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Research Article

On the Cauchy Functional Inequality in Banach Modules

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We investigate the following functional inequality: $\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|$ in Banach modules over a C^*-algebra, and prove the generalized Hyers-Ulam stability of linear mappings in Banach modules over a C^*-algebra.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. The result of Găvruţa [5] is a special case of a more general theorem, which was obtained by Forti [6].

Th. M. Rassias [7] during the 27th international symposium on functional equations asked the question whether such a theorem can also be proved for $p \ge 1$. Gajda [8], following the same approach as in Th. M. Rassias [4], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [8], as well as by Th. M. Rassias and Šemrl [9] that one cannot prove a Th. M. Rassias'-type theorem when $p = 1$.

J. M. Rassias [10] followed the innovative approach of Th. M. Rassias' theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \ne 1$.

During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [11–21]).

Gilányi [22] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\|, \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y). \tag{1.2}$$

See also [23]. Fechner [24] and Gilányi [25] proved the generalized Hyers-Ulam stability of the functional inequality (1.1). Park et al. [19] investigated the functional inequality

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\| \tag{1.3}$$

in Banach spaces, and proved the generalized Hyers-Ulam stability of the functional inequality (1.3) in Banach spaces.

Throughout this paper, let A be a unital C^*-algebra with unitary group $U(A)$ and unit e. Assume that X is a Banach A-module with norm $\|\cdot\|_X$ and that Y is a Banach A-module with norm $\|\cdot\|_Y$.

In this paper, we investigate an A-linear mapping associated with the functional inequality (1.3) and prove the generalized Hyers-Ulam stability of A-linear mappings in Banach A-modules associated with the functional inequality (1.3).

The computations in the proofs of the main theorems are special cases of the general results obtained by Forti [26].

2. Functional inequalities in Banach modules over a C^*-algebra

Lemma 2.1. *Let $f: X \to Y$ be a mapping such that*

$$\|f(x)+f(y)+uf(z)\|_Y \le \|f(x+y+uz)\|_Y \tag{2.1}$$

for all $x, y, z \in X$ and all $u \in U(A)$. Then f is A-linear.

Proof. Letting $x=y=z=0$ and $u=e\in U(A)$ in (2.1), we get

$$\|3f(0)\|_Y \le \|f(0)\|_Y. \tag{2.2}$$

So, $f(0)=0$.

Letting $z=0$ and $y=-x$ in (2.1), we get

$$\|f(x)+f(-x)\|_Y \le \|f(0)\|_Y = 0 \tag{2.3}$$

for all $x\in X$. Hence $f(-x)=-f(x)$ for all $x\in X$.

Letting $z=-x-y$ and $u=e\in U(A)$ in (2.1), we get

$$\|f(x)+f(y)-f(x+y)\|_Y = \|f(x)+f(y)+f(-x-y)\|_Y \le \|f(0)\|_Y = 0 \tag{2.4}$$

for all $x, y\in X$. Thus,

$$f(x+y)=f(x)+f(y) \tag{2.5}$$

for all $x, y\in X$.

Letting $x = -uz$ and $y = 0$ in (2.1), we get

$$\left\| -f(uz) + uf(z) \right\|_Y = \left\| f(-uz) + uf(z) \right\|_Y \le \left\| f(0) \right\|_Y = 0 \tag{2.6}$$

for all $z \in X$ and all $u \in U(A)$. Thus,

$$f(uz) = uf(z) \tag{2.7}$$

for all $u \in U(A)$ and all $z \in X$.

Now let $a \in A$ $(a \neq 0)$ and M an integer greater than $4|a|$. Then $|a/M| < 1/4 < 1 - 2/3 = 1/3$. By [27, Theorem 1], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3(a/M) = u_1 + u_2 + u_3$. So by (2.7)

$$\begin{aligned} f(ax) &= f\left(\frac{M}{3} \cdot 3\frac{a}{M}x\right) = M \cdot f\left(\frac{1}{3} \cdot 3\frac{a}{M}x\right) = \frac{M}{3} f\left(3\frac{a}{M}x\right) = \frac{M}{3} f(u_1 x + u_2 x + u_3 x) \\ &= \frac{M}{3}\left(f(u_1 x) + f(u_2 x) + f(u_3 x)\right) = \frac{M}{3}(u_1 + u_2 + u_3) f(x) = \frac{M}{3} \cdot 3\frac{a}{M} f(x) = af(x) \end{aligned} \tag{2.8}$$

for all $x \in X$. So, $f : X \to Y$ is A-linear, as desired. □

Now, we prove the generalized Hyers-Ulam stability of A-linear mappings in Banach A-modules.

Theorem 2.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\left\| f(x) + f(y) + uf(z) \right\|_Y \le \left\| f(x + y + uz) \right\|_Y + \theta\left(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r\right) \tag{2.9}$$

for all $x, y, z \in X$ and all $u \in U(A)$. Then, there exists a unique A-linear mapping $L : X \to Y$ such that

$$\left\| f(x) - L(x) \right\|_Y \le \frac{2^r + 2}{2^r - 2}\theta \|x\|_X^r \tag{2.10}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(-x) = -f(x)$ for all $x \in X$.

