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Research Article

Differences of Weighted Composition Operators on $H_\alpha^\infty(B_N)^*$

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We study the boundedness and compactness of differences of weighted composition operators on weighted Banach spaces in the unit ball of C^N.

1. Introduction

Let C^N denote the Euclidean space of complex dimension $N(N \geq 1)$. For $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ in C^N, we denote the inner product of z and w by

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_N\overline{w_N}, \tag{1.1}$$

and we write $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_N|^2}$. Let $B_N = \{z \in C^N : |z| < 1\}$ be the open unit ball of C^N and let $H(B_N)$ be the space of all holomorphic functions on B_N. For a holomorphic self-map of the unit ball $\varphi : B_N \to B_N$ and $u \in H(B_N)$, we define a weighted composition operator $W_{\varphi,u}$ by

$$W_{\varphi,u}(f) = u \cdot (f \circ \varphi) \tag{1.2}$$

for $f \in H(B_N)$. As for $u \equiv 1$, the weighted composition operator $W_{\varphi,1}$ is the usual composition operator, denoted by C_φ. When φ is the identity mapping I, the operator $W_{I,u}$ is also called the multiplication operator. During the past few decades much effort has been devoted to the research of such operators on different Banach spaces of holomorphic functions (see [1–9]). The general ideal is to explain the operator-theoretic behavior of $W_{\varphi,u}$ such as boundendness and compactness, in terms of the function-theoretic properties of the symbols φ and u. For a comprehensive overview of the field, we refer to the books by Cowen and MacCluer [10] and Shapiro [11].

The study of differences of two composition operators was first started on Hardy spaces. The primary motivation for this is to understand the topological structure of the set of composition operators on Hardy spaces. After that, such related problems have been studied on several spaces of holomorphic functions by many authors: by MacCluer et al. [12] Hosokawa et al. [13] on bounded spaces H^∞; by Moorhouse [14] on weighted Bergman spaces, and by Hosokawa and Ohno [15] and Nieminen [16] on Bloch spaces. In [1], the authors investigated the boundedness and compactness of $C_\varphi - C_\psi$ on weighted Banach spaces. In [16], Nieminen characterized the compactness of $W_{\varphi,u} - W_{\psi,v}$ when two weighted composition operators $W_{\varphi,u}$ and $W_{\psi,v}$ are bounded operators on weighted Banach spaces. Lindström and Wolf [17] generalized Nieminen's results on more general weighted Banach spaces. Furthermore, they estimated the essential norm of differences of two weighted composition operators. These works concerned with differences of weighted composition operators mainly focused on the setting of one variable. Recently, Toews [18], Gorkin et al. [19], and Aron et al. [20] extended the results of [12] to the case of several variables, respectively. In this paper, we study the boundedness and compactness of differences of weighted composition operators on weighted Banach spaces in the setting of several variables and extend some results of [16, 17]. Due to the difference between one variable and several variables, some special constructive techniques are applied. After collecting some preliminary results in the next section, we give an elegant inequality (see Lemma 3.2) which is useful to characterize the boundedness of differences of weighted composition operators on weighted Banach spaces in Section 3. In Section 4, we continue to describe the compactness of differences of weighted composition operators on these spaces and obtain some interesting corollaries.

2. Preliminaries

For $0 < \alpha < \infty$, let H_α^∞ be the weighted Banach space of holomorphic functions f on B_N satisfying

$$\|f\|_{H_\alpha^\infty} = \sup_{z\in B_N}\left(1-|z|^2\right)^\alpha |f(z)| < \infty. \tag{2.1}$$

Denote by B^α the Bloch-type space of holomorphic functions f on B_N such that

$$\sup_{z\in B_N}\left(1-|z|^2\right)^\alpha |\nabla f(z)| < \infty, \tag{2.2}$$

where $\nabla f(z) = ((\partial f/\partial z_1)(z), \ldots, (\partial f/\partial z_N)(z))$. When $\alpha = 1$, the space B^1 is the usual Bloch space. We call the function

$$K(z,z) = \frac{1}{\left(1-|z|^2\right)^{N+1}} \tag{2.3}$$

the Bergman kernel of B_N and denote the Bergman matrix by

$$B(z) = \frac{1}{N+1}\left(\frac{\partial^2}{\partial \overline{z_i}\partial z_j}\log K(z,z)\right)_{N\times N}. \tag{2.4}$$

For $f \in H(B_N)$, we define

$$Q_f(z) = \sup\left\{\frac{|\langle \nabla f(z), \overline{w}\rangle|}{\sqrt{\langle B(z)w, w\rangle}} : 0 \neq w \in C^N\right\}, \quad z \in B_N. \tag{2.5}$$

It is well known that (see [21] or [22])

$$\langle B(z)w, w\rangle = \frac{\left(1-|z|^2\right)|w|^2 + |\langle z, w\rangle|^2}{\left(1-|z|^2\right)^2}. \tag{2.6}$$

Moreover, for $\alpha > 1/2$, if a holomorphic function is $f \in B^\alpha$, then we have

$$\sup_{z\in B_N}\left(1-|z|^2\right)^{\alpha}|\nabla f(z)| \approx \sup_{z\in B_N}\left(1-|z|^2\right)^{\alpha-1}Q_f(z). \tag{2.7}$$

Here and below we use the abbreviated notation $A \approx B$ to mean that there exists a positive constant C such that $C^{-1}B \le A \le CB$. Throughout this paper, constants are denoted by C and they are positive finite quantities and not necessarily the same in each occurrence. Note that the weighted Banach space H_α^∞ can be identified with the Bloch-type space $B^{\alpha+1}$. Thus, we can easily see that if $f \in H_\alpha^\infty$ for $\alpha > 0$, then

$$\|f\|_{H_\alpha^\infty} \approx |f(0)| + \sup_{z\in B_N}\left(1-|z|^2\right)^{\alpha}Q_f(z). \tag{2.8}$$

For any point $a \in B_N - \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1-\langle z, a\rangle}, \quad z \in B_N, \tag{2.9}$$

where $s_a = \sqrt{1-|a|^2}$, P_a is the orthogonal projection from C^N onto the one-dimensional subspace $[a]$ generated by a, and $Q_a = I - P_a$ is the projection onto the orthogonal complement of $[a]$, that is

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2}a, \quad Q_a(z) = z - \frac{\langle z, a\rangle}{|a|^2}a, \quad z \in B_N. \tag{2.10}$$

When $a = 0$, we simply define $\varphi_a(z) = -z$. It is well known that each φ_a is a homeomorphism of the closed unit ball $\overline{B_N}$ onto $\overline{B_N}$. Let

$$\rho(a, z) = \left|\varphi_a(z)\right|. \tag{2.11}$$

Then ρ is a metric on B_N and is invariant under automorphisms. The metric ρ is called the pseudohyperbolic metric.

