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Research Article

Local Regularity Results for Minima of Anisotropic Functionals and Solutions of Anisotropic Equations

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This paper gives some local regularity results for minima of anisotropic functionals $I\,(u;\Omega)=\int_\Omega f(x,u,Du)dx$, $u\in W^{1,q_i}_{\text{loc}}(\Omega)$ and for solutions of anisotropic equations $-\text{div}\mathcal{A}(x,u,Du)=-\sum_{i=1}^{N}(\partial f/\partial x_i)$, $u\in W^{1,q_i}_{\text{loc}}(\Omega)$ which can be regarded as generalizations of the classical results.

1. Introduction

Let Ω be an open bounded subset of $\mathbf{R}^N$, $N\geq 2$. Let $q_i>1$, $i=1,\ldots,N$. Denote

$$q=\max_{1\leq i\leq N}q_i,\qquad p=\min_{1\leq i\leq N}q_i,\qquad \overline{q}:\frac{1}{\overline{q}}=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{q_i}. \tag{1.1}$$

Throughout this paper, we will make use of the anisotropic Sobolev space

$$W^{1,q_i}_{\text{loc}}(\Omega)=\left\{v\in L^q_{\text{loc}}(\Omega)\,:\,\frac{\partial v}{\partial x_i}\in L^{q_i}_{\text{loc}}(\Omega),\ \forall i=1,\ldots,N\right\}. \tag{1.2}$$

Let $x_0\in\Omega$ and $t>0$, we denote by B_t the ball of radius t centered at x_0. For functions u and $k>0$, let $A_k=\{x\in\Omega:|u(x)|>k\}$, $A_{k,t}=A_k\cap B_t$. Moreover, if $p>1$, then p' is always the real number $p/(p-1)$, and if $s<N$, s^* is always the real number satisfying $1/s^*=1/s-1/N$.

This paper mainly considers the functions u minimizing the anisotropic functionals

$$I(u;\Omega)=\int_\Omega f(x,u,Du)dx,\quad u\in W^{1,q_i}_{\text{loc}}(\Omega) \tag{1.3}$$

and weak solutions of the anisotropic equations

$$-\mathrm{div}\mathcal{A}(x, u, Du) = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}, \quad u \in W^{1,q_i}_{\mathrm{loc}}(\Omega). \tag{1.4}$$

We refer to the classical books by Ladyženskaya and Ural'ceva [1], Morrey [2], Gilbarg and Trudinger [3], and Giaquinta [4] for some details of isotropic cases.

For isotropic cases, global L^s-summability was proved in the 1960s by Stampacchia [5] for solutions of linear elliptic equations. This result was extended by Boccardo and Giachetti to the nonlinear case in [6]. For anisotropic cases, Giachetti and Porzio recently proved in [7] the local L^s-summability for minima of anisotropic functionals and weak solutions of anisotropic nonlinear elliptic equations. Precisely, the authors considered the minima of functionals whose prototype is (1.3), f is a Carathéodory function satisfying the growth conditions

$$a\sum_{i=1}^{N} |\xi_i|^{q_i} \le f(x,s,\xi) \le b\sum_{i=1}^{N} |\xi_i|^{q_i} + \varphi_1(x), \tag{1.5}$$

where the function $\varphi_1 \in L^r_{\mathrm{loc}}(\Omega)$ with $1 < r < N/\overline{q}$. The authors also considered the local solutions $u \in W^{1,q_i}_{\mathrm{loc}}(\Omega)$ of the anisotropic equations (1.4), where $\mathcal{A} : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function satisfying the following structural conditions:

$$\begin{gathered} \mathcal{A}(x,u,\xi)\cdot\xi \ge m_0\sum_{i=1}^{N} |\xi_i|^{q_i}, \\ \left|\mathcal{A}_j(x,u,\xi)\right| \le m_1\left(h(x) + \sum_{i=1}^{N}|\xi_i|^{q_i}\right)^{1-1/q_j}, \quad j = 1,\dots,N, \end{gathered} \tag{1.6}$$

where m_l, $l = 0, 1$ are positive constants, the function h is in $L^1_{\mathrm{loc}}(\Omega)$ and the functions f_i belong, respectively, to the spaces $L^{(q_i)'}_{\mathrm{loc}}(\Omega)$. Under the above conditions, the authors obtained some local regularity results.

