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Research Article

Maximum Principles and Boundary Value Problems for First-Order Neutral Functional Differential Equations

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We obtain the maximum principles for the first-order neutral functional differential equation $(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = f(t)$, $t \in [0,\omega]$, where $A : C_{[0,\omega]} \rightarrow L^{\infty}_{[0,\omega]}$, $B : C_{[0,\omega]} \rightarrow L^{\infty}_{[0,\omega]}$, and $S : L^{\infty}_{[0,\omega]} \rightarrow L^{\infty}_{[0,\omega]}$ are linear continuous operators, A and B are positive operators, $C_{[0,\omega]}$ is the space of continuous functions, and $L^{\infty}_{[0,\omega]}$ is the space of essentially bounded functions defined on $[0,\omega]$. New tests on positivity of the Cauchy function and its derivative are proposed. Results on existence and uniqueness of solutions for various boundary value problems are obtained on the basis of the maximum principles.

1. Preliminary

This paper is devoted to the maximum principles and their applications for first order neutral functional differential equation.

$$(Mx)(t) \equiv x'(t) - \left(Sx'\right)(t) - (Ax)(t) + (Bx)(t) = f(t), \quad t \in [0,\omega], \tag{1.1}$$

where $A : C_{[0,\omega]} \rightarrow L^{\infty}_{[0,\omega]}$, $B : C_{[0,\omega]} \rightarrow L^{\infty}_{[0,\omega]}$, and $S : L^{\infty}{}_{[0,\omega]} \rightarrow L^{\infty}_{[0,\omega]}$ are linear continuous Volterra operators, the spectral radius $\rho(S)$ of the operator S is less than one, $C_{[0,\omega]}$ is the space of continuous functions, $L^{\infty}_{[0,\omega]}$ is the space of essentially bounded functions defined on $[0,\omega]$. We consider (1.1) with the following boundary condition:

$$lx = c, \tag{1.2}$$

where $l : D_{[0,\omega]} \to R^1$ is a linear bounded functional defined on the space of absolutely continuous functions $D_{[0,\omega]}$. By solutions of (1.1) we mean functions $x : [0,\omega] \to R^1$ from the space $D_{[0,\omega]}$ which satisfy this equation almost everywhere in $[0,\omega]$ and such that $x' \in L^{\infty}_{[0,\omega]}$.

We mean the Volterra operators according to the classical Tikhonov's definition.

Definition 1.1. An operator T is called Volterra if any two functions x_1 and x_2 coinciding on an interval $[0,a]$ have the equal images on $[0,a]$, that is, $(Tx_1)(t) = (Tx_2)(t)$ for $t \in [0,a]$ and for each $0 < a \leq \omega$.

Maximum principles present one of the classical parts of the qualitative theory of ordinary and partial differential equations [1]. Although in many cases, speaking about maximum principles, authors mean quite different definitions of maximum principles (such as e.g., corresponding inequalities, boundedness of solutions and maximum boundaries principles), there exists a deep internal connection between these definitions. This connection was discussed, for example, in the recent paper [2]. Main results of our paper are based on the maximum boundaries principle, that is, on the fact that the maximal and minimal values of the solution can be achieved only at the points 0 or ω. The boundaries maximum principle in the case of the zero operator S was considered in the recent papers [2, 3]. In this paper we develop the maximum boundaries principle for neutral functional differential equation (1.1) and on this basis we obtain results on existence and uniqueness of solutions of various boundary value problems.

Although several assertions were presented as the maximum principles for delay differential equations, they can be only interpreted in a corresponding sense as analogs of classical ones for ordinary differential equations and do not imply important corollaries, reached on the basis of the finite-dimensional fundamental systems. For example, results, associated with the maximum principles in contrast with the cases of ordinary and even partial differential equations, do not add so much in problems of existence and uniqueness for boundary value problems with delay differential equations. The Azbelev's definition of the homogeneous delay differential equation [4, 5] allowed his followers to consider questions of existence, uniqueness and positivity of solutions on this basis. The first results about the maximum principles for functional differential equations, which were based on the idea of the finite-dimensional fundamental system, were presented in the paper [2].

Neutral functional differential equations have their own history. Equations in the form

$$\left(x(t) - q(t)x(\tau(t))\right)' + \sum_{i=1}^{m} b_i(t)x(h_i(t)) = f(t), \quad t \in [0,+\infty), \tag{1.3}$$

were considered in the known books [6–8] (see also the bibliography therein), where existence and uniqueness of solutions and especially stability and oscillation results for these equations were obtained. There exist problems in applications whose models can be written in the form [9]

$$x'(t) - q(t)x'(\tau(t)) + \sum_{i=1}^{m} b_i(t)x(h_i(t)) = f(t), \quad t \in [0,+\infty). \tag{1.4}$$

This equation is a particular case of (1.1).

Let us note here that the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ in (1.1) can be, for example, of the following forms:

$$(Sy)(t) = \sum_{j=1}^{m} q_j(t) y(\tau_j(t)), \quad \text{where } \tau_j(t) \leq t,\ y(\tau_j(t)) = 0 \text{ if } \tau_j(t) < 0,\ t \in [0, +\infty), \tag{1.5}$$

or

$$(Sy)(t) = \sum_{i=1}^{n} \int_0^t k_i(t,s) y(s) ds, \quad t \in [0, +\infty), \tag{1.6}$$

where $q_j(t)$ are essentially bounded measurable functions, $\tau_j(t)$ are measurable functions for $j = 1, \ldots, m$, and $k_i(t,s)$ are summable with respect to s and measurable essentially bounded with respect to t for $i = 1, \ldots, n$. All linear combinations of operators (1.5) and (1.6) and their superpositions are also allowed.

The study of the neutral functional differential equations is essentially based on the questions of the action and estimates of the spectral radii of the operators in the spaces of discontinuous functions, for example, in the spaces of summable or essentially bounded functions. Operator (1.5), which is a linear combination of the internal superposition operators, is a key object in this topic. Properties of this operator were studied by Drakhlin [10, 11]. In order to achieve the action of operator (1.5) in the space of essentially bounded functions $L^{\infty}_{[0,\infty)}$, we have for each j to assume that $\operatorname{mes}\{t : \tau_j(t) = c\} = 0$ for every constant c. Let us suppose everywhere below that this condition is fulfilled. It is known that the spectral radius of the integral operator (1.6), considered on every finite interval $t \in [0, \omega]$, is equal to zero (see, e.g., [4]). Concerning the operator (1.5), we can note the sufficient conditions of the fact that its spectral radius $\rho(S)$ is less than one. Define the set $\kappa_{\varepsilon}^{j} = \{t \in [0, \infty) : \ t - \tau_j(t) \leq \varepsilon\}$ and $\kappa_{\varepsilon} = \bigcup_{j=1}^{m} \kappa_{\varepsilon}^{j}$. If there exists such ε that mes $(\kappa_{\varepsilon}) = 0$, then on every finite interval $t \in [0, \omega]$ the spectral radius of the operator S defined by the formula (1.5) for $t \in [0, \omega]$ is zero. In the case mes $(\kappa_{\varepsilon}) > 0$, the spectral radius of the operator S defined by (1.5) on the finite interval $t \in [0, \omega]$ is less than one if $\operatorname{ess\,sup}_{t \in \kappa_{\varepsilon}} \sum_{j=1}^{m} |q_j(t)| < 1$. The inequality $\operatorname{ess\,sup}_{t \in [0,\infty)} \sum_{j=1}^{m} |q_j(t)| < 1$ implies that the spectral radius $\rho(S)$ of the operator S considered on the semiaxis $t \in [0, +\infty)$ and defined by (1.5), satisfies the inequality $\rho(S) < 1$. Usually we will also assume that τ_j are nondecreasing functions for $j = 1, \ldots, m$.

Various results on existence and uniqueness of boundary value problems for this equation and its stability were obtained in [4], where also the basic results about the representation of solutions were presented. Note also in this connection the papers in [12–15], where results on nonoscillation and positivity of Green's functions for neutral functional differential equations were obtained.

It is known [4] that the general solution of (1.1) has the representation

$$x(t) = \int_0^t C(t,s) f(s) ds + X(t) x(0), \tag{1.7}$$

where the kernel $C(t,s)$ is called the Cauchy function, and $X(t)$ is the solution of the homogeneous equation $(Mx)(t) = 0,\ t \in [0, \omega]$, satisfying the condition $X(0) = 1$. On the basis of representation (1.7), the results about differential inequalities (under corresponding conditions, solutions of inequalities are greater or less than solution of the equation) can be formulated in the following form of positivity of the Cauchy function $C(t,s)$ and the solution $X(t)$. Results about comparison of solutions for delay differential equations solved with respect to the derivative (i.e., in the case when S is the zero operator) were obtained in [2, 15, 16], where assertions on existence and uniqueness of solutions of various boundary value problems for first order functional differential equations were obtained.

