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Research Article

Schur Convexity of Generalized Heronian Means Involving Two Parameters

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The Schur convexity and Schur-geometric convexity of generalized Heronian means involving two parameters are studied, the main result is then used to obtain several interesting and significantly inequalities for generalized Heronian means.

1. Introduction

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denotes n-tuple (n-dimensional real vector), the set of vectors can be written as

$$\begin{aligned} \mathbb{R}^n &= \{\mathbf{x} = (x_1, \dots, x_n) : x_i \in \mathbb{R},\ i = 1, \dots, n\}, \\ \mathbb{R}^n_+ &= \{\mathbf{x} = (x_1, \dots, x_n) : x_i \geq 0,\ i = 1, \dots, n\}, \\ \mathbb{R}^n_{++} &= \{\mathbf{x} = (x_1, \dots, x_n) : x_i > 0,\ i = 1, \dots, n\}. \end{aligned} \tag{1.1}$$

In particular, the notations $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ denote $\mathbb{R}^1$, $\mathbb{R}^1_+$, and $\mathbb{R}^1_{++}$, respectively.

In what follows, we assume that $(a, b) \in \mathbb{R}^2_+$.

The classical Heronian means of a and b is defined as ([1], see also [2])

$$H_e(a, b) = \frac{a + \sqrt{ab} + b}{3}. \tag{1.2}$$

In [3], an analogue of Heronian means is defined by

$$\widetilde{H}(a,b)=\frac{a+4\sqrt{ab}+b}{6}. \tag{1.3}$$

Janous [4] presented a weighted generalization of the above Heronian-type means, as follows:

$$H_w(a,b)=\begin{cases}\dfrac{a+w\sqrt{ab}+b}{w+2}, & 0\le w<+\infty,\\ \sqrt{ab}, & w=+\infty.\end{cases} \tag{1.4}$$

Recently, the following exponential generalization of Heronian means was considered by Jia and Cao in [5],

$$H_p=H_p(a,b)=\begin{cases}\left[\dfrac{a^p+(ab)^{p/2}+b^p}{3}\right]^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0.\end{cases} \tag{1.5}$$

Several variants as well as interesting applications of Heronian means can be found in the recent papers [6–11].

The weighted and exponential generalizations of Heronian means motivate us to consider a unified generalization of Heronian means (1.4) and (1.5), as follows:

$$H_{p,w}(a,b)=\begin{cases}\left[\dfrac{a^p+w(ab)^{p/2}+b^p}{w+2}\right]^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0,\end{cases} \tag{1.6}$$

where $w\ge 0$.

In this paper, the Schur convexity, Schur-geometric convexity, and monotonicity of the generalized Heronian means $H_{p,w}(a,b)$ are discussed. As consequences, some interesting inequalities for generalized Heronian means are obtained.

2. Definitions and lemmas

We begin by introducing the following definitions and lemmas.

Definition 2.1 (see [12, 13]). Let $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$.

(1) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x}\prec\mathbf{y}$) if $\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]}$ for $k=1,2,\dots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[1]}\ge\cdots\ge x_{[n]}$ and $y_{[1]}\ge\cdots\ge y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(2) $\mathbf{x}\ge\mathbf{y}$ means that $x_i\ge y_i$ for all $i=1,2,\dots,n$. Let $\Omega\subset\mathbb{R}^n$, $\varphi:\Omega\to\mathbb{R}$ is said to be increasing if $\mathbf{x}\ge\mathbf{y}$ implies $\varphi(\mathbf{x})\ge\varphi(\mathbf{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

(3) Let $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function.

Definition 2.2 (see [14, 15]). Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_{++}$.

(1) Ω is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for any $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(2) Let $\Omega \subset \mathbb{R}^n_{++}$, $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function.

Lemma 2.3 (see [12, page 38]). *A function $\varphi(\mathbf{x})$ is increasing if and only if $\nabla\varphi(\mathbf{x}) \ge 0$ for $\mathbf{x} \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set, $\varphi : \Omega \to R$ is differentiable, and*

$$\nabla\varphi(\mathbf{x}) = \left(\frac{\partial\varphi(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial\varphi(\mathbf{x})}{\partial x_n}\right) \in \mathbb{R}^n. \tag{2.1}$$

Lemma 2.4 (see [12, page 58]). *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then, φ is the Schur-convex(Schur-concave) function, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \ge 0 \ (\le 0) \tag{2.2}$$

holds for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

Lemma 2.5 (see [14, page 108]). *Let $\Omega \subset \mathbb{R}^n_{++}$ is a symmetric and has a nonempty interior geometrically convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}_+$ is continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \ge 0 \ (\le 0) \tag{2.3}$$

holds for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$, then φ is the Schur-geometrically convex (Schur-geometrically concave) function.