The aim of the present paper is to prove the local regularity property for minima of the anisotropic functionals of type (1.3) with the more general growth conditions than (1.5), that is, we assume the integrand f satisfies the following growth conditions:

$$\sum_{i=1}^{N} |\xi_i|^{q_i} - b|u|^{\alpha} - \varphi_0(x) \le f(x,u,\xi) \le a\sum_{i=1}^{N} |\xi_i|^{q_i} + b|u|^{\alpha} + \varphi_1(x), \tag{1.7}$$

where

$$\begin{gathered} \varphi_0 \in L^{r_1}_{\mathrm{loc}}(\Omega), \quad \varphi_1 \in L^{r_2}_{\mathrm{loc}}(\Omega), \quad r_1, r_2 > 1, \quad a \ge 1,\ b \ge 0, \\ p \le \alpha < p^*, \quad q < \overline{q}^*, \quad \overline{q} < N, \quad 1 < \min\{r_1, r_2\} < \frac{N}{\overline{q}}. \end{gathered} \tag{1.8}$$

We also consider weak solutions of the type (1.4) with more general growth conditions than (1.6), that is, we assume the operator $\mathcal{A}$ satisfies the following coercivity and growth conditions:

$$\mathcal{A}(x,u,\xi)\cdot\xi \ge b_0\sum_{i=1}^{N} |\xi_i|^{q_i} - b_1|u|^{\alpha_1} - \varphi_2(x), \tag{1.9}$$

$$\left|\mathcal{A}(x,u,\xi)\right| \le b_2\sum_{i=1}^{N} |\xi_i|^{q_i-1} + b_3|u|^{\alpha_2} + k(x), \tag{1.10}$$

where $b_0 \geq 1$, $b_i > 0$, $i = 1,2,3$, $q < \overline{q}^*$, $\overline{q} < N$, $p \leq \alpha_1 < p^*$, $p-1 \leq \alpha_2 \leq N(p-1)/(N-p)$, $\varphi_2 \in L^{r_0}_{\mathrm{loc}}(\Omega)$ with $r_0 > 1$, $k \in L^{r_{N+1}}_{\mathrm{loc}}(\Omega)$, $f_i \in L^{r_i}_{\mathrm{loc}}(\Omega)$, $i = 1, \dots, N$.

Remark 1.1. Notice that we have confined ourselves to the case $\overline{q} < N$ because when such inequality is violated, every function in $W^{1,q_i}_{\mathrm{loc}}(\Omega)$ is trivially in $L^s_{\mathrm{loc}}(\Omega)$ (for every fixed $s < \infty$) by [7, Lemma 3.2].

Remark 1.2. Since we have assumed in (1.7), (1.9), and (1.10) that the integrand f and the operator $\mathcal{A}$ satisfy some growth conditions depending on u, in the proof of the local regularity results, we have to estimate the integral of some power of $|u|$ by means of $|Du|$. To do this, we will make use of the Sobolev inequality that has been used in [8].

2. Preliminary lemmas

In order to prove the local L^s-integrability of the local unbounded minima of the anisotropic functionals and weak solutions of anisotropic equations, we need a useful lemma from [7].

Lemma 2.1. *Let $u \in W^{1,q_i}_{\mathrm{loc}}(\Omega)$, $\phi_0 \in L^r_{\mathrm{loc}}(\Omega)$, where q, $\overline{q}$, and r satisfy*

$$1 < r < \frac{N}{\overline{q}}, \quad q < \overline{q}^*, \quad \overline{q} < N. \tag{2.1}$$

Assume that the following integral estimates hold:

$$\int_{A_{k,\tau}} \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx \leq c_0 \left[\int_{A_{k,t}} \phi_0 dx + (t-\tau)^{-\gamma} \int_{A_{k,t}} \sum_{i=1}^N |u|^{q_i} dx\right] \tag{2.2}$$

for every $k \in \mathbf{N}$ and $R_0 \leq \tau < t \leq R_1$, where c_0 is a positive constant that depends only on N, q_i, r, R_0, R_1 and $|\Omega|$ and γ is a real positive constant. Then $u \in L^s_{\mathrm{loc}}(\Omega)$, where

$$s = \frac{\overline{q}^* q}{q - \overline{q}^*(1-1/r)}. \tag{2.3}$$

One will also need a lemma from [8].