All results presented in the paper [15] and in the book [16] for equation with the difference of two positive operators are based on corresponding analogs of the following assertion [15]: *Let the operator A and the Cauchy function $C^{+}(t,s)$ of equation*

$$(M^{+}x)(t) \equiv x'(t) - (Sx')(t) + (Bx)(t) = f(t), \quad t \in [0,\omega], \tag{1.8}$$

be positive for $0 \le s \le t \le \omega$, then the Cauchy function $C(t,s)$ of (1.1) *is also positive for $0 \le s \le t \le \omega$.*

This result was extent on various boundary value problems in [16] in the form: *Let the operator A and Green's function $G^{+}(t,s)$ of problem* (1.8), (1.2) *be positive in the square $(0,\omega)\times(0,\omega)$ and the spectral radius of the operator $\Omega : C_{[0,\omega]} \to C_{[0,\omega]}$ defined by the equality*

$$(\Omega x)(t) = \int_0^{\omega} G^{+}(t,s)(Ax)(s)ds, \tag{1.9}$$

be less than one, then Green's function $G^{+}(t,s)$ of problem (1.1), (1.2) *is positive in the square $(0,\omega)\times(0,\omega)$.*

The scheme of the proof was based on the reduction of problem (1.8), (1.2) with $c = 0$, to the equivalent integral equation $x(t) = (\Omega x)(t) + \varphi(t)$, where $\varphi(t) = \int_0^{\omega} G^{+}(t,s)f(s)ds$. It is clear that the operator Ω is positive if the operator A and the Green's function $G^{+}(t,s)$ are positive. If the spectral radius $\rho(\Omega)$ or, more roughly, the norm $\|\Omega\|$ of the operator $\Omega : C_{[0,\omega]} \to C_{[0,\omega]}$ are less than one, then there exists the inverse bounded operator $(I-\Omega)^{-1} = I + \Omega + \Omega^2 + \cdots : C_{[0,\omega]} \to C_{[0,\omega]}$, which is of course positive. This implies the positivity of the Green's function $G(t,s)$ of problem (1.1), (1.2). In order to get the inequality $\rho(\Omega) < 1$, the classical theorems about estimates of the spectral radius of the operator $\Omega : C_{[0,\omega]} \to C_{[0,\omega]}$ [17] can be used. All these theorems are based on a corresponding "smallness" of the operator Ω, which is actually close to the condition $\|\Omega\| < 1$. In order to get positivity of $C^{+}(t,s)$ and $G^{+}(t,s)$ a corresponding smallness of $\|B\|$ was assumed.

Below we present another approach to this problem starting with the following question: how can one conclude about positivity of Green's function $G(t,s)$ in the cases when the spectral radius satisfies the opposite inequality $\rho(\Omega) \ge 1$ or Green's function $G^{+}(t,s)$ changes its sign? Note, that in the case, when the operator $S : L^{\infty}{}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ is positive and its spectral radius is less than one, the positivity of the Cauchy function $C^{+}(t,s)$ of (1.8) follows from the nonoscillation of the homogeneous equation $M^{+}x = 0$, and in the case of the zero operator S, the positivity of $C^{+}(t,s)$ is even equivalent to nonoscillation [15]. This allows us to formulate our question also in the form: how can we make the conclusions about nonoscillation of the equation $Mx = 0$ or about positivity of the Cauchy function $C(t,s)$ of (1.1) without assumption about nonoscillation of the equation $M^{+}x = 0$? In this paper we obtain assertions allowing to make such conclusions. Our assertions are based on the assumption that the operator A is a dominant among two operators A and B.

We assume that the spectral radius of the operator $S : L^{\infty}{}_{[0,\omega]} \to L^{\infty}_{[0,\omega]}$ is less than one. In this case we can rewrite (1.1) in the equivalent form

$$(Nx)(t) \equiv x'(t) - (I-S)^{-1}(A-B)x(t) = (I-S)^{-1}f(t), \quad t \in [0,\omega], \tag{1.10}$$

and its general solution can be written in the form

$$x(t) = \int_0^t C_0(t,s)(I-S)^{-1}f(s)ds + X(t)x(0), \tag{1.11}$$

where $C_0(t,s)$ is the Cauchy function of (1.10) [4]. Note that this approach in the study of neutral equations was first used in the paper [14]. Below in the paper we use the fact that the Cauchy function $C_0(t,s)$ coincides with the fundamental function of (1.10). It is also clear that

$$\int_0^t C(t,s)f(s)ds = \int_0^t C_0(t,s)(I-S)^{-1}f(s)ds. \tag{1.12}$$

2. About Maximum Boundaries Principles in the Case of Difference of Two Positive Volterra Operators

In this paragraph we consider the equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = f(t), \quad t \in [0,+\infty), \tag{2.1}$$

where $A : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$, $B : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ and $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ are positive linear continuous Volterra operators and the spectral radius $\rho(S)$ of the operator S is less than one.

These operators A and B are u-bounded operators and according to [18], they can be written in the form of the Stieltjes integrals

$$(Ax)(t) = \int_0^t x(\xi)d_{\xi}a(t,\xi), \quad (Bx)(t) = \int_0^t x(\xi)d_{\xi}b(t,\xi), \quad t \in [0,+\infty), \tag{2.2}$$

respectively, where the functions $a(\cdot,\xi)$ and $b(\cdot,\xi) : [0,\omega] \to R^1$ are measurable for $\xi \in [0,\omega]$, $a(t,\cdot)$ and $b(t,\cdot) : [0,\omega] \to R^1$ has the bounded variation for almost all $t \in [0,\omega]$ and $\bigvee_{\xi=0}^{t} a(t,\xi)$, $\bigvee_{\xi=0}^{t} b(t,\xi)$ are essentially bounded.

Consider for convenience (2.1) in the following form:

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - \int_0^t x(\xi)d_{\xi}a(t,\xi) + \int_0^t x(\xi)d_{\xi}b(t,\xi) = f(t), \quad t \in [0,+\infty). \tag{2.3}$$

We can study properties of solution of (2.3) on each finite interval $[0,\omega]$ since every solution $x(t)$ of (2.1) satisfies also the equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - \int_0^t x(\xi)d_\xi a(t,\xi) + \int_0^t x(\xi)d_\xi b(t,\xi) = f(t), \quad t \in [0,\omega]. \tag{2.4}$$

Consider also the homogeneous equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = 0, \quad t \in [0,+\infty), \tag{2.5}$$

and the following auxiliary equations (which are analogs of the so-called s-trancated equations defined first in [5])

$$(M_s x)(t) \equiv x'(t) - (S_s x')(t) - (A_s x)(t) + (B_s x)(t) = 0, \quad t \in [s,+\infty), \ s \geq 0, \tag{2.6}$$

where the operators $A_s : C_{[s,\infty)} \to L^\infty_{[s,\infty)}$ and $B_s : C_{[s,\infty)} \to L^\infty_{[s,\infty)}$ are defined by the formulas

$$(A_s x)(t) = \int_s^t x(\xi)d_\xi a(t,\xi), \quad (B_s x)(t) = \int_s^t x(\xi)d_\xi b(t,\xi), \quad t \in [s,+\infty), \tag{2.7}$$

and the operator $S_s : L^\infty_{[s,\infty)} \to L^\infty_{[s,\infty)}$ is defined by the equality $(S_s y_s)(t) = (Sy)(t)$, where $y_s(t) = y(t)$ for $t \geq s$ and $y(t) = 0$ for $t < s$. We have

$$(S_s y)(t) = \sum_{j=1}^m q_j(t) y(\tau_j(t)), \quad \text{where } \tau_j(t) \leq t, \ \ y(\tau_j(t)) = 0 \text{ if } \tau_j(t) < s, \ t \in [s,+\infty), \tag{2.8}$$

for the operator described by formula (1.5), and

$$(S_s y)(t) = \sum_{i=1}^n \int_s^t k_i(t,s) y(s) ds, \quad t \in [s,+\infty), \tag{2.9}$$

for the operator described by formula (1.6). It is clear that $\rho(S_s) < 1$ for every $s \in [0,+\infty)$ if $\rho(S) < 1$.

Functions from the space $D_{[s,\infty)}$ of absolutely continuous functions $x : [s,+\infty) \to R^1$, $x' \in L^\infty_{[s,\infty)}$, satisfy (2.6) almost everywhere in $[s,+\infty)$, we call solutions of this equation.

It was noted above that the general solution of (2.1) has the representation [4]

$$x(t) = \int_0^t C(t,s) f(s) ds + X(t) x(0), \tag{2.10}$$

where the function $C(t,s)$ is the Cauchy function of (2.1).We use also formula (1.12) connecting $C(t,s)$ and the Cauchy function $C_0(t,s)$ of (1.10). Note that $C_0(t,s)$ is a solution of (2.6) as a function of the first argument t for every fixed s and satisfies also the equation

$$(N_s x)(t) \equiv x'(t) + (I - S_s)^{-1}[-(A_s x)(t) + (B_s x)](t) = 0, \quad t \in [s, +\infty). \tag{2.11}$$

Let us formulate our results about positivity of the Cauchy function $C(t,s)$ and the maximum boundaries principle in the case when the condition $C^+(t,s) > 0$ is not assumed.