For any two points z and w in B_N, let $\gamma(t) = (\gamma_1(t), \ldots, \gamma_N(t)) : [0,1] \to B_N$ be a smooth curve to connect z and w. Define

$$l(\gamma) = \int_0^1 \sqrt{\langle B(\gamma(t))\gamma'(t), \gamma'(t)\rangle}dt. \tag{2.12}$$

The infimum of the set consisting of all $l(\gamma)$ is denoted by $\beta(z, w)$, where γ is a smooth curve in B_N from z and w. We call β the Bergman metric on B_N. It is known that

$$\beta(z, w) = \frac{1}{2}\log\frac{1+\rho(z, w)}{1-\rho(z, w)}. \tag{2.13}$$

3. The Boundedness of $W_{\varphi,u} - W_{\psi,v}$

In this section, we will characterize the boundedness of the operator $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$. For this purpose, we state some useful lemmas.

Lemma 3.1. *For z and w in B_N , then*

$$\frac{1-\rho(z, w)}{1+\rho(z, w)} \leq \frac{1-|z|^2}{1-|w|^2} \leq \frac{1+\rho(z, w)}{1-\rho(z, w)}. \tag{3.1}$$

Proof. Set $\varphi_w(z) = a$. Since φ_w is an involution, it follows that $\varphi_w(a) = z$. Thus, from the identity

$$1 - \left|\varphi_w(a)\right|^2 = 1 - |z|^2 = \frac{\left(1-|w|^2\right)\left(1-|a|^2\right)}{|1-\langle w, a\rangle|^2}, \tag{3.2}$$

we have

$$\frac{1-|z|^2}{1-|w|^2} = \frac{1-|a|^2}{|1-\langle w,a\rangle|^2}. \tag{3.3}$$

On the other hand,

$$\frac{1-|a|}{1+|a|} \leq \frac{1-|a|^2}{|1-\langle w,a\rangle|^2} \leq \frac{1+|a|}{1-|a|}. \tag{3.4}$$

Therefore, it follows that

$$\frac{1-\left|\varphi_w(z)\right|}{1+\left|\varphi_w(z)\right|} \leq \frac{1-|z|^2}{1-|w|^2} \leq \frac{1+\left|\varphi_w(z)\right|}{1-\left|\varphi_w(z)\right|}, \tag{3.5}$$

which, together with (2.11), yields the desired estimate. □

The following lemma can be found in [16, 17] for the one variable case.

Lemma 3.2. *Let $f \in H_\alpha^\infty$. Then*

$$\left|\left(1-|z|^2\right)^\alpha f(z) - \left(1-|w|^2\right)^\alpha f(w)\right| \leq C\|f\|_{H_\alpha^\infty}\rho(z,w) \tag{3.6}$$

for all z, w in B_N.

Proof. Fix any two points z and w in B_N. Let $\gamma = \gamma(t)(0 \leq t \leq 1)$ be a smooth curve in B_N from w to z. Then

$$\begin{aligned}\left(1-|z|^2\right)^\alpha f(z) - \left(1-|w|^2\right)^\alpha f(w) &= \int_0^1 d\left(1-\left|\gamma(t)\right|^2\right)^\alpha f(\gamma(t)) \\ &= \int_0^1 f(\gamma(t))\, d\left(1-\left|\gamma(t)\right|^2\right)^\alpha + \int_0^1 \left(1-\left|\gamma(t)\right|^2\right)^\alpha df(\gamma(t)).\end{aligned} \tag{3.7}$$

Since $f \in H_\alpha^\infty$, we get

$$\begin{aligned}
\left|\int_0^1 f(\gamma(t))\,d\left(1-|\gamma(t)|^2\right)^\alpha\right| &= \left|\int_0^1 -\alpha f(\gamma(t))\left(1-|\gamma(t)|^2\right)^{\alpha-1}\sum_{k=1}^N\left[\gamma_k(t)\overline{\gamma_k'(t)}+\overline{\gamma_k(t)}\gamma_k'(t)\right]dt\right| \\
&\le 2\alpha\int_0^1|f(\gamma(t))|\left(1-|\gamma(t)|^2\right)^{\alpha-1}|\langle\gamma(t),\gamma'(t)\rangle|\,dt \\
&\le C\|f\|_{H_\alpha^\infty}\int_0^1\frac{|\langle\gamma(t),\gamma'(t)\rangle|}{1-|\gamma(t)|^2}dt \\
&\le C\|f\|_{H_\alpha^\infty}\int_0^1\sqrt{\langle B(\gamma(t))\gamma'(t),\gamma'(t)\rangle}\,dt,
\end{aligned} \tag{3.8}$$

where the last inequality comes from (2.6).