Lemma 2.2. *Let $f(t)$ be a nonnegative bounded function defined for $0 \leq T_0 \leq t \leq T_1$. Suppose that for $T_0 \leq t < s \leq T_1$,*

$$f(t) \leq A(s-t)^{-\gamma} + B + \theta f(s), \tag{2.4}$$

where A, B, γ, θ are nonnegative constants, and $\theta < 1$. Then there exists a constant c, depending only on γ and θ such that for every ϱ, R, $T_0 \leq \varrho < R \leq T_1$, one has

$$f(\varrho) \leq c\left[A(R-\varrho)^{-\gamma} + B\right]. \tag{2.5}$$

3. Minima of anisotropic functionals

In this section, we prove a local regularity result for minima of anisotropic functionals.

Definition 3.1. By a local minimum of the anisotropic functional I in (1.3), we mean a function $u \in W^{1,q_i}_{\mathrm{loc}}(\Omega)$, such that for every function $\varphi \in W^{1,q_i}(\Omega)$ with $\operatorname{supp}\varphi \subset\subset \Omega$, it holds that

$$I(u;\operatorname{supp}\varphi) \le I(u+\varphi;\operatorname{supp}\varphi). \tag{3.1}$$

Theorem 3.2. *Assume that the functional I satisfies the conditions* (1.7). *If u is a local minimum of I, then it belongs to $L^s_{\mathrm{loc}}(\Omega)$, where*

$$s = \frac{\overline{q}^* q}{q - \overline{q}^*\left(1 - 1/\min\left\{r_1, r_2\right\}\right)}. \tag{3.2}$$

Proof. Owing to Lemma 2.1, it is sufficient to prove that u satisfies the integral estimates (2.2) with $\gamma = q$ and $\phi_0 = \varphi_0 + \varphi_1$. Let $B_{R_1} \subset\subset \Omega$ and $0 \le R_0 \le \tau < t \le R_1$ be arbitrarily but fixed. It is no loss of generality to assume that $R_1 - R_0 < 1$. For $k > 0$, let

$$A_k^+ = \left\{x \in \Omega : u(x) > k\right\}, \qquad A_k^- = \left\{x \in \Omega : u(x) < -k\right\}. \tag{3.3}$$

It is obvious that $A_k = A_k^+ \cup A_k^-$. Denote $A_{k,t}^+ = A_k^+ \cap B_t$ and $A_{k,t}^- = A_k^- \cap B_t$. Let $w = \max(u-k, 0)$. Choose $\varphi = -\eta w$ in (3.1), where η is a cut-off function such that

$$\operatorname{supp}\eta \subset B_t, \quad 0 \le \eta \le 1,\ \eta = 1 \text{ in } B_\tau, \quad |D\eta| \le 2(t-\tau)^{-1}. \tag{3.4}$$

We obtain from the minimality of u that

$$\begin{aligned}\int_{B_t} f(x,u,Du)dx &\le \int_{B_t} f(x,u+\varphi,Du+D\varphi)dx \\ &= \int_{A_{k,t}^+} f\left(x,u-\eta w,Du-D(\eta w)\right)dx + \int_{B_t \cap \{u \le k\}} f(x,u,Du)dx.\end{aligned} \tag{3.5}$$

This implies that

$$\int_{A_{k,t}^+} f(x,u,Du)dx \le \int_{A_{k,t}^+} f\left(x,u-\eta w,Du-D(\eta w)\right)dx. \tag{3.6}$$