Lemma 2.6 (see [12, page 5]). *Let $\mathbf{x} \in \mathbb{R}^n$ and $\overline{\mathbf{x}} = (1/n)\sum_{i=1}^n x_i$. Then,*

$$(\overline{\mathbf{x}}, \ldots, \overline{\mathbf{x}}) \prec \mathbf{x}. \tag{2.4}$$

Lemma 2.7 (see [16, page 43]). *The generalized logarithmic means (Stolarsky's means) of two positive numbers a and b is defined as follows*

$$S_p(a,b) = \begin{cases} \left(\dfrac{b^p - a^p}{p(b-a)}\right)^{1/(p-1)}, & p \neq 0,1,\ a \neq b, \\ e^{-1}\left(\dfrac{a^a}{b^b}\right)^{1/(a-b)}, & p = 1,\ a \neq b, \\ \dfrac{b-a}{\ln b - \ln a}, & p = 0,\ a \neq b, \\ b, & a = b, \end{cases} \tag{2.5}$$

when $a \neq b$, $S_p(a,b)$ is a strictly increasing function for $p \in \mathbb{R}$.

Lemma 2.8 (see [17]). *Let $a,b > 0$ and $a \neq b$. If $x > 0$, $y \leq 0$ and $x + y \geq 0$, then,*

$$\frac{b^{x+y} - a^{x+y}}{b^x - a^x} \leq \frac{x+y}{x}(ab)^{y/2}. \tag{2.6}$$

3. Main results and their proofs

Our main results are stated in Theorems 3.1 and 3.2 below.

Theorem 3.1. *For fixed $(p,w) \in \mathbb{R}^2$,*

(1) *$H_{p,w}(a,b)$ is increasing for $(a,b) \in \mathbb{R}^2_+$;*

(2) *if $(p,w) \in \{p \leq 1, w \geq 0\} \cup \{1 < p \leq 3/2, w \geq 1\} \cup \{3/2 < p \leq 2, w \geq 2\}$, then, $H_{p,w}(a,b)$ is Schur concave for $(a,b) \in \mathbb{R}^2_+$;*

(3) *if $p \geq 2$, $0 \leq w \leq 2$, then, $H_{p,w}(a,b)$ is Schur convex for $(a,b) \in \mathbb{R}^2_+$.*

Proof. Let

$$\varphi(a,b) = \frac{a^p + w(ab)^{p/2} + b^p}{w+2}, \tag{3.1}$$

when $p \neq 0$ and $w \geq 0$, we have $H_{p,w}(a,b) = \varphi^{1/p}(a,b)$. It is clear that $H_{p,w}(a,b)$ is symmetric with $(a,b) \in \mathbb{R}^2_+$.

Since

$$\begin{aligned} \frac{\partial H_{p,w}(a,b)}{\partial a} &= \frac{1}{w+2}\left[a^{p-1} + \frac{wb}{2}(ab)^{p/2-1}\right]\varphi^{1/p-1} \quad (a,b) \geq 0, \\ \frac{\partial H_{p,w}(a,b)}{\partial b} &= \frac{1}{w+2}\left[b^{p-1} + \frac{wa}{2}(ab)^{p/2-1}\right]\varphi^{1/p-1} \quad (a,b) \geq 0, \end{aligned} \tag{3.2}$$

we deduce from Lemma 2.3 that $H_{p,w}(a,b)$ is increasing for $(a,b) \in \mathbb{R}^2_+$.

Let

$$\Lambda := (b-a)\left(\frac{\partial H_{p,w}(a,b)}{\partial b} - \frac{\partial H_{p,w}(a,b)}{\partial a}\right), \tag{3.3}$$

when $a = b$, then $\Lambda = 0$. We assume $a \neq b$ below.

Let $\Lambda = ((b-a)^2/(w+2))\varphi^{1/p-1}(a,b)Q$, where

$$Q = \frac{b^{p-1} - a^{p-1}}{b-a} - \frac{w}{2}(ab)^{p/2-1}. \tag{3.4}$$

We consider the following four cases.

Case 1. If $p \leq 1,\ w \geq 0$, then $(b^{p-1} - a^{p-1})/(b-a) \leq 0$, which implies that $\Lambda \leq 0$. It follows from Lemma 2.4 that $H_{p,w}(a,b)$ is Schur concave.

Case 2. If $1 < p \leq 3/2,\ w \geq 1$, then $p - 1 \leq 1/2 \leq w/2$.