On the other hand,

$$\left|\int_0^1\left(1-|\gamma(t)|^2\right)^\alpha df(\gamma(t))\right| = \left|\int_0^1\left(1-|\gamma(t)|^2\right)^\alpha\sum_{k=1}^N\gamma_k'(t)\frac{\partial f}{\partial z_k}(\gamma(t))\,dt\right|. \tag{3.9}$$

From the definition of Q_f we see that

$$\left|\sum_{k=1}^N\gamma_k'(t)\frac{\partial f}{\partial z_k}(\gamma(t))\right| \le Q_f(\gamma(t))\sqrt{\langle B(\gamma(t))\gamma'(t),\gamma'(t)\rangle}. \tag{3.10}$$

Thus, by (2.8) it follows that

$$\begin{aligned}
\left|\int_0^1\left(1-|\gamma(t)|^2\right)^\alpha df(\gamma(t))\right| &\le \int_0^1\left(1-|\gamma(t)|^2\right)^\alpha Q_f(\gamma(t))\sqrt{\langle B(\gamma(t))\gamma'(t),\gamma'(t)\rangle}\,dt \\
&\le C\|f\|_{H_\alpha^\infty}\int_0^1\sqrt{\langle B(\gamma(t))\gamma'(t),\gamma'(t)\rangle}\,dt.
\end{aligned} \tag{3.11}$$

Therefore, we have proved that

$$\left|\left(1-|z|^2\right)^\alpha f(z)-\left(1-|w|^2\right)^\alpha f(w)\right| \le C\|f\|_{H_\alpha^\infty}\int_0^1\sqrt{\langle B(\gamma(t))\gamma'(t),\gamma'(t)\rangle}\,dt. \tag{3.12}$$

Since $\gamma = \gamma(t)$ $(0 \le t \le 1)$ is an arbitrary smooth curve in B_N from w to z, by the definition of $\beta(z,w)$, we have

$$\left|\left(1-|z|^2\right)^\alpha f(z)-\left(1-|w|^2\right)^\alpha f(w)\right| \le C\|f\|_{H_\alpha^\infty}\beta(z,w). \tag{3.13}$$

If $\rho(z,w) < 1/2$, routine estimates show that $\beta(z,w) \le \rho(z,w)$. If $\rho(z,w) \ge 1/2$, then $4\rho(z,w) \ge 2$. Meanwhile, note that

$$\left|\left(1-|z|^2\right)^{\alpha} f(z) - \left(1-|w|^2\right)^{\alpha} f(w)\right| \le 2\|f\|_{H_\alpha^\infty}. \tag{3.14}$$

Thus, we have

$$\left|\left(1-|z|^2\right)^{\alpha} f(z) - \left(1-|w|^2\right)^{\alpha} f(w)\right| \le C\|f\|_{H_\alpha^\infty}\rho(z,w). \tag{3.15}$$

□

Remark 3.3. From the proof of Lemma 3.2, it is not hard to see that for any $z,w \in rB_N = \{z \in B_N : |z| < r < 1\}$, then

$$\left|\left(1-|z|^2\right)^{\alpha} f(z) - \left(1-|w|^2\right)^{\alpha} f(w)\right| \le C\|f_r\|_{H_\alpha^\infty}\rho(z,w) \tag{3.16}$$

for any $f \in H_\alpha^\infty$, where $\|f_r\|_{H_\alpha^\infty} = \sup_{z\in rB_N}(1-|z|^2)^{\alpha}|f(z)|$.

Now we provide a characterization of the boundedness of $W_{\varphi,u} - W_{\psi,v}$ from H_α^∞ to H_β^∞. For that purpose, consider the following three conditions:

$$\sup_{z\in B_N} \frac{\left(1-|z|^2\right)^{\beta}|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\rho(\varphi(z),\psi(z)) < \infty, \tag{3.17}$$

$$\sup_{z\in B_N} \frac{\left(1-|z|^2\right)^{\beta}|v(z)|}{\left(1-|\psi(z)|^2\right)^{\alpha}}\rho(\varphi(z),\psi(z)) < \infty, \tag{3.18}$$

$$\sup_{z\in B_N} \left|\frac{\left(1-|z|^2\right)^{\beta}u(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha}} - \frac{\left(1-|z|^2\right)^{\beta}v(z)}{\left(1-|\psi(z)|^2\right)^{\alpha}}\right| < \infty. \tag{3.19}$$

Theorem 3.4. *The following statements are equivalent.*

(i) $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ *is bounded.*

(ii) *The conditions* (3.17) *and* (3.19) *hold.*

(iii) *The conditions* (3.18) *and* (3.19) *hold.*

Proof. First, we prove the implication (ii) $\to$ (iii). Assume that the conditions (3.17) and (3.19) hold. Note that the pseudohyperbolic metric ρ is less then 1. Then we have

$$\frac{\left(1-|z|^2\right)^{\beta}|v(z)|}{\left(1-|\psi(z)|^2\right)^{\alpha}}\rho(\varphi(z),\psi(z)) \le \frac{\left(1-|z|^2\right)^{\beta}|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha}}\rho(\varphi(z),\psi(z)) + \left|\frac{\left(1-|z|^2\right)^{\beta}u(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha}} - \frac{\left(1-|z|^2\right)^{\beta}v(z)}{\left(1-|\psi(z)|^2\right)^{\alpha}}\right|\rho(\varphi(z),\psi(z)), \tag{3.20}$$

which implies that (3.18) holds. □

Next, we show the implication (iii) $\to$ (i). Let $f \in H_\alpha^\infty$. Assume that the conditions (3.18) and (3.19) hold; by Lemma 3.2, we have

$$\begin{aligned}
&\left(1-|z|^2\right)^{\beta}\left|(W_{\varphi,u}-W_{\psi,v})f(z)\right| \\
&\quad= \left(1-|z|^2\right)^{\beta}\left|f(\varphi(z))u(z)-f(\psi(z))v(z)\right| \\
&\quad= \Bigg|\left(1-|\varphi(z)|^2\right)^{\alpha}f(\varphi(z))\left[\frac{\left(1-|z|^2\right)^{\beta}u(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha}} - \frac{\left(1-|z|^2\right)^{\beta}v(z)}{\left(1-|\psi(z)|^2\right)^{\alpha}}\right] \\
&\qquad+ \frac{\left(1-|z|^2\right)^{\beta}v(z)}{\left(1-|\psi(z)|^2\right)^{\alpha}}\left[\left(1-|\varphi(z)|^2\right)^{\alpha}f(\varphi(z)) - \left(1-|\psi(z)|^2\right)^{\alpha}f(\psi(z))\right]\Bigg| \\
&\quad\le \|f\|_{H_\alpha^\infty}\left|\frac{\left(1-|z|^2\right)^{\beta}u(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha}} - \frac{\left(1-|z|^2\right)^{\beta}v(z)}{\left(1-|\psi(z)|^2\right)^{\alpha}}\right| + C\|f\|_{H_\alpha^\infty}\rho(\varphi(z),\psi(z))\frac{\left(1-|z|^2\right)^{\beta}|v(z)|}{\left(1-|\psi(z)|^2\right)^{\alpha}} \\
&\quad\le C\|f\|_{H_\alpha^\infty},
\end{aligned} \tag{3.21}$$

from which it follows that $W_{\varphi,u}-W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is bounded.