By (1.7), we obtain

$$\begin{aligned}&\int_{A_{k,t}^+} \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx \\ &\quad \le b\int_{A_{k,t}^+} u^\alpha dx + \int_{A_{k,t}^+} \varphi_0 dx + a\int_{A_{k,t}^+} \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i} - \frac{\partial(\eta w)}{\partial x_i}\right|^{q_i} dx + b\int_{A_{k,t}^+} (u-\eta w)^\alpha dx + \int_{A_{k,t}^+} \varphi_1 dx.\end{aligned} \tag{3.7}$$

We first estimate the 3rd term on the right-hand side of (3.7). Using the elementary inequality

$$(a+b)^q \le 2^{q-1}(a^q+b^q), \quad a,b\ge 0,\ q\ge 1, \tag{3.8}$$

we obtain

$$\begin{aligned}
a\int_{A^+_{k,t}}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}-\frac{\partial(\eta w)}{\partial x_i}\right|^{q_i}dx &= a\int_{A^+_{k,t}\setminus A^+_{k,\tau}}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}-\frac{\partial(\eta w)}{\partial x_i}\right|^{q_i}dx\\
&\le 2^{q-1}a\int_{A^+_{k,t}\setminus A^+_{k,\tau}}\sum_{i=1}^N\left[(1-\eta)^{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}+\left|\frac{\partial \eta}{\partial x_i}\right|^{q_i}w^{q_i}\right]dx\\
&\le 2^{q-1}a\int_{A^+_{k,t}\setminus A^+_{k,\tau}}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx+\frac{2^{2q-1}a}{(t-\tau)^{q_i}}\int_{A^+_{k,t}\setminus A^+_{k,\tau}}\sum_{i=1}^N w^{q_i}dx\\
&\le 2^{q-1}a\int_{A^+_{k,t}\setminus A^+_{k,\tau}}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx+\frac{2^{2q-1}a}{(t-\tau)^{q}}\int_{A^+_{k,t}\setminus A^+_{k,\tau}}\sum_{i=1}^N u^{q_i}dx
\end{aligned} \tag{3.9}$$

since $w^{q_i}\le u^{q_i}$ in $A^+_{k,t}$ and $t-\tau<1$. The summation of the 1st and the 4th terms on the right-hand side of (3.7) can be estimated as

$$b\int_{A^+_{k,t}}u^\alpha dx+b\int_{A^+_{k,t}}(u-\eta w)^\alpha dx\le 2b\int_{A^+_{k,t}}u^\alpha dx. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.7) yields

$$\begin{aligned}
&\int_{A^+_{k,t}}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx\\
&\quad\le\int_{A^+_{k,t}}(\varphi_0+\varphi_1)dx+2b\int_{A^+_{k,t}}u^\alpha dx+2^{q-1}a\int_{A^+_{k,t}\setminus A^+_{k,\tau}}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx+\frac{2^{2q-1}a}{(t-\tau)^q}\int_{A^+_{k,t}\setminus A^+_{k,\tau}}\sum_{i=1}^N u^{q_i}dx.
\end{aligned} \tag{3.11}$$

We know from [8] that if $\tilde u\in W^{1,p}(B_t)$ and $|\operatorname{supp}\tilde u|\le(1/2)|B_t|$, we then have the Sobolev inequality

$$\left(\int_{B_t}\tilde u^{p^*}dx\right)^{p/p^*}\le c_1(N,p)\int_{B_t}|D\tilde u|^p dx. \tag{3.12}$$

Let

$$\tilde u=\begin{cases}u, & x\in A^+_{k,t},\\ 0, & x\in\Omega\setminus A^+_{k,t}.\end{cases} \tag{3.13}$$