In Lemma 2.8, letting $x = 1,\ y = p-2$, which implies $x > 0,\ y < 0,\ x + y > 0$. By Lemma 2.8 we have

$$\frac{b^{p-1} - a^{p-1}}{b-a} \leq (p-1)(ab)^{(p-2)/2} \leq \frac{w}{2}(ab)^{p/2-1}. \tag{3.5}$$

We conclude that $\Lambda \leq 0$. Therefore, $H_{p,w}(a,b)$ is Schur concave.

Case 3. If $3/2 < p \leq 2,\ w \geq 2$, then $p - 1 \leq 1 \leq w/2$.

In Lemma 2.8, letting $x = 1,\ y = p-2$, which implies $x > 0,\ y \leq 0,\ x + y > 0$. By Lemma 2.8 we have

$$\frac{b^{p-1} - a^{p-1}}{b-a} \leq (p-1)(ab)^{(p-2)/2} \leq \frac{w}{2}(ab)^{p/2-1}, \tag{3.6}$$

it follows that $\Lambda \leq 0$. Therefore, $H_{p,w}(a,b)$ is Schur concave.

Case 4. If $p \geq 2,\ 0 \leq w \leq 2$. Note that

$$Q = (p-1)\left[S_{p-1}(a,b)\right]^{p-2} - \frac{w}{2}\left[S_{-1}(a,b)\right]^{p-2}. \tag{3.7}$$

By Lemma 2.7, we obtain that $S_p(a,b)$ is increasing for $p \in \mathbb{R}$. Thus, we conclude that $[S_{p-1}(a,b)]^{p-2} \geq [S_{-1}(a,b)]^{p-2}$. Then, using $p - 1 \geq 1 \geq w/2$, we have $\Lambda \geq 0$. Therefore, $H_{p,w}(a,b)$ is Schur convex.

This completes the proof of Theorem 3.1. □

Theorem 3.2. *For fixed* $(p, w) \in \mathbb{R}^2$,

(1) *if* $p < 0,\ w \geq 0$, *then* $H_{p,w}(a,b)$ *is Schur-geometrically concave for* $(a,b) \in \mathbb{R}^2_{++}$;

(2) *if* $p > 0,\ w \geq 0$, *then* $H_{p,w}(a,b)$ *is Schur-geometrically convex for* $(a,b) \in \mathbb{R}^2_{++}$.

Proof. Since

$$\begin{aligned} a\frac{\partial H_{p,w}(a,b)}{\partial a} &= \frac{1}{w+2}\left[a^p + \frac{wb}{2}(ab)^{p/2}\right]\varphi^{1/p-1}(a,b), \\ b\frac{\partial H_{p,w}(a,b)}{\partial b} &= \frac{1}{w+2}\left[b^p + \frac{wa}{2}(ab)^{p/2}\right]\varphi^{1/p-1}(a,b), \end{aligned} \tag{3.8}$$

we have

$$\Delta := (\ln b - \ln a)\left(a\frac{\partial H_{p,w}(a,b)}{\partial b} - b\frac{\partial H_{p,w}(a,b)}{\partial a}\right) = \frac{(\ln b - \ln a)(b^p - a^p)}{w+2}\varphi^{1/p-1}(a,b), \tag{3.9}$$

when $p < 0,\ w \geq 0$, then $(\ln b - \ln a)(b^p - a^p) \leq 0$, which implies that $\Delta \leq 0$. Therefore, $H_{p,w}(a,b)$ is Schur-geometrically concave.

When $p > 0,\ w \geq 0$, then $(\ln b - \ln a)(b^p - a^p) \geq 0$, which implies that $\Delta \geq 0$. Therefore, $H_{p,w}(a,b)$ is Schur-geometrically convex.

The proof of Theorem 3.2 is complete. □

4. Some applications

In this section, we provide several interesting applications of Theorems 3.1 and 3.2.

Theorem 4.1. *Let* $0 < a \leq b,\ A(a,b) = (a+b)/2,\ u(t) = tb + (1-t)a,\ v(t) = ta + (1-t)b$, *and let* $1/2 \leq t_2 \leq t_1 \leq 1$ *or* $0 \leq t_1 \leq t_2 \leq 1/2$. *If* $(p,w) \in \{p \leq 1, w \geq 0\} \cup \{1 < p \leq 3/2, w \geq 1\} \cup \{3/2 < p \leq 2, w \geq 2\}$, *then,*

$$A(a,b) \geq H_{p,w}(u(t_2), v(t_2)) \geq H_{p,w}(u(t_1), v(t_1)) \geq H_{p,w}(a,b). \tag{4.1}$$

If $p \geq 2,\ 0 \leq w \leq 2$, *then each of the inequalities in* (4.1) *is reversed.*

Proof. When $1/2 \leq t_2 \leq t_1 \leq 1$. From $0 < a \leq b$, it is easy to see that $u(t_1) \geq v(t_1),\ u(t_2) \geq v(t_2),\ b \geq u(t_1) \geq u(t_2)$, and $u(t_2) + v(t_2) = u(t_1) + v(t_1) = a + b$.