Consider the equation

$$x'(t) - (Sx')(t) - \int_{g_1(t)}^{g_2(t)} x(\xi) d_\xi a(t,\xi) + \int_{h_1(t)}^{h_2(t)} x(\xi) d_\xi b(t,\xi) = f(t), \quad t \in [0, +\infty). \tag{2.12}$$

Theorem 2.1. *Let $S : L^\infty_{[0,\infty)} \to L^\infty_{[0,\infty)}$ be a positive Volterra operator, $\rho(S) < 1$, $0 \le h_1(t) \le h_2(t) \le g_1(t) \le g_2(t) \le t$, let the functions $a(t,\xi)$ and $b(t,\xi)$ be nondecreasing functions with respect to ξ for almost every t, and let the following inequality be fulfilled:*

$$\bigvee_{\xi=h_1(t)}^{h_2(t)} b(t,\xi) \le \bigvee_{\xi=g_1(t)}^{g_2(t)} a(t,\xi), \quad t \in [0, +\infty), \tag{2.13}$$

then the Cauchy function $C(t,s)$ of (2.12) *and its derivative satisfy the inequalities $C(t,s) > 0$, $C'_t(t,s) \ge 0$ for $0 \le s \le t < \infty$.*

Consider now the equation

$$x'(t) - (Sx')(t) + \sum_{i=1}^{m} \left\{ -\int_{g_{1i}(t)}^{g_{2i}(t)} x(\xi) d_\xi a_i(t,\xi) + \int_{h_{1i}(t)}^{h_{2i}(t)} x(\xi) d_\xi b_i(t,\xi) \right\} = f(t), \quad t \in [0, +\infty). \tag{2.14}$$

Theorem 2.2. *Let $S : L^\infty_{[0,\infty)} \to L^\infty_{[0,\infty)}$ be a positive Volterra operator, $\rho(S) < 1$, $0 \le h_{1i}(t) \le h_{2i}(t) \le g_{1i}(t) \le g_{2i}(t) \le t$, let the functions $a_i(t,\xi)$ and $b_i(t,\xi)$ be nondecreasing functions with respect to ξ for almost every t and let the following inequalities*

$$\bigvee_{\xi=h_{1i}(t)}^{h_{2i}(t)} b_i(t,\xi) \le \bigvee_{\xi=g_{1i}(t)}^{g_{2i}(t)} a_i(t,\xi), \quad t \in [0, +\infty), \ i = 1, \ldots, m, \tag{2.15}$$

be fulfilled, then the Cauchy function $C(t,s)$ of (2.14) *and its derivative $C'_t(t,s)$ satisfy the inequalities $C(t,s) > 0$ and $C'_t(t,s) \ge 0$ for $0 \le s \le t < \infty$.*

Consider the delay equation

$$x'(t) - (Sx')(t) - \sum_{i=1}^{m} a_i(t) x(g_i(t)) + \sum_{i=1}^{m} b_i(t) x(h_i(t)) = f(t), \quad t \in [0, +\infty), \tag{2.16}$$

where

$$x(\xi) = 0 \quad \text{for } \xi < 0, \tag{2.17}$$

with $a_i, b_i \in L^{\infty}_{[0,\infty)}$ and measurable functions g_i and h_i $(i = 1, \ldots, m)$. This equation is a particular case of (2.14).

Theorem 2.3. *Let $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ be a positive Volterra operator, $\rho(S) < 1$, $h_i(t) \le g_i(t) \le t$ and $0 \le b_i(t) \le a_i(t)$ for $t \in [0, +\infty)$, $i = 1, \ldots, m$, then the Cauchy function $C(t,s)$ of* (2.16) *and its derivative $C'_t(t,s)$ satisfy the inequalities $C(t,s) > 0$ and $C'_t(t,s) \ge 0$ for $0 \le s \le t < \infty$.*

Example 2.4. The inequality on deviating argument $h_i(t) \le g_i(t)$ is essential as the following equation

$$x'(t) - x(0) + b_1(t)x(h_1(t)) = 0, \quad t \in [0, +\infty), \tag{2.18}$$

demonstrates. This is a particular case of (2.16), where S is the zero operator, $m = 1$, $g_1(t) \equiv 0$, $a_1(t) \equiv 1$,

$$h_1(t) = \begin{cases} 0, & 0 \le t < 2, \\ 2, & 2 \le t, \end{cases} \qquad b_1(t) = \begin{cases} 0, & 0 \le t < 2, \\ \dfrac{1}{2}, & 2 \le t. \end{cases} \tag{2.19}$$

The function

$$x(t) \equiv C(t,0) = \begin{cases} t+1, & 0 \le t < 2, \\ 4 - \dfrac{1}{2}t, & 2 \le t, \end{cases} \tag{2.20}$$

is a nontrivial solution of (2.18) and its Cauchy function $C(t,s)$ satisfies the equality

$$C(t,s) = \begin{cases} 1, & 0 < s \le 2,\ 0 \le t < 2, \\ 1 - \dfrac{1}{2}(t-2), & 0 < s \le 2,\ 2 \le t, \end{cases} \tag{2.21}$$

that is, $C(t,0) > 0$ for $0 \le t < 8$, $C(t,0) < 0$ for $t > 8$, $C(t,s) > 0$ for $0 < s \le 2$, $0 \le t < 4$, $C(t,s) < 0$ for $0 < s \le 2$, $t > 4$. We see that each interval $[0,\omega]$, where $\omega < 8$, is a nonoscillation one for this equation, but $C(t,s)$ changes its sign for $0 < s \le 2$, $4 < t$.

Consider the integrodifferential equation

$$x'(t) - (Sx')(t) - \sum_{i=1}^{m} \int_{g_{1i}(t)}^{g_{2i}(t)} m_i(t,\xi)x(\xi)d\xi + \sum_{i=1}^{m} \int_{h_{1i}(t)}^{h_{2i}(t)} k_i(t,\xi)x(\xi)d\xi = f(t), \quad t \in [0,+\infty),$$
$$x(\xi) = 0 \quad \text{for } \xi < 0, \tag{2.22}$$

as a particular case of (2.14).

Let us define the functions $h_{ji}^0(t) = \max\{0, h_{ji}(t)\}$ and $g_{ji}^0(t) = \max\{0, g_{ji}(t)\}$, where $j = 1, 2$.

Theorem 2.5. *Let $S : L^\infty_{[0,\infty)} \to L^\infty_{[0,\infty)}$ be a positive Volterra operator, $\rho(S) < 1$, $h_{1i}(t) \leq h_{2i}(t) \leq g_{1i}(t) \leq g_{2i}(t) \leq t$, $k_i(t,\xi) \geq 0$, $m_i(t,\xi) \geq 0$ for $t, \xi \in [0,+\infty)$, $i = 1,\ldots,m$, and the following inequalities be fulfilled:*

$$\int_{h_{1i}^0(t)}^{h_{2i}^0(t)} k_i(t,\xi)d\xi \leq \int_{g_{1i}^0(t)}^{g_{2i}^0(t)} m_i(t,\xi)d\xi, \quad t \in [0,+\infty),\ i = 1,\ldots,m, \tag{2.23}$$

then the Cauchy function of (2.22) *and its derivative $C_t'(t,s)$ satisfy the inequalities $C(t,s) > 0$ and $C_t'(t,s) \geq 0$ for $0 \leq s \leq t < \infty$.*

Consider the equation

$$x'(t) - (Sx')(t) - \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi + b(t)x(h(t)) = f(t), \quad t \in [0,+\infty).$$
$$x(\xi) = 0 \quad \text{for } \xi < 0. \tag{2.24}$$

Let us define

$$\chi(t,s) = \begin{cases} 0, & t < s, \\ 1, & t \geq s. \end{cases} \tag{2.25}$$

In the following assertion the integral term is dominant.

Theorem 2.6. *Let $S : L^\infty_{[0,\infty)} \to L^\infty_{[0,\infty)}$ be a positive Volterra operator, $\rho(S) < 1$, $h(t) \leq g_1(t) \leq g_2(t) \leq t$, $m(t,\xi) \geq 0$, $b(t) \geq 0$ for $t, \xi \in [0,+\infty)$ and the following inequalities be fulfilled:*

$$b(t)\chi(h(t),0) \leq \int_{g_1^0(t)}^{g_2^0(t)} m(t,\xi)d\xi, \quad t \in [0,+\infty), \tag{2.26}$$

then the Cauchy function of (2.24) *and its derivative $C_t'(t,s)$ satisfy the inequalities $C(t,s) > 0$ and $C_t'(t,s) \geq 0$ for $0 \leq s \leq t < \infty$.*

Consider the equation

$$x'(t) - (Sx')(t) - a(t)x(g(t)) + \int_{h_1(t)}^{h_2(t)} k(t,\xi)x(\xi)d\xi = f(t), \quad t \in [0,+\infty),$$
$$x(\xi) = 0 \quad \text{for } \xi < 0. \tag{2.27}$$

In the following assertion the term $a(t)x(g(t))$ is a dominant one.

Theorem 2.7. *Let $S : L^\infty_{[0,\infty)} \to L^\infty_{[0,\infty)}$ be a positive Volterra operator, $\rho(S) < 1, k(t,\xi) \geq 0,\ h_1(t) \leq h_2(t) \leq g(t) \leq t,\ a(t) \geq 0$ for $t,\xi \in [0,+\infty)$ and the following inequalities be fulfilled:*

$$\int_{h_1^0(t)}^{h_2^0(t)} k(t,\xi)d\xi \leq a(t), \quad t \in [0,+\infty), \tag{2.28}$$

then the Cauchy function of (2.27) *and its derivative $C'_t(t,s)$ satisfy the inequalities $C(t,s) > 0$ and $C'_t(t,s) \geq 0$ for $0 \leq s \leq t < \infty$.*

Consider now the equation

$$x'(t) - (Sx')(t) - a(t)x(g(t)) + b(t)x(h(t)) - \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi + \int_{h_1(t)}^{h_2(t)} k(t,\xi)x(\xi)d\xi$$
$$= f(t), \quad t \in [0,+\infty),$$
$$x(\xi) = 0 \quad \text{for } \xi < 0. \tag{2.29}$$

In the following assertion we do not assume inequalities $k(t,\xi) \leq m(t,\xi)$ or $b(t) \leq a(t)$. Here the sum $a(t)x(g(t)) + \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi$ is a dominant term.