By assumption, $p \le \alpha < p^*$, which implies

$$\begin{aligned}\int_{A_{k,t}^+} u^\alpha dx &= \int_{B_t} \tilde{u}^\alpha dx \le \|\tilde{u}\|_{p^*}^{\alpha-p} \left|B_t\right|^{1-\alpha/p^*} \left(\int_{B_t} \tilde{u}^{p^*} dx \right)^{p/p^*} \\ &\le c_1 \|\tilde{u}\|_{p^*}^{\alpha-p} \left|B_t\right|^{1-\alpha/p^*} \int_{B_t} |D\tilde{u}|^p dx \\ &\le c_1 \|\tilde{u}\|_{p^*}^{\alpha-p} \left|B_t\right|^{1-\alpha/p^*} \max\left\{1, 2^{p/2-1}\right\} \int_{B_t} \sum_{i=1}^N \left| \frac{\partial \tilde{u}}{\partial x_i} \right|^{q_i} dx \\ &= c_1 \|\tilde{u}\|_{p^*}^{\alpha-p} \left|B_t\right|^{1-\alpha/p^*} \max\left\{1, 2^{p/2-1}\right\} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx,\end{aligned} \tag{3.14}$$

provided that $|\operatorname{supp} \tilde{u}|_{B_t}| \le (1/2)|B_t|$. We can choose T so small that for $t \le T$ we get

$$c_1 \|\tilde{u}\|_{p^*}^{\alpha-p} \left|B_t\right|^{1-\alpha/p^*} \max\left\{1, 2^{p/2-1}\right\} \le \frac{1}{4b}. \tag{3.15}$$

It is obvious that

$$k^{p^*} \left|A_k^+\right| \le \|\tilde{u}\|_{p^*,\Omega}^{p^*}, \tag{3.16}$$

and therefore, there exists a constant k_0, such that for $k \ge k_0$, we have

$$\left|A_k^+\right| \le \frac{1}{2} \left|B_{T/2}\right|. \tag{3.17}$$

For such values of k we then have $|\operatorname{supp} \tilde{u}| < (1/2)|B_{T/2}|$ and therefore, if $T/2 \le t \le T$,

$$\int_{A_{k,t}^+} u^\alpha dx \le \frac{1}{4b} \int_{A_{K,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx. \tag{3.18}$$

Thus, from (3.11) and, we get

$$\begin{aligned}&\int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\quad \le 2 \int_{A_{k,t}^+} (\varphi_0 + \varphi_1) dx + 2^q a \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{2^{2q} a}{(t-\tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N |u|^{qi} dx.\end{aligned} \tag{3.19}$$

Suppose now $T/2 \le \varrho \le \tau < t \le R \le T$, we get

$$\begin{aligned}&\int_{A_{k,\varrho}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\quad \le 2 \int_{A_{k,R}^+} (\varphi_0 + \varphi_1) dx + 2^q a \int_{A_{k,t}^+ \setminus A_{k,\varrho}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{2^{2q} a}{(t-\tau)^q} \int_{A_{k,R}^+} \sum_{i=1}^N |u|^{q_i} dx.\end{aligned} \tag{3.20}$$

Adding to both sides $2^q a$ times the left-hand side, we get eventually

$$\begin{aligned}\int_{A_{k,\varrho}^+}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx \\ \le \frac{2}{2^q a+1}\int_{A_{k,R}^+}(\varphi_0+\varphi_1)dx+\frac{2^q a}{2^q a+1}\int_{A_{k,t}^+}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx+\frac{2^{2q}a}{2^q a+1}\cdot\frac{1}{(t-\tau)^q}\int_{A_{k,R}^+}\sum_{i=1}^N|u|^{q_i}dx,\end{aligned} \tag{3.21}$$

we can now apply Lemma 2.2 to conclude that

$$\int_{A_{k,\tau}^+}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx\le c\left\{\frac{2}{2^q a+1}\int_{A_{k,t}^+}(\varphi_0+\varphi_1)dx+\frac{2^{2q}a}{2^q a+1}\cdot\frac{1}{(t-\tau)^q}\int_{A_{k,t}^+}\sum_{i=1}^N|u|^{q_i}dx\right\}, \tag{3.22}$$

where c depends only on q and a.