We thus conclude that

$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a,b). \tag{4.2}$$

When $0 \leq t_1 \leq t_2 \leq 1/2$, then $1/2 \leq 1 - t_2 \leq 1 - t_1 \leq 1$, it follows that

$$(u(1-t_2), v(1-t_2)) \prec (u(1-t_1), v(1-t_1)) \prec (a,b). \tag{4.3}$$

Since $u(1-t_2)=v(t_2)$, $v(1-t_2)=u(t_2)$, $u(1-t_1)=v(t_1)$, $v(1-t_1)=u(t_1)$, we also have

$$(u(t_2),v(t_2)) \prec (u(t_1),v(t_1)) \prec (a,b). \tag{4.4}$$

On the other hand, it follows from Lemma 2.6 that $((a+b)/2,(a+b)/2) \prec (u(t_2),v(t_2))$. Applying Theorem 3.1 gives the inequalities asserted by Theorem 4.1. □

Theorem 4.1 enables us to obtain a large number of refined inequalities by assigning appropriate values to the parameters p, w, t_1, and t_2, for example, putting $p=1/2$, $w=1$, $t_1=3/4$, $t_2=1/2$ in (4.1), we obtain

$$\frac{a+b}{2} \geq \left(\frac{\sqrt{a+3b}+\sqrt[4]{(a+3b)(3a+b)}+\sqrt{3a+b}}{6}\right)^2 \geq \left(\frac{\sqrt{a}+\sqrt[4]{ab}+\sqrt{b}}{3}\right)^2. \tag{4.5}$$

Putting $p=2$, $w=1$, $t_1=3/4$, $t_2=1/2$ in (4.1), we get

$$\frac{a+b}{2} \leq \sqrt{\frac{(a+3b)^2+(a+3b)(3a+b)+(3a+b)^2}{48}} \leq \sqrt{\frac{a^2+ab+b^2}{3}}. \tag{4.6}$$

Theorem 4.2. *Let $0<a\leq b$, $c\geq 0$. If $(p,w)\in\{p\leq 1, w\geq 0\}\cup\{1<p\leq 3/2, w\geq 1\}\cup\{3/2<p\leq 2, w\geq 2\}$, then*

$$\frac{H_{p,w}(a+c,b+c)}{a+b+2c} \geq \frac{H_{p,w}(a,b)}{a+b}. \tag{4.7}$$

If $p\geq 2$, $0\leq w\leq 2$, then the inequality (4.7) *is reversed.*

Proof. From the hypotheses $0\leq a\leq b$, $c\geq 0$, we deduce that

$$\begin{gathered}\frac{a+c}{a+b+2c} \leq \frac{b+c}{a+b+2c}, \qquad \frac{a}{a+b} \leq \frac{b}{a+b}, \qquad \frac{b+c}{a+b+2c} \leq \frac{b}{a+b}, \\ \frac{a+c}{a+b+2c}+\frac{b+c}{a+b+2c} = \frac{a}{a+b}+\frac{b}{a+b} = 1.\end{gathered} \tag{4.8}$$

We hence have

$$\left(\frac{a+c}{a+b+2c},\frac{b+c}{a+b+2c}\right) \prec \left(\frac{a}{a+b},\frac{b}{a+b}\right). \tag{4.9}$$

Using Theorem 3.1 yields the inequalities asserted by Theorem 4.2. □

Theorem 4.3. *Let* $0 < a \le b,\ G(a,b) = \sqrt{ab},\ \tilde{u}(t) = b^t a^{1-t},\ \tilde{v}(t) = a^t b^{1-t}$, *and let* $1/2 \le t_2 \le t_1 \le 1$ *or* $0 \le t_1 \le t_2 \le 1/2$. *If* $p > 0,\ w \ge 0$, *then*

$$G(a,b) \le H_{p,w}(\tilde{u}(t_2),\tilde{v}(t_2)) \le H_{p,w}(\tilde{u}(t_1),\tilde{v}(t_1)) \le H_{p,w}(a,b). \tag{4.10}$$

If $p < 0,\ w \ge 0$, *then each of the inequalities in* (4.10) *is reversed.*

Proof. From the hypotheses $0 < a \le b$, $1/2 \le t_2 \le t_1 \le 1$ (or $0 \le t_1 \le t_2 \le 1/2$), it is easy to verify that

$$(\ln\tilde{u}(t_2),\ln\tilde{v}(t_2)) \prec (\ln\tilde{u}(t_1),\ln\tilde{v}(t_1)) \prec (\ln a,\ln b). \tag{4.11}$$

In addition, from Lemma 2.6 we have $(\ln\sqrt{ab},\ln\sqrt{ab}) \prec (\ln\tilde{u}(t_2),\ln\tilde{v}(t_2))$.