Letting $u = e \in U(A)$, $y = x$ and $z = -2x$ in (2.9), we get

$$\left\| 2f(x) - f(2x) \right\|_Y = \left\| 2f(x) + f(-2x) \right\|_Y \le \left(2 + 2^r\right)\theta \|x\|_X^r \tag{2.11}$$

for all $x \in X$. So,

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_Y \le \frac{2 + 2^r}{2^r}\theta \|x\|_X^r \tag{2.12}$$

for all $x \in X$. Hence,

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_Y &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_Y \\ &\le \frac{2+2^r}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|_X^r\end{aligned} \tag{2.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.13) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So, one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.14}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get (2.10).

It follows from (2.9) that

$$\begin{aligned}\|L(x) + L(y) + uL(z)\|_Y &= \lim_{n\to\infty} 2^n\left\|f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + uf\left(\frac{z}{2^n}\right)\right\|_Y \\ &\le \lim_{n\to\infty} 2^n\left\|f\left(\frac{x+y+uz}{2^n}\right)\right\|_Y + \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}\left(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r\right) \\ &= \|L(x+y+uz)\|_Y\end{aligned} \tag{2.15}$$

for all $x, y, z \in X$ and all $u \in U(A)$. So,

$$\|L(x) + L(y) + uL(z)\|_Y \le \|L(x+y+uz)\|_Y \tag{2.16}$$

for all $x, y, z \in X$ and all $u \in U(A)$. By Lemma 2.1, the mapping $L : X \to Y$ is A-linear.

Now, let $T : X \to Y$ be another A-linear mapping satisfying (2.10). Then, we have

$$\begin{aligned}\|L(x) - T(x)\|_Y &= 2^n\left\|L\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_Y \\ &\le 2^n\left(\left\|L\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_Y + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_Y\right) \\ &\le \frac{2(2^r+2)2^n}{(2^r-2)2^{nr}}\theta\|x\|_X^r,\end{aligned} \tag{2.17}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So, we can conclude that $L(x) = T(x)$ for all $x \in X$. This proves the uniqueness of L. Thus, the mapping $L : X \to Y$ is a unique A-linear mapping satisfying (2.10). □

Theorem 2.3. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (2.9). *Then, there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{2+2^r}{2-2^r}\theta\|x\|_X^r \tag{2.18}$$

for all $x \in X$.

Proof. It follows from (2.11) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \leq \frac{2+2^r}{2}\theta\|x\|_X^r \tag{2.19}$$

for all $x \in X$. Hence,

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\|_Y &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\|_Y \\ &\leq \frac{2+2^r}{2}\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\theta\|x\|_X^r\end{aligned} \tag{2.20}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.20) that the sequence $\{(1/2^n)f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{(1/2^n)f(2^n x)\}$ converges. So, one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x) \tag{2.21}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.20), we get (2.18).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.4. *Let $r > 1/3$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|f(x) + f(y) + uf(z)\|_Y \leq \|f(x+y+uz)\|_Y + \theta\cdot\|x\|_X^r\cdot\|y\|_X^r\cdot\|z\|_X^r \tag{2.22}$$

for all $x, y, z \in X$ and all $u \in U(A)$. Then, there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{2^r\theta}{8^r - 2}\|x\|_X^{3r} \tag{2.23}$$

for all $x \in X$.

Proof. Since f is an odd mapping, $f(-x) = -f(x)$ for all $x \in X$.

Letting $u = e \in U(A)$, $y = x$ and $z = -2x$ in (2.22), we get

$$\|2f(x) - f(2x)\|_Y = \|2f(x) + f(-2x)\|_Y \leq 2^r\theta\|x\|_X^{3r} \tag{2.24}$$

for all $x \in X$. So,

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_Y \leq \frac{2^r}{8^r}\theta\|x\|_X^{3r} \tag{2.25}$$

for all $x \in X$. Hence,

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_Y &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_Y \\ &\le \frac{2^r}{8^r}\sum_{j=l}^{m-1}\frac{2^j}{8^{rj}}\theta\|x\|_X^{3r}\end{aligned} \tag{2.26}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.26) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So, one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.27}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.26), we get (2.23).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.5. *Let $r < 1/3$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (2.22). *Then, there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \le \frac{2^r\theta}{2 - 8^r}\|x\|_X^{3r} \tag{2.28}$$

for all $x \in X$.

Proof. It follows from (2.24) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \le \frac{2^r}{2}\theta\|x\|_X^{3r} \tag{2.29}$$

for all $x \in X$. Hence,

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\|_Y &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\|_Y \\ &\le \frac{2^r}{2}\sum_{j=l}^{m-1}\frac{8^{rj}}{2^j}\theta\|x\|_X^{3r}\end{aligned} \tag{2.30}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.30) that the sequence $\{(1/2^n)f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{(1/2^n)f(2^n x)\}$ converges. So, one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x) \tag{2.31}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.30), we get (2.28).

The rest of the proof is similar to the proof of Theorem 2.2. □

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