Since $-u$ minimizes the functional

$$\widetilde{F}(v;\Omega)=\int_\Omega \widetilde{f}(x,v,Dv)dx, \tag{3.23}$$

where $\widetilde{f}(x,v,p) = f(x,-v,-p)$ satisfies the same growth conditions (1.7), inequality (3.22) holds with u replaced by $-u$. We then conclude that

$$\int_{A_{k,\tau}^-}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx\le c\left\{\frac{2}{2^q a+1}\int_{A_{k,t}^-}(\varphi_0+\varphi_1)dx+\frac{2^{2q}a}{2^q a+1}\cdot\frac{1}{(t-\tau)^q}\int_{A_{k,t}^-}\sum_{i=1}^N|u|^{q_i}dx\right\}. \tag{3.24}$$

Adding (3.22) and (3.24) yields

$$\int_{A_{k,\tau}}\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx\le c\left\{\frac{2}{2^q a+1}\int_{A_{k,t}}(\varphi_0+\varphi_1)dx+\frac{2^{2q}a}{2^q a+1}\cdot\frac{1}{(t-\tau)^q}\int_{A_{k,t}}\sum_{i=1}^N|u|^{q_i}dx\right\}. \tag{3.25}$$

This shows that u satisfies estimates (2.2) with $\gamma = q$ and $\phi_0 = \varphi_0 + \varphi_1$. Theorem 3.2 follows from Lemma 2.1. □

4. Local solutions of anisotropic equations

In this section, we prove a local regularity result for weak solutions of anisotropic equations. Let $u \in W^{1,q_i}_{\mathrm{loc}}(\Omega)$ be a local solution of the anisotropic equation (1.4), where $\mathcal{A} : \Omega\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$ is a Carathéodory function satisfying the structural conditions (1.9) and (1.10).

Definition 4.1. By a weak solution of (1.4) we mean a function $u \in W^{1,q_i}_{\mathrm{loc}}(\Omega)$, such that for every function $\psi \in W^{1,q_i}(\Omega)$ with $\operatorname{supp}\psi \subset\subset \Omega$ it holds

$$\int_{\operatorname{supp}\psi}\mathcal{A}(x,u,Du)\cdot D\psi\,dx=\int_{\operatorname{supp}\psi}f\cdot D\psi\,dx, \tag{4.1}$$

where $f = (f_1, f_2, \ldots, f_N)$.

Theorem 4.2. *Under the previous assumptions* (1.9) *and* (1.10), *if one assumes that* $\varphi_2 \in L^{r_0}_{\text{loc}}(\Omega)$, $f_i \in L^{r_i}_{\text{loc}}(\Omega)$, $i = 1, 2, \ldots, N$, $k \in L^{r_{N+1}}_{\text{loc}}(\Omega)$, *and* r_i, $i = 0, \ldots, N+1$ *satisfy*

$$1 < r = \min_{1 \le i \le N} \left\{ \frac{r_i}{q'_i}, r_0, \frac{r_{N+1}}{p'} \right\} < \frac{N}{\overline{q}}, \tag{4.2}$$

then $u \in L^s_{\text{loc}}(\Omega)$, *where*

$$s = \frac{\overline{q}^* q}{q - \overline{q}^*(1 - 1/r)}. \tag{4.3}$$

Proof. By virtue of Lemma 2.1, it is sufficient to prove that u satisfies the integral estimates (2.2) with $\gamma = q$ and $\phi_0 = \varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q'_i}$. Let $B_{R_1} \subset\subset \Omega$ and $0 \le R_0 \le \tau < t \le R_1$ be arbitrarily but fixed. Assume again that $R_1 - R_0 < 1$. Let $w = \max\{u - k, 0\}$. Choose $\varphi = \eta w$ as a test function in (4.1), where the cut-off function η satisfies the conditions (3.4). We obtain from Definition 4.1 that

$$\int_{A^+_{k,t}} \mathcal{A}(x, u, Du) \cdot D(\eta w) dx = \int_{A^+_{k,t}} f \cdot D(\eta w) dx. \tag{4.4}$$