Theorem 2.8. *Let $S : L^\infty_{[0,\infty)} \to L^\infty_{[0,\infty)}$ be a positive Volterra operator, $\rho(S) < 1,\ h(t) \leq g_1(t) \leq g_2(t) \leq t, h_1(t) \leq h_2(t) \leq g(t) \leq t,\ k(t,\xi) \geq 0,\ m(t,\xi) \geq 0,\ a(t) \geq 0,\ b(t) \geq 0$ for $t,\xi \in [0,+\infty)$ and let the following inequalities be fulfilled:*

$$b(t)\chi(h(t),0) \leq \int_{g_1^0(t)}^{g_2^0(t)} m(t,\xi)d\xi,$$
$$\int_{h_1^0(t)}^{h_2^0(t)} k(t,\xi)d\xi \leq a(t), \tag{2.30}$$

for $t \in [0,+\infty)$, then the Cauchy function $C(t,s)$ of (2.29) *and its derivative $C'_t(t,s)$ satisfy the inequalities $C(t,s) > 0$ and $C'_t(t,s) \geq 0$ for $0 \leq s \leq t < +\infty$.*

The proofs of Theorems 2.1–2.8 are based on the following auxiliary lemmas.

Lemma 2.9 ([2]). *Let S be the zero operator. Then the following two assertions are equivalent:*

(1) *for every positive s there exists a positive function $v_s \in D_{[s,\infty)}$ such that $(M_s v_s)(t) \leq 0$ for $t \in [s, +\infty)$,*

(2) *the Cauchy function $C(t,s)$ of (2.1) is positive for $0 \leq s \leq t < +\infty$.*

Lemma 2.10. *Let $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ be a positive Volterra operator, $\rho(S) < 1$ and for every positive s there exist a positive function $v_s \in D_{[s,\infty)}$ such that $(M_s v_s)(t) \leq 0$ for $t \in [s, +\infty)$, then the Cauchy function $C_0(t,s)$ of (1.10) is positive for $0 \leq s \leq t < +\infty$.*

Proof. Using the condition $\rho(S) < 1$, we can write (1.1) in the form (1.10). The positivity of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ implies that the inequality $(I - S)^{-1} f \geq 0$ follows from the inequality $f \geq 0$. It is clear that the inequality $(M_s v_s)(t) \leq 0$ *for* $t \in [s, +\infty)$ implies the inequality $(N_s v_s)(t) \leq 0$ *for* $t \in [s, +\infty)$. Now according to Lemma 2.9, we obtain that $C_0(t,s) > 0$ for $0 \leq s \leq t < +\infty$. □

Lemma 2.11. *Let $A : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$, $B : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ and $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ be positive Volterra operators, $\rho(S) < 1$ and for every $s \in [0, +\infty)$ the inequality*

$$(A_s 1)(t) \geq (B_s 1)(t) \quad \text{for } t \in [s, +\infty), \tag{2.31}$$

be fulfilled. Then $C_0(t,s) > 0$ for $0 \leq s \leq t < +\infty$.

In order to prove Lemma 2.11 we set $v_s(t) \equiv 1,\ t \in [s, +\infty)$ for every $s \in [0, +\infty)$ in the assertion 1 of Lemma 2.10.

Remark 2.12. The condition

$$(A1)(t) \geq (B1)(t) \quad \text{for } t \in [0, \omega] \tag{2.32}$$

cannot be set instead of condition (2.31) in Lemma 2.11 as Example 2.4 demonstrates. It is clear that this inequality is fulfilled for (2.18), where $(Ax)(t) = x(0)$ and $(Bx)(t) = b_1(t)x(h_1(t))$, the functions h_1 and b_1 are defined by formula (2.19) respectively. The operator A_s is the zero one for every $s > 0$ and consequently $(A_s 1)(t) = 0$ for $t \in [s, +\infty)$, $(B_2 1)(t) = 1/2$ for $t \in [2, +\infty)$ and condition (2.31) is not fulfilled for $s = 2$.

Lemma 2.13. *Let $A : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$, $B : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ and $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ be positive Volterra operators, $\rho(S) < 1$ and inequality (2.31) be fulfilled for every $s \in [0, +\infty)$. Then $C(t,s) > 0$, $(\partial/\partial t)C_0(t,s) \geq 0$ and $(\partial/\partial t)C(t,s) \geq 0$ for $0 \leq s \leq t < \infty$.*

Proof. According to Lemma 2.11, we have $C_0(t,s) > 0$ for $0 \leq s \leq t < \infty$. The positivity of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ and formula (1.12) implies now that $C(t,s) \geq C_0(t,s) > 0$ for $0 \leq s \leq t < \infty$.

Now let us prove that $(\partial/\partial t)C_0(t,s) \geq 0$ for $0 \leq s \leq t < \infty$. We use the fact that the function $C_0(t,s)$, as a function of the argument t for each fixed s, satisfies (2.11) and the condition $x(s) = 1$.

The following integral equation

$$x(t) = \int_s^t \sum_{n=0}^{\infty} (S_s^n (A_s - B_s)x)(\xi) d\xi + 1 \tag{2.33}$$

is equivalent to (2.11) with the condition $x(s) = 1$.

The spectral radius of the operator $T : C_{[s,\omega]} \to C_{[s,\omega]}$, defined by the equality

$$(Tx)(t) = \int_s^t \sum_{n=0}^{\infty} (S_s^n (A_s - B_s)x)(\xi) d\xi, \quad t \in [s, \omega], \tag{2.34}$$

is zero for every positive number ω [4]. Let us build the sequence

$$x_{m+1}(t) = \int_s^t \sum_{n=0}^{\infty} (S_s^n (A_s - B_s)x_m)(\xi) d\xi + 1, \tag{2.35}$$

where the iterations start with the constant $x_0(t) \equiv 1$ for $t \in [s, \omega]$.

The sequence of functions $x_m(t)$ converges in the space $C_{[s,\omega]}$ to the unique solution $x(t)$ of (2.33) on the interval $[s, \omega]$. It is clear that this solution is absolutely continuous. It follows from the fact that all operators are Volterra ones, that the solution $y(t)$ of (2.11) with the initial condition $y(s) = 1$ and the solution $x(t)$ of (2.33) coincide for $t \in [s, \omega]$.

Positivity of the operator S, the inequalities $\rho(S) < 1$ and (2.31) imply nonnegativity of the derivatives

$$x'_{m+1}(t) = \sum_{n=0}^{\infty} (S_s^n (A_s - B_s)x_m)(t), \quad t \in [s, \omega]. \tag{2.36}$$

Let us prove now that the sequence x_m of nondecreasing functions converges to the nondecreasing function x. Assume in the contrary that there exist two points $t_1 < t_2$, such that $x(t_1) > x(t_2)$. Let us choose $\varepsilon < (x(t_1) - x(t_2))/2$. There exists a number $N_1(\varepsilon)$ such that $|x(t_1) - x_m(t_1)| < \varepsilon$ for $m \geq N_1(\varepsilon)$, and there exists $N_2(\varepsilon)$ such that $|x_m(t_2) - x(t_2)| < \varepsilon$ for $m \geq N_2(\varepsilon)$. It is clear that $x_m(t_1) > x_m(t_2)$ for $m \geq \max\{N_1(\varepsilon),\ N_2(\varepsilon)\}$. This contradicts to the fact that $x_m(t)$ nondecreases.

We have proven that for every positive ω, the solution x of (2.33) is nondecreasing for $t \in [s, \omega]$. It means that the solution x of (2.11) is nondecreasing for every $t \in [s, +\infty)$ and consequently $(\partial/\partial t)C_0(t, s) \geq 0$ for $0 \leq s \leq t < \infty$.

Positivity of the operator S, the inequality $\rho(S) < 1$ and formula (1.12) imply now the inequality $(\partial/\partial t)C(t, s) \geq 0$ for $0 \leq s \leq t < \infty$. □

To prove Theorems 2.1–2.8 it is sufficient to note that the conditions of each theorem imply inequality (2.31).

Remark 2.14. The space of solutions of the homogeneous equation

$$(Mx)(t) \equiv x'(t) - (Sx')(t) - (Ax)(t) + (Bx)(t) = 0, \quad t \in [0, +\infty), \tag{2.37}$$

in the case $\rho(S) < 1$ is one dimensional. All nontrivial solutions of (2.37) are proportional to $C_0(t,0)$. One of the assertions of Lemma 2.11 claims that $(\partial/\partial t)C_0(t,0) \geq 0$ for $0 \leq t < \infty$, that is, all nontrivial positive solutions do not decrease. This allows us to consider Lemma 2.13 as the maximum boundaries principle for (2.1). Theorems 2.1–2.8 present the sufficient conditions of this maximum principle for the equations (2.12), (2.14), (2.16), (2.22), (2.24), (2.27) and (2.29) respectively.

