Carlitz's q-Bernoulli number of higher order using an integral by the q-analogue μ_q of the ordinary p-adic invariant measure. He gave a different proof of complete sums of products of higher order q-Bernoulli polynomials. In [21], Jang et al. gave complete sums of products of Bernoulli polynomials and Frobenious Euler polynomials. In [14], Simsek et al. gave complete sums of products of (h, q)-Bernoulli polynomials and numbers.

Theorem 3.6. *Let $n \in \mathbb{Z}^+$. Then*

$$E_{n,w}^{(h,v)}(z + y, q) = \sum_{l=0}^{n} \binom{n}{l} E_{l,w}^{(h,v)}(y, q) z^{n-l}. \tag{3.9}$$

Proof. Assume

$$\begin{aligned} E_{n,w}^{(h,v)}(z + y, q) &= \left(E_w^{(h,v)}(q) + z + y\right)^n \\ &= \sum_{l=0}^{n} \binom{n}{l} E_{l,w}^{(h,v)}(q)(y + z)^{n-l} \end{aligned} \tag{3.10}$$

with usual convention of symbolically replacing $E_w^{l(h,v)}$ by $E_{l,w}^{(h,v)}(q)$. By using (2.10) in the above, we have

$$E_{n,w}^{(h,v)}(z + y, q) = \sum_{m=0}^{n} \binom{n}{m} E_{m,w}^{(h,v)}(y, q) z^{n-m}. \tag{3.11}$$

Thus the proof is completed. □

From Theorems 3.4 and 3.6, after some elementary calculations, we arrive at the following interesting result.

Corollary 3.7. *Let $n \in \mathbb{Z}^+$. Then*

$$\sum_{m=0}^{n} \binom{n}{m} E_{m,w}^{(h,v)}(y_1, q) y_2^{n-m} = \sum_{\substack{l_1,l_2 \geq 0 \\ l_1+l_2=n}} \binom{n}{l_1,l_2} E_{l_1,w}^{(h)}(y_1, q) B_{l_2,w}^{(h)}(y_2, q). \tag{3.12}$$

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