Finally, we prove the implication (i) $\to$ (ii). Assume that $W_{\varphi,u} - W_{\psi,v}$ is bounded. Fix $w \in B_N$; consider the function f_w defined by

$$f_w(z) = \frac{1}{(1-\langle z, \varphi(w)\rangle)^{\alpha}} \cdot \frac{\langle \varphi_{\varphi(w)}(z), \varphi_{\varphi(w)}(\psi(w))\rangle}{|\varphi_{\varphi(w)}(\psi(w))|} \tag{3.22}$$

for $z \in B_N$. It is easy to check that $f_w \in H_\alpha^\infty$ with $\|f_w\|_{H_\alpha^\infty} \le 2^\alpha$. Note that

$$f_w(\varphi(w)) = \frac{\rho(\varphi(w), \psi(w))}{\left(1-|\varphi(w)|^2\right)^{\alpha}}, \quad f_w(\psi(w)) = 0. \tag{3.23}$$

By the boundedness of $W_{\varphi,u} - W_{\psi,v}$, we have

$$\begin{aligned}
\infty > C\|f_w\|_{H_\alpha^\infty} &\ge \|(W_{\varphi,u} - W_{\psi,v}) f_w\|_{H_\beta^\infty} \\
&= \sup_{z \in B_N} \left(1-|z|^2\right)^{\beta} |f_w(\varphi(z))u(z) - f_w(\psi(z))v(z)| \\
&\ge \left(1-|w|^2\right)^{\beta} |f_w(\varphi(w))u(w) - f_w(\psi(w))v(w)| \\
&= \frac{\left(1-|w|^2\right)^{\beta} \rho(\varphi(w), \psi(w)) |u(w)|}{\left(1-|\varphi(w)|^2\right)^{\alpha}}
\end{aligned} \tag{3.24}$$

for any $w \in B_N$. This proves (3.17). Now we prove that (3.19) is also true. For given $w \in B_N$ instead of the function f_w, we consider the function g_w given by

$$g_w(z) = \frac{1}{(1-\langle z, \psi(w)\rangle)^{\alpha}}. \tag{3.25}$$

Clearly, $g_w \in H_\alpha^\infty$ with $\|g_w\|_{H_\alpha^\infty} \le 2^\alpha$. Thus, we have

$$\begin{aligned}
\infty > \|(W_{\varphi,u} - W_{\psi,v}) g_w\|_{H_\beta^\infty} &\ge \left(1-|w|^2\right)^{\beta} |g_w(\varphi(w))u(w) - g_w(\psi(w))v(w)| \\
&= |I(w) + J(w)| ,
\end{aligned} \tag{3.26}$$

where

$$I(w)=\left(1-\left|\psi(w)\right|^2\right)^{\alpha}g_w(\psi(w))\left[\frac{\left(1-|w|^2\right)^{\beta}u(w)}{\left(1-\left|\varphi(w)\right|^2\right)^{\alpha}}-\frac{\left(1-|w|^2\right)^{\beta}v(w)}{\left(1-\left|\psi(w)\right|^2\right)^{\alpha}}\right]$$
$$=\frac{\left(1-|w|^2\right)^{\beta}u(w)}{\left(1-\left|\varphi(w)\right|^2\right)^{\alpha}}-\frac{\left(1-|w|^2\right)^{\beta}v(w)}{\left(1-\left|\psi(w)\right|^2\right)^{\alpha}}, \tag{3.27}$$
$$J(w)=\frac{\left(1-|w|^2\right)^{\beta}u(w)}{\left(1-\left|\varphi(w)\right|^2\right)^{\alpha}}\left[\left(1-\left|\varphi(w)\right|^2\right)^{\alpha}g_w(\varphi(w))-\left(1-\left|\psi(w)\right|^2\right)^{\alpha}g_w(\psi(w))\right].$$

By (3.17) that has been proved as before and Lemma 3.2, we conclude that $|J(w)|<\infty$ for all $w\in B_N$, which implies that $|I(w)|<\infty$ for all $w\in B_N$. Thus, the condition (3.19) is proved. The whole proof is complete.

Corollary 3.5. *The weighted composition operator $W_{\varphi,u}:H_\alpha^\infty\to H_\beta^\infty$ is bounded if and only if*

$$\sup_{z\in B_N}\frac{\left(1-|z|^2\right)^{\beta}|u(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^{\alpha}}<\infty. \tag{3.28}$$

Proof. The result follows from the simple case in which $v\equiv 0$ of Theorem 3.4. □

Corollary 3.6. *The operator $uC_\varphi-uC_\psi:H_\alpha^\infty\to H_\beta^\infty$ is bounded if and only if the following two conditions hold:*

$$\sup_{z\in B_N}\frac{\left(1-|z|^2\right)^{\beta}|u(z)|}{\left(1-\left|\varphi(z)\right|^2\right)^{\alpha}}\rho(\varphi(z),\psi(z))<\infty,$$
$$\sup_{z\in B_N}\frac{\left(1-|z|^2\right)^{\beta}|u(z)|}{\left(1-\left|\psi(z)\right|^2\right)^{\alpha}}\rho(\varphi(z),\psi(z))<\infty. \tag{3.29}$$