By applying Theorem 3.2, we obtain the desired inequalities in Theorem 4.3. □

Combining the inequalities (4.1) and (4.10), we obtain the following refinement of arithmetic-geometric means inequality.

Theorem 4.4. *Let* $0 < a \le b,\ u(t) = tb+(1-t)a,\ v(t) = ta+(1-t)b,\ \tilde{u}(t) = b^t a^{1-t},\ \tilde{v}(t) = a^t b^{1-t}$, *and let* $1/2 \le t_2 \le t_1 \le 1$ *or* $0 \le t_1 \le t_2 \le 1/2$. *If* $(p,w) \in \{0 < p \le 1, w \ge 0\} \cup \{1 < p \le 3/2, w \ge 1\} \cup \{3/2 < p \le 2, w \ge 2\}$, *then*

$$\begin{aligned} G(a,b) &\le H_{p,w}(\tilde{u}(t_2),\tilde{v}(t_2)) \\ &\le H_{p,w}(\tilde{u}(t_1),\tilde{v}(t_1)) \\ &\le H_{p,w}(a,b) \\ &\le H_{p,w}(u(t_1),v(t_1)) \\ &\le H_{p,w}(u(t_2),v(t_2)) \\ &\le A(a,b). \end{aligned} \tag{4.12}$$

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References

[1] H. Alzer and W. Janous, "Solution of problem 8*," *Crux Mathematicorum*, vol. 13, pp. 173–178, 1987.
[2] P. S. Bullen, D. S. Mitrinvić, and P. M. Vasić, *Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.
[3] Q.-J. Mao, "Dual means, logarithmic and Heronian dual means of two positive numbers," *Journal of Suzhou College of Education*, vol. 16, pp. 82–85, 1999.
[4] W. Janous, "A note on generalized Heronian means," *Mathematical Inequalities & Applications*, vol. 4, no. 3, pp. 369–375, 2001.

[5] G. Jia and J. Cao, "A new upper bound of the logarithmic mean," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, article 80, 4 pages, 2003.
[6] K. Guan and H. Zhu, "The generalized Heronian mean and its inequalities," *Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta. Serija Matematika*, vol. 17, pp. 60–75, 2006.
[7] Z. Zhang and Y. Wu, "The generalized Heron mean and its dual form," *Applied Mathematics E-Notes*, vol. 5, pp. 16–23, 2005.
[8] Z. Zhang, Y. Wu, and A. Zhao, "The properties of the generalized Heron means and its dual form," *RGMIA Research Report Collection*, vol. 7, no. 2, article 1, 2004.
[9] Z. Liu, "Comparison of some means," *Journal of Mathematical Research and Exposition*, vol. 22, no. 4, pp. 583–588, 2002.
[10] N.-G. Zheng, Z.-H. Zhang, and X.-M. Zhang, "Schur-convexity of two types of one-parameter mean values in n variables," *Journal of Inequalities and Applications*, vol. 2007, Article ID 78175, 10 pages, 2007.
[11] H.-N. Shi, S.-H. Wu, and F. Qi, "An alternative note on the Schur-convexity of the extended mean values," *Mathematical Inequalities & Applications*, vol. 9, no. 2, pp. 219–224, 2006.
[12] B.-Y. Wang, *Foundations of Majorization Inequalities*, Beijing Normal University Press, Beijing, China, 1990.
[13] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, vol. 143 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1979.
[14] X.-M. Zhang, *Geometrically Convex Functions*, Anhui University Press, Hefei, China, 2004.
[15] C. P. Niculescu, "Convexity according to the geometric mean," *Mathematical Inequalities & Applications*, vol. 3, no. 2, pp. 155–167, 2000.
[16] J.-C. Kuang, *Applied Inequalities*, Shandong Science and Technology Press, Jinan, China, 3rd edition, 2004.
[17] Z. Liu, "A note on an inequality," *Pure and Applied Mathematics*, vol. 17, no. 4, pp. 349–351, 2001.