Remark 2.15. The condition $\rho(S) < 1$ about the spectral radius of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ is essential as the following example demonstrates.

Example 2.16. Consider the equation

$$x'(t) - x'\left(\frac{t}{2}\right) = f(t), \quad t \in [0,+\infty). \tag{2.38}$$

The spectral radius of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$, defined by the formula $(Sy)(t) = y(t/2)$, is equal to one. All other conditions of Theorems 2.1–2.8 for the zero operators A and B are fulfilled. The space of solutions of this neutral homogeneous equation is infinitely dimensional. Every linear functions $x = 1 - ct$ satisfy the homogeneous equation

$$x'(t) - x'\left(\frac{t}{2}\right) = 0, \quad t \in [0,+\infty). \tag{2.39}$$

If $c > 0$, the solutions x are decreasing.

3. About Nondecreasing Solutions of Neutral Equations

Let us consider the equation

$$(Mx)(t) \equiv x'(t) + (Sx')(t) - (Ax)(t) + (Bx)(t) = f(t), \quad t \in [0,+\infty), \tag{3.1}$$

where $A : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ and $B : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ are positive linear continuous Volterra operators, and the spectral radius $\rho(S)$ of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ is less than one. If the operator S is positive, then $(I+S)^{-1} = I - S + S^2 - S^3 + \cdots$ is not generally speaking a positive operator. This is the main difficulty in the study of positivity of the solution x and its derivative x'. All previous results about the positivity of solutions for this equation assumed the negativity of the operator S (see, e.g., [12, 13, 15]). In this paragraph we propose results about positivity of solutions in the case of the positive operator S defined by the equality

$$(Sy)(t) = q(t)y(r(t)), \quad \text{where } r(t) \leq t, \ y(r(t)) = 0 \text{ if } r(t) < 0, \ t \in [0,+\infty), \tag{3.2}$$

Let us start with the equation

$$\begin{gathered} x'(t) + q(t)x'(r(t)) - a(t)x(g(t)) + b(t)x(h(t)) = f(t), \quad t \in [0,+\infty), \\ x(\xi) = x'(\xi) = 0 \quad \text{for } \xi < 0. \end{gathered} \tag{3.3}$$

Theorem 3.1. *Assume that the spectral radius* $\rho(S)$ *of the operator* $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ *defined by equality* (3.2) *is less than one,* $r(t)$*,* $h(t)$ *and* $g(t)$ *are nondecreasing functions, and the coefficients satisfy the inequalities* $a(t) \geq 0$*,* $b(t) \geq 0$*,* $q(t) \geq 0$*,* $g(t) \geq h(t)$ *and* $a(t) - b(t)\chi(h(t),0) - q(t)a(r(t))\chi(g(r(t)),0) \geq 0$ *for* $t \in [0,+\infty)$*, then the solution* x *of the equation*

$$x'(t) + q(t)x'(r(t)) - a(t)x(g(t)) + b(t)x(h(t)) = 0, \quad t \in [0,+\infty),$$
$$x(\xi) = x'(\xi) = 0 \quad \textit{for } \xi < 0, \tag{3.4}$$

such that $x(0) > 0$*, satisfies the inequalities* $x(t) \geq 0$*,* $x'(t) \geq 0$ *for* $t \in [0,+\infty)$ *and in the case, when there exists* ε *such that* $0 \leq q(t) \leq \varepsilon < 1$*, the solution* x *of* (3.3) *is nonnegative and nondecreasing for every positive nondecreasing function* $f \in L^{\infty}_{[0,\infty)}$*.*

Consider the equation

$$x'(t) + q(t)x'(r(t)) + \sum_{i=1}^{m}\left\{-\int_{g_{1i}(t)}^{g_{2i}(t)} x(\xi)d_{\xi}a_i(t,\xi) + \int_{h_{1i}(t)}^{h_{2i}(t)} x(\xi)d_{\xi}b_i(t,\xi)\right\} = f(t), \quad t \in [0,+\infty),$$
$$x'(\xi) = 0 \quad \text{for } \xi < 0. \tag{3.5}$$

Theorem 3.2. *Let the spectral radius* $\rho(S)$ *of the operator* $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ *defined by equality* (3.2) *be less than one,* $r(t)$ *be a nondecreasing function and the functions* $a_i(t,\xi)$ *and* $b_i(t,\xi)$ *be nondecreasing functions with respect to* ξ*,* $0 \leq h_{1i}(t) \leq h_{2i}(t) \leq g_{1i}(t) \leq g_{2i}(t) \leq t$*,* $q(t) \geq 0$*, and the following inequalities be fulfilled*

$$\bigvee_{\xi=h_{1i}(t)}^{h_{2i}(t)} b_i(t,\xi) + q(t)\chi(r(t),0) \bigvee_{\xi=g_{1i}(r(t))}^{g_{2i}(r(t))} a_i(r(t),\xi) \leq \bigvee_{\xi=g_{1i}(t)}^{g_{2i}(t)} a_i(t,\xi), \tag{3.6}$$

for $t \in [0,+\infty)$*,* $i = 1,\ldots,m$*, then the solution* x *of the equation*

$$x'(t) + q(t)x'(r(t)) + \sum_{i=1}^{m}\left\{-\int_{g_{1i}(t)}^{g_{2i}(t)} x(\xi)d_{\xi}a_i(t,\xi) + \int_{h_{1i}(t)}^{h_{2i}(t)} x(\xi)d_{\xi}b_i(t,\xi)\right\} = 0, \quad t \in [0,+\infty),$$
$$x'(\xi) = 0 \quad \textit{for } \xi < 0. \tag{3.7}$$

such that $x(0) > 0$*, satisfies the inequalities* $x(t) \geq 0$*,* $x'(t) \geq 0$ *for* $t \in [0,+\infty)$ *and in the case, when there exists* ε *such that* $0 \leq q(t) \leq \varepsilon < 1$*, the solution* x *of* (3.5) *is nonnegative and nondecreasing for every positive nondecreasing function* $f \in L^{\infty}_{[0,\infty)}$*.*

Consider the integrodifferential equation

$$x'(t) + q(t)x'(r(t)) - \sum_{i=1}^{m}\int_{g_{1i}(t)}^{g_{2i}(t)} m_i(t,\xi)x(\xi)d\xi + \sum_{i=1}^{m}\int_{h_{1i}(t)}^{h_{2i}(t)} k_i(t,\xi)x(\xi)d\xi = f(t), \quad t \in [0,+\infty),$$
$$x(\xi) = x'(\xi) = 0 \quad \text{for } \xi < 0, \tag{3.8}$$

Theorem 3.3. *Let the spectral radius $\rho(S)$ of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ defined by equality* (3.2) *be less than one, $r(t)$ be a nondecreasing function and $h_{1i}(t) \le h_{2i}(t) \le g_{1i}(t) \le g_{2i}(t) \le t$, $q(t) \ge 0$, $k_i(t,\xi) \ge 0$, $m_i(t,\xi) \ge 0$ for $t,\xi \in [0,+\infty)$, and the following inequalities be fulfilled*

$$\int_{h_{1i}(t)}^{h_{2i}(t)} k_i(t,\xi)d\xi + q(t)\chi(r(t),0)\int_{g_{1i}(r(t))}^{g_{2i}(r(t))} m_i(t,\xi)d\xi \le \int_{g_{1i}(t)}^{g_{2i}(t)} m_i(t,\xi)d\xi, \tag{3.9}$$

$t \in [0,+\infty)$, $i = 1,\dots,m$, then the solution x of the equation

$$x'(t) + q(t)x'(r(t)) - \sum_{i=1}^{m}\int_{g_{1i}(t)}^{g_{2i}(t)} m_i(t,\xi)x(\xi)d\xi + \sum_{i=1}^{m}\int_{h_{1i}(t)}^{h_{2i}(t)} k_i(t,\xi)x(\xi)d\xi = 0, \quad t \in [0,+\infty),$$
$$x(\xi) = x'(\xi) = 0 \quad \text{for } \xi < 0, \tag{3.10}$$

such that $x(0) > 0$, satisfies the inequalities $x(t) \ge 0$, $x'(t) \ge 0$ for $t \in [0,+\infty)$ and in the case, when there exists ε such that $0 \le q(t) \le \varepsilon < 1$, the solution x of (3.8) *is nonnegative and nondecreasing for every positive nondecreasing function $f \in L^{\infty}_{[0,\infty)}$.*

Consider the equation

$$x'(t) + q(t)x'(r(t)) - \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi + b(t)x(h(t)) = f(t), \quad t \in [0,+\infty),$$
$$x(\xi) = x'(\xi) = 0 \quad \text{for } \xi < 0. \tag{3.11}$$

In the following assertion the integral term is dominant.