Proof. The necessity is clear by Theorem 3.4. To prove the sufficiency, we only need to show that if the conditions (3.29) hold, then

$$\left|\frac{\left(1-|z|^2\right)^{\beta}u(z)}{\left(1-\left|\varphi(z)\right|^2\right)^{\alpha}}-\frac{\left(1-|z|^2\right)^{\beta}u(z)}{\left(1-\left|\psi(z)\right|^2\right)^{\alpha}}\right|<\infty \tag{3.30}$$

for all $z \in B_N$. In fact, we easily see that (3.30) holds for $z \in B_N$ satisfying $\rho(\varphi(z), \psi(z)) > 1/2$ by (3.29). If $z \in B_N$ such that $\rho(\varphi(z), \psi(z)) \le 1/2$, by Lemma 3.1 we have

$$\begin{aligned}
\left| \frac{\left(1-|z|^2\right)^\alpha u(z)}{\left(1-|\varphi(z)|^2\right)^\alpha} - \frac{\left(1-|z|^2\right)^\alpha u(z)}{\left(1-|\psi(z)|^2\right)^\alpha} \right| &= \frac{\left(1-|z|^2\right)^\alpha |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha} \left| 1 - \left(\frac{1-|\varphi(z)|^2}{1-|\psi(z)|^2} \right)^\alpha \right| \\
&\le \frac{\left(1-|z|^2\right)^\alpha |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha} \left| 1 - \left(\frac{1+\rho(\varphi(z),\psi(z))}{1-\rho(\varphi(z),\psi(z))} \right)^\alpha \right| \\
&\le C \frac{\left(1-|z|^2\right)^\alpha |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha} \rho(\varphi(z),\psi(z)),
\end{aligned} \tag{3.31}$$

which implies that (3.30) holds for $z \in B_N$ satisfying $\rho(\varphi(z), \psi(z)) \le 1/2$. The proof is complete. □

Example 3.7. We give a nontrivial example such that the weighted composition operators $W_{\varphi,u}$ and $W_{\psi,v}$ are unbounded on H_α^∞ while the operator $W_{\varphi,u} - W_{\psi,v}$ is bounded on H_α^∞.

Choose the analytic functions $\varphi(z) = (1+z)/2$ and $\psi(z) = \varphi(z) + t(z-1)^3$ in the unit disk, where t is real and positive and t is so small that ψ maps the unit disk D into D. Let $u(z) = v(z) = (1-z)^{-1}$, $\alpha = \beta > 0$. It is easy to see that when $0 < r < 1$,

$$\frac{(1-r^2)^\alpha u(r)}{(1-\varphi^2(r))^\alpha} \longrightarrow \infty, \qquad \frac{(1-r^2)^\alpha v(r)}{(1-\psi^2(r))^\alpha} \longrightarrow \infty \tag{3.32}$$

as $r \to 1$. This shows that $W_{\varphi,u}$ and $W_{\psi,v}$ are unbounded on H_α^∞ by Corollary 3.5. On the other hand, we know that $\rho(\varphi(z), \psi(z)) \le Ct|1-z|$ when t is small enough (see [12, Example 1]). By the Schwarz-Pick lemma, we get

$$\begin{aligned}
&\sup_{z\in D} \frac{\left(1-|z|^2\right)^\alpha |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha} \rho(\varphi(z),\psi(z)) \le \sup_{z\in D} \frac{Ct|1-z|}{|1-z|} < \infty, \\
&\sup_{z\in D} \frac{\left(1-|z|^2\right)^\alpha |v(z)|}{\left(1-|\psi(z)|^2\right)^\alpha} \rho(\varphi(z),\psi(z)) \le \sup_{z\in D} \frac{Ct|1-z|}{|1-z|} < \infty.
\end{aligned} \tag{3.33}$$

So, it follows that $W_{\varphi,u} - W_{\psi,v}$ is bounded on H_α^∞ from Corollary 3.6.

4. The Compactness of $W_{\varphi,u} - W_{\psi,v}$

In this section, we give a characterization of the compactness of the operator $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$. Before doing this, we need the following lemma whose proof is an easy modification of that of [10, Proposition 3.11].

Lemma 4.1. *The operator $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is compact if and only if whenever $\{f_n\}$ is a bounded sequence in H_α^∞ with $f_n \to 0$ uniformly on compact subsets of B_N, and then $\|(W_{\varphi,u} - W_{\psi,v})f_n\|_{H_\beta^\infty} \to 0$.*

Here we consider the following conditions:

$$\frac{\left(1-|z|^2\right)^\beta u(z)}{\left(1-|\varphi(z)|^2\right)^\alpha}\rho(\varphi(z),\psi(z)) \longrightarrow 0 \quad \text{as } |\varphi(z)| \longrightarrow 1, \tag{4.1}$$

$$\frac{\left(1-|z|^2\right)^\beta v(z)}{\left(1-|\psi(z)|^2\right)^\alpha}\rho(\varphi(z),\psi(z)) \longrightarrow 0 \quad \text{as } |\psi(z)| \longrightarrow 1, \tag{4.2}$$

$$\frac{\left(1-|z|^2\right)^\beta u(z)}{\left(1-|\varphi(z)|^2\right)^\alpha} - \frac{\left(1-|z|^2\right)^\beta v(z)}{\left(1-|\psi(z)|^2\right)^\alpha} \longrightarrow 0 \quad \text{as } |\varphi(z)| \longrightarrow 1,\ |\psi(z)| \longrightarrow 1. \tag{4.3}$$

In one complex variable case, Nieminen [16] characterized the compactness of $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ under the assumption that $W_{\varphi,u}$ and $W_{\psi,v}$ are bounded from H_α^∞ to H_β^∞. Here, a necessary and sufficient condition for the compactness of $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is completely obtained in the case of several variables without any assumption.

Theorem 4.2. *The operator $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is compact if and only if $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is bounded and the conditions* (4.1)–(4.3) *hold.*

Proof. First, we prove the sufficiency. Assume that $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is bounded and the conditions (4.1)–(4.3) hold. Then the conditions (3.17)–(3.19) hold by Theorem 3.4. For $\varepsilon > 0$, there exists $0 < r < 1$ such that

$$\frac{\left(1-|z|^2\right)^\beta |v(z)|}{\left(1-|\psi(z)|^2\right)^\alpha}\rho(\varphi(z),\psi(z)) \le \varepsilon \quad \text{when } |\psi(z)| > r, \tag{4.4}$$

$$\frac{\left(1-|z|^2\right)^\beta |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha}\rho(\varphi(z),\psi(z)) \le \varepsilon \quad \text{when } |\varphi(z)| > r, \tag{4.5}$$

$$\left|\frac{\left(1-|z|^2\right)^\beta u(z)}{\left(1-|\varphi(z)|^2\right)^\alpha} - \frac{\left(1-|z|^2\right)^\beta v(z)}{\left(1-|\psi(z)|^2\right)^\alpha}\right| \le \varepsilon \quad \text{when } |\varphi(z)| > r,\ |\psi(z)| > r. \tag{4.6}$$