We now estimate the integrals in (4.4). Applying the assumption (1.9), we deduce from (4.4) that

$$\begin{aligned} b_0 \int_{A^+_{k,\tau}} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \le{}& b_1 \int_{A^+_{k,t}} |u|^{\alpha_1} dx + \int_{A^+_{k,t}} \varphi_2 dx + \int_{A^+_{k,t}} f \cdot Du\, dx + \frac{2}{t - \tau} \int_{A^+_{k,t}} |f| w\, dx \\ &+ \frac{2}{t - \tau} \int_{A^+_{k,t} \setminus A^+_{k,\tau}} |\mathcal{A}(x, u, Du)| w\, dx. \end{aligned} \tag{4.5}$$

The 3rd term on the right-hand side of the above inequality can be estimated as

$$\int_{A^+_{k,t}} f \cdot Du\, dx = \int_{A^+_{k,t}} \sum_{i=1}^N f_i \cdot \frac{\partial u}{\partial x_i} dx \le \varepsilon \int_{A^+_{k,t}} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \sum_{i=1}^N C(\varepsilon, q_i) \int_{A^+_{k,t}} |f_i|^{q'_i} dx. \tag{4.6}$$

By Young's inequality, the 4th term on the right-hand side of inequality (4.5) can be estimated as

$$\frac{2}{t - \tau} \int_{A^+_{k,t}} |f| w\, dx \le \frac{2^q}{(t - \tau)^q} \int_{A^+_{k,t}} \sum_{i=1}^N (u - k)^{q^i} dx + \sum_{i=1}^N \int_{A^+_{k,t}} |f_i|^{q'_i} dx. \tag{4.7}$$

By (1.10), the last term on the right-hand side of (4.5) can be estimated as

$$\frac{2}{t - \tau} \int_{A^+_{k,t} \setminus A^+_{k,\tau}} |\mathcal{A}(x, u, Du)| w\, dx \le \frac{2}{t - \tau} \int_{A^+_{k,t} \setminus A^+_{k,\tau}} \left[b_2 \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i - 1} + b_3 |u|^{\alpha_2} + k \right] w\, dx = I_1 + I_2 + I_3. \tag{4.8}$$

By Young's inequality, we derive that

$$I_1 \le b_2 \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{b_2 2^q}{(t-\tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^{N} (u-k)^{q_i} dx. \tag{4.9}$$

Hölder's inequality and Young's inequality yield

$$\begin{aligned} I_2 &\le b_3 \varepsilon \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} |u|^{\alpha_2 p'} dx + \frac{C(\varepsilon, p) 2^p}{(t-\tau)^p} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} (u-k)^p dx \\ &\le b_3 \varepsilon \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} |u|^{\alpha_2 p'} dx + \frac{C(\varepsilon, p) 2^p}{N(t-\tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^{N} (u-k)^{q_i} dx, \end{aligned} \tag{4.10}$$

where ε is a positive constant to be determined later. Further,

$$I_3 \le \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} |k|^{p'} dx + \frac{2^q}{N(t-\tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^{N} (u-k)^{q_i} dx. \tag{4.11}$$

Combining (4.6)–(4.11) with (4.5) yields

$$\begin{aligned} &b_0 \int_{A_{k,\tau}^+} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\quad \le \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^{N} \left(C(\varepsilon, q_i) + 1 \right) |f_i|^{q_i'} \right) dx + b_1 \int_{A_{k,t}^+} |u|^{\alpha_1} dx + b_3 \varepsilon \int_{A_{k,t}^+} |u|^{\alpha_2 p'} dx \\ &\qquad + \varepsilon \int_{A_{k,t}^+} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q^i} dx + b_2 \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q^i} dx \\ &\qquad + \left(b_2 + C(\varepsilon, p) + 2 \right) \frac{2^q}{(t-\tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^{N} (u-k)^{q^i} dx. \end{aligned} \tag{4.12}$$

Since $p \le \alpha_1 < p^*$, then as in the proof of Theorem 3.2, we know that there exist a sufficiently small T and a sufficiently large k_0, such that for all $T/2 \le t \le T$ and $k \ge k_0$, we have

$$\int_{A_{k,t}^+} |u|^{\alpha_1} dx \le \frac{1}{2b_1} \int_{A_{k,t}^+} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx. \tag{4.13}$$