Theorem 3.4. *Let the spectral radius $\rho(S)$ of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ defined by equality* (3.2) *be less than one, $r(t)$ be a nondecreasing function, $q(t) \ge 0$, $b(t) \ge 0$, $m(t,\xi) \ge 0$, $h(t) \le g_1(t) \le g_2(t) \le t$ for $t,\xi \in [0,+\infty)$, and the following inequality be fulfilled*

$$b(t)\chi(h(t),0) + q(t)\chi(r(t),0)\int_{g_1(r(t))}^{g_2(r(t))} m(t,\xi)d\xi \le \int_{g_1(t)}^{g_2(t)} m(t,\xi)d\xi, \quad t \in [0,+\infty), \tag{3.12}$$

then the solution x of the equation

$$x'(t)+q(t)x'(r(t))-\int_{g_1(t)}^{g_2(t)}m(t,\xi)x(\xi)d\xi+b(t)x(h(t))=0,\quad t\in[0,+\infty),$$
$$x(\xi)=x'(\xi)=0\quad \textit{for } \xi<0, \tag{3.13}$$

such that $x(0)>0$, satisfies the inequalities $x(t)\geq 0, x'(t)\geq 0$ for $t\in[0,\infty)$ and in the case, when there exists ε such that $0\leq q(t)\leq\varepsilon<1$, the solution x of (3.11) *is nonnegative and nondecreasing for every positive nondecreasing function $f\in L^{\infty}_{[0,\infty)}$.*

Consider the equation

$$x'(t)+q(t)x'(r(t))-a(t)x(g(t))+\int_{h_1(t)}^{h_2(t)}k(t,\xi)x(\xi)d\xi=f(t),\quad t\in[0,+\infty),$$
$$x(\xi)=x'(\xi)=0\quad \text{for } \xi<0. \tag{3.14}$$

In the following assertion the term $a(t)x(g(t))$ is dominant.

Theorem 3.5. *Let the spectral radius $\rho(S)$ of the operator $S:L^{\infty}_{[0,\infty)}\to L^{\infty}_{[0,\infty)}$ defined by equality* (3.2) *be less than one, $r(t)$ be a nondecreasing function, $h_1(t)\leq h_2(t)\leq g(t)\leq t$, $q(t)\geq 0$, $k(t,\xi)\geq 0$, $a(t)\geq 0$ for $t,\xi\in[0,+\infty)$, and the following inequality*

$$\int_{h_1^0(t)}^{h_2^0(t)}k(t,\xi)d\xi+q(t)a(r(t))\chi(r(t),0)\leq a(t),\quad t\in[0,+\infty), \tag{3.15}$$

be fulfilled, then the solution x of the equation

$$x'(t)+q(t)x'(r(t))-a(t)x(g(t))+\int_{h_1(t)}^{h_2(t)}k(t,\xi)x(\xi)d\xi=0,\quad t\in[0,+\infty),$$
$$x(\xi)=x'(\xi)=0\quad \textit{for } \xi<0. \tag{3.16}$$

such that $x(0)>0$, satisfies the inequalities $x(t)\geq 0$, $x'(t)\geq 0$ for $t\in[0,\infty)$ and in the case, when there exists ε such that $0\leq q(t)\leq\varepsilon<1$, the solution x of (3.14) *is nonnegative and nondecreasing for every positive nondecreasing $f\in L^{\infty}_{[0,\infty)}$.*

Consider now the equation

$$x'(t)+q(t)x'(r(t))-a(t)x(g(t))+b(t)x(h(t))-\int_{g_1(t)}^{g_2(t)}m(t,\xi)x(\xi)d\xi$$
$$+\int_{h_1(t)}^{h_2(t)}k(t,\xi)x(\xi)d\xi=f(t),\quad t\in[0,+\infty), \tag{3.17}$$
$$x(\xi)=x'(\xi)=0\quad \text{for } \xi<0.$$

In the following assertion we do not assume inequalities $k(t,\xi) \leq m(t,\xi)$ or $b(t) \leq a(t)$. Here the sum $a(t)x(g(t)) + \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi$ is a dominant term.

Theorem 3.6. *Let the spectral radius $\rho(S)$ of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ defined by equality* (3.2) *be less than one, $r(t)$ be a nondecreasing function, $k(t,\xi) \geq 0$, $m(t,\xi) \geq 0$, $a(t) \geq 0$, $b(t) \geq 0$, $q(t) \geq 0$, $h(t) \leq g_1(t) \leq g_2(t) \leq t$, $h_1(t) \leq h_2(t) \leq g(t) \leq t$ for $t,\xi \in [0,+\infty)$ and the following inequalities be fulfilled*

$$b(t)\chi(h(t),0) + q(t)\chi(r(t),0)\int_{g_1^0(r(t))}^{g_2^0(r(t))} m(t,\xi)d\xi \leq \int_{g_1^0(t)}^{g_2^0(t)} m(t,\xi)d\xi,$$
$$\int_{h_1^0(t)}^{h_2^0(t)} k(t,\xi)d\xi + q(t)a(r(t))\chi(r(t),0) \leq a(t), \tag{3.18}$$

for $t \in [0,+\infty)$, then the solution x of the equation

$$x'(t) + q(t)x'(r(t)) - a(t)x\big(g(t)\big) + b(t)x(h(t)) - \int_{g_1(t)}^{g_2(t)} m(t,\xi)x(\xi)d\xi$$
$$+ \int_{h_1(t)}^{h_2(t)} k(t,\xi)x(\xi)d\xi = 0, \quad t \in [0,+\infty), \tag{3.19}$$

$$x(\xi) = x'(\xi) = 0 \quad \text{for } \xi < 0, \tag{3.20}$$

such that $x(0) > 0$, satisfies the inequalities $x(t) \geq 0$, $x'(t) \geq 0$ for $t \in [0,+\infty)$ and in the case, when there exists ε such that $0 \leq q(t) \leq \varepsilon < 1$, the solution x of (3.17) *is nonnegative and nondecreasing for every positive nondecreasing $f \in L^{\infty}_{[0,\infty)}$.*

Let us write (3.1) in the form

$$(I + S)x'(t) = Ax(t) - Bx(t) + f(t). \tag{3.21}$$

The spectral radius $\rho(S)$ of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ is less than one, then there exists the bounded operator $(I+S)^{-1} : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ and we can write (3.21) in the form

$$(Nx)(t) \equiv x'(t) - \sum_{n=0}^{\infty}(-1)^n(S^n(A-B)x)(t) = \sum_{n=0}^{\infty}(-1)^n\big(S^n f\big)(t). \tag{3.22}$$

Denote by $C_0(t,s)$ the Cauchy function of the equation $Nx = 0$, which is also the fundamental function of (3.1).

Proofs of Theorems 3.1–3.6 are based on the following auxiliary assertions.

Lemma 3.7. *Let* $A : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$, $B : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ *and* $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ *be positive Volterra operators, the spectral radius* $\rho(S)$ *of the operator* S *be less than one and*

$$(A_s 1)(t) \geq (B_s 1)(t) + (S_s A_s 1)(t), \quad t \in [s, +\infty), \tag{3.23}$$

for every nonnegative s. *Then* $C_0(t,s) > 0$ *and* $(\partial/\partial t)C_0(t,s) \geq 0$ *for* $0 \leq s \leq t < +\infty$.

Proof. Lemma 2.9 is true for (3.22). Let us set $v_s(t) \equiv 1$, $t \in [s, +\infty)$ in the assertion 1 of Lemma 2.9. Condition (3.23) implies, according to Lemma 2.9, that $C_0(t,s) > 0$ for $0 \leq s \leq t < +\infty$.

Now let us prove that $(\partial/\partial t)C_0(t,s) \geq 0$ for $0 \leq s \leq t < +\infty$. We use the fact that the function $C_0(t,s)$, as a function of argument t for each fixed positive s, satisfies the equation

$$(N_s x)(t) \equiv x'(t) - \sum_{n=0}^{\infty} (-1)^n \left(S_s^n (A_s - B_s) x\right)(t) = 0, \quad t \in [s, +\infty), \tag{3.24}$$

and the condition $x(s) = 1$.

The following integral equation

$$x(t) = \int_s^t \sum_{n=0}^{\infty} (-1)^n \left(S_s^n (A_s - B_s) x\right)(\xi) d\xi + 1 \tag{3.25}$$

is equivalent to (3.24) with the condition $x(s) = 1$.

The spectral radius of the operator $T : C_{[s,\omega]} \to C_{[s,\omega]}$, defined by the equality

$$(Tx)(t) = \int_s^t \sum_{n=0}^{\infty} (-1)^n \left(S_s^n (A_s - B_s) x\right)(\xi) d\xi, \quad t \in [s, \omega], \tag{3.26}$$

is zero for every positive number ω [4]. Let us build the sequence

$$x_{m+1}(t) = \int_s^t \sum_{n=0}^{\infty} (-1)^n \left(S_s^n (A_s - B_s) x_m\right)(\xi) d\xi + 1, \tag{3.27}$$

where the iterations start with the constant $x_0(t) \equiv 1$ for $t \in [s, \omega]$.

The sequence of functions $x_m(t)$ converges in the space $C_{[s,\omega]}$ to the unique solution $x(t)$ of (3.25) on the interval $[s, \omega]$. It is clear that this solution is absolutely continuous. It follows from the fact that all operators are Volterra ones, that the solution $y(t)$ of (3.24) with the initial condition $y(s) = 1$ and the solution $x(t)$ of (3.25) coincide for $t \in [s, \omega]$.