To prove that $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is compact, we will apply Lemma 4.1. Suppose that $\{f_n\}$ is a sequence in H_α^∞ such that $\|f_n\|_{H_\alpha^\infty} \le 1$ and $f_n \to 0$ uniformly on compact subsets of B_N. We need only to show that $\|(W_{\varphi,u} - W_{\psi,v})f_n\|_{H_\beta^\infty} \to 0$. We write

$$\left(1-|z|^2\right)^\beta \left|f_n(\varphi(z))u(z) - f_n(\psi(z))v(z)\right| = |I_n(z) + J_n(z)|, \tag{4.7}$$

where

$$\begin{aligned} I_n(z) &= \left(1-\left|\varphi(z)\right|^2\right)^\alpha f_n(\varphi(z)) \left[\frac{\left(1-|z|^2\right)^\beta u(z)}{\left(1-\left|\varphi(z)\right|^2\right)^\alpha} - \frac{\left(1-|z|^2\right)^\beta v(z)}{\left(1-\left|\psi(z)\right|^2\right)^\alpha}\right], \\ J_n(z) &= \frac{\left(1-|z|^2\right)^\beta v(z)}{\left(1-\left|\psi(z)\right|^2\right)^\alpha} \left[\left(1-\left|\varphi(z)\right|^2\right)^\alpha f_n(\varphi(z)) - \left(1-\left|\psi(z)\right|^2\right)^\alpha f_n(\psi(z))\right]. \end{aligned} \tag{4.8}$$

We divide the argument into a few cases.

Case 1 ($|\psi(z)| \le r$ and $|\varphi(z)| \le r$). Clearly, by (3.19), $I_n(z)$ converges to 0 uniformly for all z with $|\varphi(z)| \le r$ and $|\psi(z)| \le r$. On the other hand, from Remark 3.3 and (3.18), we have

$$|J_n(z)| \le C \frac{\left(1-|z|^2\right)^\beta |v(z)|}{\left(1-\left|\psi(z)\right|^2\right)^\alpha} \rho(\varphi(z), \psi(z)) \sup_{z \in rB_N} \left(1-|z|^2\right)^\alpha \left|f_n(z)\right| \le \varepsilon \tag{4.9}$$

for sufficiently large n.

Case 2 ($|\psi(z)| > r$ and $|\varphi(z)| \le r$). As in the proof of Case 1, $I_n(z) \to 0$ uniformly. As regards $J_n(z)$, by Lemma 3.2 and inequality (4.4), we have $|J_n(z)| \le \varepsilon$ for sufficiently large n.

Case 3 ($|\psi(z)| > r$ and $|\varphi(z)| > r$). The inequality (4.6) implies that $|I_n(z)| \le \varepsilon$ for n sufficiently large. Meanwhile, $J_n(z) \to 0$ uniformly by Lemma 3.2 and inequality (4.4).

Case 4 ($|\psi(z)| \le r$ and $|\varphi(z)| > r$). Then we rewrite

$$\left(1-|z|^2\right)^\beta \left|f_n(\varphi(z))u(z) - f_n(\psi(z))v(z)\right| = |P_n(z) + Q_n(z)|, \tag{4.10}$$

where

$$P_n(z) = \left(1 - |\varphi(z)|^2\right)^{\alpha} f_n(\varphi(z)) \left[\frac{\left(1 - |z|^2\right)^{\beta} u(z)}{\left(1 - |\varphi(z)|^2\right)^{\alpha}} - \frac{\left(1 - |z|^2\right)^{\beta} v(z)}{\left(1 - |\psi(z)|^2\right)^{\alpha}} \right],$$
$$Q_n(z) = \frac{\left(1 - |z|^2\right)^{\beta} u(z)}{\left(1 - |\varphi(z)|^2\right)^{\alpha}} \left[\left(1 - |\varphi(z)|^2\right)^{\alpha} f_n(\varphi(z)) - \left(1 - |\psi(z)|^2\right)^{\alpha} f_n(\psi(z)) \right]. \tag{4.11}$$

The desired result follows by an argument analogous to that in the proof of Case 2. Thus, together with the above cases, we conclude

$$\left\| (W_{\varphi,u} - W_{\psi,v}) f_n \right\|_{H_\beta^\infty} = \sup_{z \in B_N} \left(1 - |z|^2\right)^{\beta} \left| f_n(\varphi(z)) u(z) - f_n(\psi(z)) v(z) \right| \le \varepsilon \tag{4.12}$$

for sufficiently large n.

Now we will prove the necessity. Suppose that $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is compact. Since the compactness implies the boundedness, we only need to show that (4.1)–(4.3) hold. Let $\{z_n\}$ be a sequence of points in B_N such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Consider the functions f_n defined by

$$f_n(z) = \frac{1 - |\varphi(z_n)|^2}{(1 - \langle z, \varphi(z_n) \rangle)^{\alpha+1}} \cdot \frac{\langle \varphi_{\psi(z_n)}(z), \varphi_{\psi(z_n)}(\varphi(z_n)) \rangle}{\left| \varphi_{\psi(z_n)}(\varphi(z_n)) \right|}. \tag{4.13}$$

Clearly, f_n converges to 0 uniformly on compact subsets of B_N as $n \to \infty$ and $f_n \in H_\alpha^\infty$ with $\|f_n\|_{H_\alpha^\infty} \le C$ for all n. Moreover,

$$f_n(\varphi(z_n)) = \frac{\rho(\varphi(z_n), \psi(z_n))}{\left(1 - |\varphi(z_n)|^2\right)^{\alpha}}, \quad f_n(\psi(z_n)) = 0. \tag{4.14}$$