Similarly, since $p-1 \le \alpha_2 \le N(p-1)/(n-p)$, then $p \le \alpha_2 p' \le p^*$, therefore

$$\int_{A_{k,t}^+} |u|^{\alpha_2 p'} dx \le C \int_{A_{k,t}^+} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx. \tag{4.14}$$

Thus, from (4.12)–(4.14) we can derive that

$$b_0 \int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \le \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N (C(\varepsilon, q_i) + 1) \left| f_i \right|^{q_i'} \right) dx$$
$$+ \left(\frac{1}{2} + (Cb_3 + 1)\varepsilon \right) \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{qi} dx + b_2 \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \qquad (4.15)$$
$$+ (b_2 + C(\varepsilon, p) + 2) \frac{2^q}{(t-\tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u-k)^{q_i} dx.$$

Adding to both sides

$$b_2 \int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx, \qquad (4.16)$$

we get eventually

$$\int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \le \frac{1}{b_0 + b_2} \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N (C(\varepsilon, q_i) + 1) |f_i|^{q_i'} \right) dx$$
$$+ \left(\frac{1}{2} + (Cb_3 + 1)\varepsilon \right) \frac{1}{b_0 + b_2} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{b_2}{b_0 + b_2} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx$$
$$+ (b_2 + C(\varepsilon, p) + 2) \frac{1}{b_0 + b_2} \frac{2^q}{(t-\tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u-k)^{q_i} dx. \qquad (4.17)$$

Choosing ε small enough, such that

$$\theta = \frac{1/2 + (Cb_3 + 1)\varepsilon + b_2}{b_0 + b_2} < 1, \qquad (4.18)$$

(4.17) implies that

$$\int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \le C \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i'} \right) dx + \theta \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{C}{(t-\tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u-k)^{q_i} dx. \qquad (4.19)$$

Suppose now that $T/2 \le \varrho \le \tau < t \le R \le T$, we get

$$\int_{A_{k,\varrho}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \le C \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i'} \right) dx$$
$$+ \theta \int_{A_{k,R}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{C}{(R-\varrho)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u-k)^{q_i} dx. \qquad (4.20)$$

Applying Lemma 2.2, we conclude that

$$\int_{A_{k,\tau}^{+}} \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx \le cC \int_{A_{k,t}^{+}} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^{N} \left|f_i\right|^{q_{i'}}\right) dx + \frac{cC}{(t-\tau)^q} \int_{A_{k,t}^{+}} \sum_{i=1}^{N} (u-k)^{q_i} dx. \tag{4.21}$$

Since $-u$ is a weak solution of

$$-\mathrm{div}\, \widetilde{\mathcal{A}}(x,u,Du) = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}, \tag{4.22}$$

where $\widetilde{\mathcal{A}}(x,s,\xi) = \mathcal{A}(x,-s,-\xi)$ satisfies the same conditions (1.9) and (1.10), inequality (4.21) holds with u replaced by $-u$. We then conclude that

$$\int_{A_{k,\tau}^{-}} \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx \le cC \int_{A_{k,t}^{-}} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^{N} \left|f_i\right|^{q_i'}\right) dx + \frac{cC}{(t-\tau)^q} \int_{A_{k,t}^{-}} \sum_{i=1}^{N} (u-k)^{q_i} dx. \tag{4.23}$$

Adding (4.21) with (4.23) yields

$$\int_{A_{k,\tau}} \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx \le cC \int_{A_{k,t}} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^{N} \left|f_i\right|^{q_i'}\right) dx + \frac{cC}{(t-\tau)^q} \int_{A_{k,t}} \sum_{i=1}^{N} (u-k)^{q_i} dx. \tag{4.24}$$

Thus, u satisfies (2.2) with $\phi_0 = \varphi_2 + |k|^{p'} + \sum_{i=1}^{N} \left|f_i\right|^{q_i'}$ and $\alpha = q$. Theorem 4.2 follows from Lemma 2.1. □

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