Positivity of the operator S, the inequalities $\rho(S) < 1$ and (3.23) imply nonnegativity of the derivatives

$$x'_{m+1}(t) = \sum_{n=0}^{\infty} (-1)^n \left(S_s^n (A_s - B_s) x_m\right)(t), \quad t \in [s, \omega]. \tag{3.28}$$

Repeating the argumentation used in the proof of Lemma 2.13, we obtain that this sequence of nondecreasing functions x_m converges to the nondecreasing solution x, that is, $(\partial/\partial t)C_0(t,s) \geq 0$ for $0 \leq s \leq t < +\infty$. □

Concerning nonhomogeneous (3.1) we propose the following assertion.

Lemma 3.8. *Let $A : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$, $B : C_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ and $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ be positive Volterra operators, the spectral radius $\rho(S)$ of the operator S be less than one and inequality* (3.23) *be fulfilled for every nonnegative s. Then the solution x of the homogeneous equation*

$$(Mx)(t) \equiv x'(t) + (Sx')(t) - (Ax)(t) + (Bx)(t) = 0, \quad t \in [0,+\infty), \tag{3.29}$$

such that $x(0) \geq 0$, satisfies inequalities $x(t) \geq 0$, $x'(t) \geq 0$ for $t \in [0,+\infty)$. If in addition the nonnegative function $f \in L^{\infty}_{[0,\infty)}$ satisfies the inequality $f(t) \geq (Sf)(t)$ for $t \in [0,+\infty)$, then the solution x of (3.1) *is nonnegative and nondecreasing for every positive nondecreasing f.*

Remark 3.9. The inequality $f(t) \geq (Sf)(t)$ *for* $t \in [0,+\infty)$ is fulfilled if a nonnegative function f is nondecreasing and the norm of the operator $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ is less than one.

Proof of Lemma 3.8. Assertions about nonnegativity of solution x of the homogeneous equation $Mx = 0$ and its derivative follows from the equalities $x(t) = C_0(t,0)$ and $x'(t) = (\partial/\partial t)C_0(t,0)$ and Lemma 3.7. From the representation of solutions of (3.22) we can write

$$\begin{aligned} x(t) &= \int_0^t C_0(t,s)\sum_{n=0}^{+\infty}(-1)^n\left(S^n f\right)(s)ds + x(0)C_0(t,0), \\ x'(t) &= \sum_{n=0}^{+\infty}(-1)^n\left(S^n f\right)(t) + \int_0^t \frac{\partial}{\partial t}C_0(t,s)\sum_{n=0}^{+\infty}(-1)^n\left(S^n f\right)(s)ds + x(0)\frac{\partial}{\partial t}C_0(t,0). \end{aligned} \tag{3.30}$$

It is clear now that the inequality$f(t) \geq (Sf)(t)$ for $t \in [0,+\infty)$, positivity of S and nonnegativity of $(\partial/\partial t)C_0(t,s)$ for $0 \leq s \leq t < +\infty$ imply the inequalities $x(t) \geq 0$, $x'(t) \geq 0$ for $t \in [0,+\infty)$. □

The proofs of Theorems 3.1–3.6 follows from the fact that conditions of every theorem imply the conditions of Lemma 3.8 for corresponding equations.

Remark 3.10. The condition

$$(A1)(t) \geq (B1)(t) + (SA1)(t), \quad t \in [0,+\infty), \tag{3.31}$$

cannot be set instead of condition (3.23) as Example 2.4 demonstrates. In this example, the operator S is the zero one, $(Ax)(t) = x(0)$ and $(A1)(t) = 1, (B1)(t) = 0$ for $t \in [0,2)$ and $(B1)(t) = 1/2$ for $t \geq 2$, condition (3.31) fulfilled, the Cauchy function $C(t,s)$ of (2.18) and its derivative change their signs. We noted that the inequality on delays avoids this situation.

Remark 3.11. In the case of the neutral equation

$$x'(t) + x'(t-1) - x(g(t)) + x(h(t)) = 0, \quad t \in [0,+\infty), \tag{3.32}$$

where

$$h(t) = \begin{cases} 0, & 0 \le t < 3, \\ 2, & 3 \le t, \end{cases}$$
$$g(t) = \begin{cases} 0, & 0 \le t < 2, \\ 2, & 2 \le t, \end{cases} \tag{3.33}$$

the inequality $h(t) \le g(t)$ for $t \in [0, +\infty)$ does not avoid the changes of signs of $C_0(t, s)$ and its derivative:

$$C_0(t, 2) = \begin{cases} t - 1, & 2 \le t < 3, \\ 5 - t, & 3 \le t, \end{cases} \tag{3.34}$$

and $\partial C_0(t, 2)/\partial t = -1 < 0$ for $t > 3$ and $C_0(t, 2) < 0$ for $t > 5$.

This example demonstrates that we cannot set very natural inequality

$$a(t) - b(t)\chi(h(t), 0) - q(t)a(r(t))\chi\big(g(r(t)), 0\big) + q(t)b(r(t))\chi(h(r(t)), 0) \ge 0, \quad t \in [0, +\infty), \tag{3.35}$$

instead of

$$a(t) - b(t)\chi(h(t), 0) - q(t)a(r(t))\chi\big(g(r(t)), 0\big) \ge 0, \quad t \in [0, +\infty), \tag{3.36}$$

in Theorem 3.1 even in the case when $h(t) \le g(t)$ for $t \in [0, +\infty)$.

4. Maximum Boundaries Principles in Existence and Uniqueness of Boundary Value Problems

Consider the boundary value problems of the following type

$$(Mx)(t) \equiv x'(t) - \big(Sx'\big)(t) - (Ax)(t) + (Bx)(t) = f(t), \quad t \in [0, \omega], \tag{4.1}$$

$$lx = c, \tag{4.2}$$

where $l : D_{[0,\omega]} \to R^1$ is a linear bounded functional and $c \in R^1$.

It was explained in Remark 2.14 that Lemma 2.13 can be considered as the maximum boundaries principle for (4.1), and Theorems 2.1–2.8 present sufficient conditions of the maximum boundaries principles for equations (2.12), (2.14), (2.16), (2.22), (2.24), (2.27) and (2.29) respectively (i.e., under these conditions the modulus of nontrivial solutions of the corresponding homogeneous equations does not decrease). This allows us to obtain various results about existence and uniqueness of solutions of boundary value problems for these equations without the standard assumption about smallness of the norms of the operators A and B.

The assertions about existence and uniqueness are based on the known Fredholm alternative for functional differential equations.

Lemma 4.1 ([4]). *Let the spectral radius of the operator* $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ *be less than one, then boundary value problem* (4.1), (4.2) *is uniquely solvable for each* $f \in L_{[0,\omega]},\ \ c \in R^1$ *if and only if the homogeneous problem*

$$(Mx)(t) \equiv x'(t) - \left(Sx'\right)(t) - (Ax)(t) + (Bx)(t) = 0, \quad t \in [0,\omega],\ lx = 0, \tag{4.3}$$

has only the trivial solution.

Theorem 4.2. *Let* $S : L^{\infty}_{[0,\infty)} \to L^{\infty}_{[0,\infty)}$ *be a positive Volterra operator and its spectral radius satisfy the inequality* $\rho(S) < 1$, *and for each* $s \in 0,\omega)$ *the inequality*

$$(A_s 1)(t) \geq (B_s 1)(t) \quad \text{for } t \in [s,\omega], \tag{4.4}$$

be fulfilled. Then the following assertions are true:

(1) *If* $l : C_{[0,\omega]} \to R^1$ *is a linear nonzero positive functional, then boundary value problem* (4.1), (4.2) *is uniquely solvable for each* $f \in L_{[0,\omega]},\ \ c \in R^1$.

(2) *The boundary value problem* (4.1), (4.5), *where*

$$lx \equiv x(\omega) - mx = c, \tag{4.5}$$

and the norm of the linear functional $m : C_{[0,\omega]} \to R^1$ *is less than one is uniquely solvable for each* $f \in L_{[0,\omega]},\ \ c \in R^1$;

(3) *The boundary value problem* (4.1), (4.6), *where*

$$\sum_{j=1}^{2k} \alpha_j x(t_j) = c, \quad 0 \leq t_1 < t_2 < \cdots < t_{2k} \leq \omega, \tag{4.6}$$

with $0 \leq -\alpha_{2j-1} \leq \alpha_{2j},\ j = 1,\ldots,k$, *and there exists an index* i *such that* $-\alpha_{2i-1} < \alpha_{2i}$, *is uniquely solvable for each* $f \in L_{[0,\omega]},\ \ c \in R^1$.

(4) *The boundary value problem* (4.1), (4.7), *where*

$$\sum_{j=1}^{2k} \int_{t_{j-1}}^{t_j} \alpha(t) x(t) dt = c, \quad 0 = t_0 \leq t_1 < t_2 < \cdots < t_{2k} \leq \omega, \tag{4.7}$$

in the case when $\alpha(t) \leq 0$ *for* $t \in [t_{2j-2}, t_{2j-1}]$, $\alpha(t) \geq 0$ *for* $t \in [t_{2j-1}, t_{2j}]$, $\int_{t_{2j-2}}^{t_{2j}} \alpha(t)dt \geq 0$, $j = 1,\ldots,k$, *and there exists* j *such that* $\int_{t_{2j-2}}^{t_{2j}} \alpha(t)dt > 0$, *is uniquely solvable for each* $f \in L_{[0,\omega]},\ \ c \in R^1$.