By the compactness of $W_{\varphi,u} - W_{\psi,v}$ and Lemma 4.1, it follows $\|(W_{\varphi,u} - W_{\psi,v}) f_n\|_{H_\beta^\infty} \to 0$. On the other hand, we have

$$\begin{aligned} \left\| (W_{\varphi,u} - W_{\psi,v}) f_n \right\|_{H_\beta^\infty} &= \sup_{z \in B_N} \left(1 - |z|^2\right)^{\beta} \left| f_n(\varphi(z)) u(z) - f_n(\psi(z)) v(z) \right| \\ &\ge \left(1 - |z_n|^2\right)^{\beta} \left| f_n(\varphi(z_n)) u(z_n) - f_n(\psi(z_n)) v(z_n) \right| \\ &= \frac{\left(1 - |z_n|^2\right)^{\beta} \rho(\varphi(z_n), \psi(z_n)) |u(z_n)|}{\left(1 - |\varphi(z_n)|^2\right)^{\alpha}}, \end{aligned} \tag{4.15}$$

which shows that $(1-|z_n|^2)^{\beta}\rho(\varphi(z_n),\psi(z_n))|u(z_n)|/(1-|\varphi(z_n)|^2)^{\alpha} \to 0$ as $|\varphi(z_n)| \to 1$. This proves (4.1). The condition (4.2) also holds by similar arguments. Now we show that the condition (4.3) holds. Assume that $\{z_n\}$ is a sequence of points in B_N such that $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$ as $n \to \infty$. Define the function

$$g_n(z) = \frac{1-|\psi(z_n)|^2}{(1-\langle z,\psi(z_n)\rangle)^{\alpha+1}}. \tag{4.16}$$

It is easy to check that g_n converges to 0 uniformly on compact subsets of B_N as $n \to \infty$ and $g_n \in H_{\alpha}^{\infty}$ with $\|g_n\|_{H_{\alpha}^{\infty}} \le C$ for all n. Furthermore, $g_n(\psi(z_n)) = (1-|\psi(z_n)|^2)^{-\alpha}$. By Lemma 4.1 we have $\|(W_{\varphi,u}-W_{\psi,v})g_n\|_{H_{\beta}^{\infty}} \to 0$. Meanwhile, we have

$$\|(W_{\varphi,u}-W_{\psi,v})g_n\|_{H_{\beta}^{\infty}} \ge \left(1-|z_n|^2\right)^{\beta}\left|g_n(\varphi(z_n))u(z_n)-g_n(\psi(z_n))v(z_n)\right| = |I(z_n)+J(z_n)|, \tag{4.17}$$

where

$$\begin{aligned} I(z_n) &= \left(1-|\psi(z_n)|^2\right)^{\alpha} g_n(\psi(z_n))\left[\frac{\left(1-|z_n|^2\right)^{\beta}u(z_n)}{\left(1-|\varphi(z_n)|^2\right)^{\alpha}}-\frac{\left(1-|z_n|^2\right)^{\beta}v(z_n)}{\left(1-|\psi(z_n)|^2\right)^{\alpha}}\right] \\ &= \frac{\left(1-|z_n|^2\right)^{\beta}u(z_n)}{\left(1-|\varphi(z_n)|^2\right)^{\alpha}}-\frac{\left(1-|z_n|^2\right)^{\beta}v(z_n)}{\left(1-|\psi(z_n)|^2\right)^{\alpha}}, \\ J(z_n) &= \frac{\left(1-|z_n|^2\right)^{\beta}u(z_n)}{\left(1-|\varphi(z_n)|^2\right)^{\alpha}}\left[\left(1-|\varphi(z_n)|^2\right)^{\alpha}g_n(\varphi(z_n))-\left(1-|\psi(z_n)|^2\right)^{\alpha}g_n(\psi(z_n))\right]. \end{aligned} \tag{4.18}$$

By Lemma 3.2 and the condition (4.1) that has been proved, we conclude $J(z_n) \to 0$, which implies that $I(z_n) \to 0$ as $n \to \infty$. This shows that (4.3) is true. The whole proof is complete. □

Corollary 4.3. *The weighted composition operator $W_{\varphi,u} : H_{\alpha}^{\infty} \to H_{\beta}^{\infty}$ is compact if and only if $W_{\varphi,u}$ is bounded and*

$$\frac{\left(1-|z|^2\right)^{\beta}u(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha}} \longrightarrow 0 \quad \text{as } |\varphi(z)| \longrightarrow 1. \tag{4.19}$$

Corollary 4.4. *If $\beta \geq \alpha$, then the composition operator $C_\varphi : H_\alpha^\infty \to H_\beta^\infty$ is bounded. If $\beta > \alpha$, then $C_\varphi : H_\alpha^\infty \to H_\beta^\infty$ is compact.*

Proof. By the Schwarz-Pick lemma in the unit ball (see [23])

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} \leq \frac{1-|\varphi(0)|^2}{|1-\langle\varphi(z),\varphi(0)\rangle|^2} \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}, \tag{4.20}$$

we obtain

$$\frac{\left(1-|z|^2\right)^\beta}{\left(1-|\varphi(z)|^2\right)^\alpha} \leq C\left(1-|\varphi(z)|^2\right)^{\beta-\alpha}. \tag{4.21}$$

Therefore, the desired results follow from Corollaries 3.5 and 4.3. □

The following result appears in [1] when $u \equiv 1$ in one-dimensional case.

Corollary 4.5. *The operator $uC_\varphi - uC_\psi : H_\alpha^\infty \to H_\beta^\infty$ is compact if and only if $uC_\varphi - uC_\psi$ is bounded and the following conditions hold:*

$$\frac{\left(1-|z|^2\right)^\beta u(z)}{\left(1-|\varphi(z)|^2\right)^\alpha}\rho(\varphi(z),\psi(z)) \longrightarrow 0 \quad \text{as } |\varphi(z)| \longrightarrow 1, \tag{4.22}$$

$$\frac{\left(1-|z|^2\right)^\beta u(z)}{\left(1-|\psi(z)|^2\right)^\alpha}\rho(\varphi(z),\psi(z)) \longrightarrow 0 \quad \text{as } |\psi(z)| \longrightarrow 1. \tag{4.23}$$