Proof. If we suppose in the contrary that the assertions 1–4 are not true, then according to Lemma 4.1, the nontrivial solution x of homogeneous problem (4.3) exists. According to Lemma 2.13 (see also Remark 2.14), the maximum boundaries principle is true, moreover, solutions of the homogeneous equation $Mx = 0$ does not decrease on $[0, \omega]$. Conditions of each of the assertions 1–4 lead us to the inequality $lx \neq 0$ which contradicts to the existence of the nontrivial solution x of the homogeneous problem $Mx = 0$, $lx = 0$. □

Theorem 4.3. *Conditions of each of Theorems 2.1–2.8 imply assertions 1–4 of Theorem 4.2 for equations* (2.12), (2.14), (2.16), (2.22), (2.24), (2.27) *and* (2.29) *respectively.*

In order to prove Theorem 4.3 we have only to note that conditions of Theorem 4.2 for corresponding equations follow from the conditions of each of Theorems 2.1–2.8.

Remark 4.4. The condition

$$(A1)(t) \geq (B1)(t) \quad \text{for } t \in [0, \omega], \tag{4.8}$$

cannot be set instead of condition (4.4) as the following example demonstrates.

Example 4.5. Consider the boundary value problem

$$\begin{gathered} x'(t) - x(0) + b_1(t)x(h_1(t)) = f(t), \quad t \in [0, 7], \\ x(7) - \frac{1}{2}x(0) = c, \end{gathered} \tag{4.9}$$

where $h_1(t)$ and $b_1(t)$ are defined by formula (2.19) respectively. Here the operator $A : C_{[0,\omega]} \to L_{[0,\omega]}$ is defined as $(Ax)(t) = x(0)$ and inequality (4.8) is fulfilled, but the operator A_s is a zero operator for $s > 0$, and inequality (4.4) is not true. Formula (2.20) defines the nontrivial solution of the homogeneous problem

$$x'(t) - x(0) + b_1(t)x(h_1(t)) = 0, \qquad x(7) - \frac{1}{2}x(0) = 0, \quad t \in [0, 7]. \tag{4.10}$$

According to Lemma 4.1, problem (4.9) cannot be uniquely solvable for each $f \in L_{[0,\omega]}$, $c \in R^1$.

Remark 4.6. The periodic problem

$$x'(t) = f(t), \quad t \in [0, \omega], \qquad x(\omega) - x(0) = c, \tag{4.11}$$

where $\|m\| = 1$, demonstrates that the condition $\|m\| < 1$ in the assertion 2 of Theorem 4.2 is essential: the function $x(t) \equiv 1$ for $t \in [0, \omega]$ is a nontrvial solution of the homogeneous boundary value problem

$$x'(t) = 0, \quad t \in [0, \omega], \qquad x(\omega) - x(0) = 0, \tag{4.12}$$

and reference to Lemma 4.1 completes this example.

Remark 4.7. According to Lemma 4.1, the problem

$$x'(t) = f(t), \quad t \in [0,\omega], \qquad \sum_{j=1}^{2k} \alpha_j x(t_j) = c, \quad 0 \le t_1 < t_2 < \cdots < t_{2k} \le \omega, \tag{4.13}$$

demonstrates that the condition about existence of such i that $-\alpha_{2i-1} < \alpha_{2i}$ in the assertion 3 is essential, and the problem

$$x'(t) = f(t), \quad t \in [0,\omega], \quad \sum_{j=1}^{2k} \int_{t_{2j-2}}^{t_{2j}} \alpha(t)x(t)dt = c, \quad 0 = t_0 \le t_1 < t_2 < \cdots < t_{2k} \le \omega, \tag{4.14}$$

demonstrates that the condition about existence of such i that $\int_{t_{2j-2}}^{t_{2j}} \alpha(t)dt > 0$ in the assertion 4 of Theorem 4.2 is essential. If we suppose that such i does not exist, then the function $x(t) \equiv 1$ for $t \in [0,\omega]$ is a nontrvial solution of each of the homogeneous boundary value problems

$$\begin{gathered} x'(t) = 0, \quad t \in [0,\omega], \qquad \sum_{j=1}^{2k} \alpha_j x(t_j) = 0, \quad 0 \le t_1 < t_2 < \cdots < t_{2k} \le \omega, \\ x'(t) = 0, \quad t \in [0,\omega], \qquad \sum_{j=1}^{2k} \int_{t_{2j-2}}^{t_{2j}} \alpha(t)x(t)dt = 0, \quad 0 = t_0 \le t_1 < t_2 < \cdots < t_{2k} \le \omega. \end{gathered} \tag{4.15}$$

Remark 4.8. The condition $\alpha(t) \le 0$ for $t \in [t_{2j-2}, t_{2j-1}]$ in the assertion 4 of Theorem 4.2 cannot be omitted that follows from the example of one of the reviewers: the function $x = 1 + t$ is a nontrivial solution of the boundary value problem

$$x'(t) = x(0), \quad t \in [0,2], \qquad \int_0^2 \alpha(t)x(t)dt = 0, \tag{4.16}$$

where $\alpha(t) = 10$ for $0 \le t < 1/2, \alpha(t) = -10$ for $1/2 \le t < 1, \alpha(t) = 1$ for $1 \le t \le 2$. In this case $t_0 = 0,\ t_1 = 1,\ t_2 = 2,\ \int_0^2 \alpha(t)dt = 1$ and consequently all other conditions of the assertion 4 of Theorem 4.2 are fulfilled.

Remark 4.9. Let us define the set

$$E = \{t \in [0,\omega] : (A1)(t) > (B1)(t)\}. \tag{4.17}$$

and the following condition:

(a) there exists a set E of nonzero measure (i.e., mes$E > 0$).

Instead of the condition $\|m\| < 1$ in the assertion 2 of Theorem 4.2 we can assume that the inequality $\|m\| \le 1$ and the condition (a) are fulfilled.

Remark 4.10. Instead of the condition about existence of such i that $-\alpha_{2i-1} < \alpha_{2i}$ in the assertion 3 of Theorem 4.2 we can assume that

$$\alpha_{2k} > 0, \quad \text{mes}\{[0, t_{2k}] \cap E\} > 0. \tag{4.18}$$

Condition (4.18) is essential as the following example demonstrates.

Example 4.11. The homogeneous boundary value problem

$$x'(t) - x\left(\frac{t}{2}\right) + b_1(t)x\left(\frac{t}{3}\right) = 0, \quad t \in [0,2], \tag{4.19}$$

$$x(1) - x\left(\frac{1}{2}\right) = 0, \tag{4.20}$$

where

$$b_1(t) = \begin{cases} 1, & 0 \le t \le 1, \\ \frac{1}{2}, & 1 < t, \end{cases} \tag{4.21}$$

has a nontrivial solution

$$x(t) = \begin{cases} 1, & 0 \le t \le 1, \\ \frac{1}{2}(t+1), & 1 < t. \end{cases} \tag{4.22}$$

In this case we have $k = 1,\ t_1 = 1/2,\ t_2 = 1,\ E = [1,2],\ \text{mes}\{[0,1] \cap E\} = 0$.

Remark 4.12. Let us denote as E_1 the following set: $E_1 = \{t : \alpha(t) > 0\}$. Instead of the condition about existence of such $i$that $\int_{t_{2j-2}}^{t_{2j}} \alpha(t)dt > 0$, in the assertion 4, we can assume the following: $\text{mes}\{[0, t_{2k}] \cap E \cap E_1\} > 0$.

Example 4.13. Consider (4.19), where the coefficient $b_1(t)$ is defined by (4.21), with the boundary condition

$$-\int_0^1 x(t) \sin 2\pi t dt = 0. \tag{4.23}$$

Homogeneous problem (4.19), (4.23) has a nontrivial solution defined by (4.22). In this case we have $k = 1,\ t_1 = 0,\ t_2 = 1,\ E = [1,2],\ \text{mes}\{[0,1] \cap E \cap E_1\} = 0$.

Consider now (3.1) with the opposite sign near the neutral term $(Sx')(t)$.

Theorem 4.14. *Let the conditions of Lemma 3.8 be fulfilled for* (3.1)*, then assertions 1–4 of Theorem 4.2 are true for* (3.1).

Proof follows from the fact that according to Lemma 3.8, solution x of the homogeneous equation

$$x'(t) + \left(Sx'\right)(t) - (Ax)(t) + (Bx)(t) = 0, \quad t \in [0, +\infty). \tag{4.24}$$

does not decrease. Conditions of each of the assertions 1–4 of Theorem 4.2 lead us to the maximum boundaries principle and consequently to the inequality $lx \neq 0$ which contradicts to the existence of the nontrivial solution x of the homogeneous problem $Mx = 0$, $lx = 0$.

Theorem 4.15. *Conditions of each of Theorems 3.1–3.6 imply assertions 1–4 of Theorem 4.2 for equations* (3.3), (3.5), (3.8), (3.11), (3.14) *and* (3.17) *respectively.*

Proof follows from the fact that conditions of each of Theorems 3.1–3.6 imply that the conditions of Lemma 3.8 are fulfilled.

Note that Remarks 4.4–4.12 are relevant also for Theorems 4.14–4.15.

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