Proof. This is the case in which $u \equiv v$ of Theorem 4.2. We need only show that the two conditions (4.22) and (4.23) imply that

$$\frac{\left(1-|z|^2\right)^\beta u(z)}{\left(1-|\varphi(z)|^2\right)^\alpha} - \frac{\left(1-|z|^2\right)^\beta u(z)}{\left(1-|\psi(z)|^2\right)^\alpha} \longrightarrow 0 \quad \text{as } |\varphi(z)| \longrightarrow 1,\ |\psi(z)| \longrightarrow \frac{a}{b}1. \tag{4.24}$$

Suppose that (4.24) is not true. Then there exist $\varepsilon_0 > 0$ and a sequence of points $\{z_n\} \subset B_N$ such that $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$ as $n \to \infty$ and

$$\left|\frac{\left(1-|z_n|^2\right)^\beta u(z_n)}{\left(1-|\varphi(z_n)|^2\right)^\alpha} - \frac{\left(1-|z_n|^2\right)^\beta u(z_n)}{\left(1-|\psi(z_n)|^2\right)^\alpha}\right| \geq \varepsilon_0. \tag{4.25}$$

We deduce that $\rho(\varphi(z_n), \psi(z_n)) \to 0$ as $n \to \infty$. If not, there exists a subsequence $\{z_{n_k}\}$ in $\{z_n\}$ such that $\rho(\varphi(z_{n_k}), \psi(z_{n_k})) \to a > 0$. By (4.22) and (4.23), we obtain

$$\frac{\left(1-|z_{n_k}|^2\right)^{\beta} u(z_{n_k})}{\left(1-\left|\varphi(z_{n_k})\right|^2\right)^{\alpha}} \longrightarrow 0, \qquad \frac{\left(1-|z_{n_k}|^2\right)^{\beta} u(z_{n_k})}{\left(1-\left|\psi(z_{n_k})\right|^2\right)^{\alpha}} \longrightarrow 0, \tag{4.26}$$

which contradicts (4.25). Thus, we may assume $\rho(\varphi(z_n), \psi(z_n)) \leq 1/2$ for all n. Therefore, by Lemma 3.1 and (4.22) we obtain

$$\begin{aligned}\left|\frac{\left(1-|z_n|^2\right)^{\beta} u(z_n)}{\left(1-\left|\varphi(z_n)\right|^2\right)^{\alpha}} - \frac{\left(1-|z_n|^2\right)^{\beta} u(z_n)}{\left(1-\left|\psi(z_n)\right|^2\right)^{\alpha}}\right| &= \frac{\left(1-|z_n|^2\right)^{\beta} |u(z_n)|}{\left(1-\left|\varphi(z_n)\right|^2\right)^{\alpha}}\left|\left(\frac{1-\left|\varphi(z_n)\right|^2}{1-\left|\psi(z_n)\right|^2}\right)^{\alpha} - 1\right| \\ &\leq C\frac{\left(1-|z_n|^2\right)^{\beta} |u(z_n)| \rho(\varphi(z_n), \psi(z_n))}{\left(1-\left|\varphi(z_n)\right|^2\right)^{\alpha}} \longrightarrow 0,\end{aligned} \tag{4.27}$$

which contradicts (4.25). The proof is complete. □

Corollary 4.6. *Let λ be a complex number and $\lambda \neq 0, 1$. Suppose the operator $C_\varphi - C_\psi$ is bounded from H_α^∞ to H_β^∞. Then $C_\varphi - \lambda C_\psi : H_\alpha^\infty \to H_\beta^\infty$ is compact if and only if C_φ and C_ψ are compact from H_α^∞ to H_β^∞.*

Proof. Assume that C_φ and C_ψ are compact from H_α^∞ to H_β^∞. It is clear that $C_\varphi - \lambda C_\psi : H_\alpha^\infty \to H_\beta^\infty$ is compact. Conversely, by Theorem 4.2, we can see that (4.22) and (4.23) hold for $u \equiv 1$ if $C_\varphi - \lambda C_\psi : H_\alpha^\infty \to H_\beta^\infty$ is compact. Thus, it follows that $C_\varphi - C_\psi$ is compact from Corollary 4.5. So, we conclude that $C_\psi = (1/(1-\lambda))[(C_\varphi - \lambda C_\psi) - (C_\varphi - C_\psi)]$ is also compact, which implies the compactness of $C_\varphi = (C_\varphi - C_\psi) + C_\psi$. This completes of the proof. □

Example 4.7. We give an example of noncompact composition operators such that their difference is compact. Choose two analytic functions $\varphi(z)$ and $\psi(z)$ in the unit disk, as previously in Example 3.7. Let $\beta \geq \alpha > 0$, $u(z) = v(z) = (1-z)^{\alpha-\beta}$. Clearly, when $0 < r < 1$,

$$\frac{(1-r^2)^{\beta} u(r)}{(1-\varphi^2(r))^{\alpha}} \longrightarrow 2^{\beta}, \qquad \frac{(1-r^2)^{\beta} v(r)}{(1-\psi^2(r))^{\alpha}} \longrightarrow 2^{\beta} \tag{4.28}$$

as $r \to 1$. Therefore, from Corollary 4.3 it follows that $W_{\varphi,u}$ and $W_{\psi,v}$ are not compact from H_α^∞ to H_β^∞. By the Schwarz-Pick lemma, we see that $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is bounded from Corollary 3.6. Note that $\rho(\varphi(z), \psi(z)) \to 0$ as $z \to 1$, and so we have

$$\begin{aligned}
&\frac{\left(1-|z|^2\right)^\beta |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha}\rho(\varphi(z),\psi(z)) \le C\rho(\varphi(z),\psi(z)) \longrightarrow 0 \quad \text{as } |\varphi(z)| \longrightarrow 1, \\
&\frac{\left(1-|z|^2\right)^\beta |v(z)|}{\left(1-|\psi(z)|^2\right)^\alpha}\rho(\varphi(z),\psi(z)) \le C\rho(\varphi(z),\psi(z)) \longrightarrow 0 \quad \text{as } |\psi(z)| \longrightarrow 1.
\end{aligned} \tag{4.29}$$

Thus, by Corollary 4.5 we conclude that $W_{\varphi,u} - W_{\psi,v} : H_\alpha^\infty \to H_\beta^\infty$ is compact.

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