Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 156167, 12 pages
doi:10.1155/2009/156167

Research Article

Investigation of the Stability via Shadowing Property

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Received 25 November 2008; Revised 16 February 2009; Accepted 19 May 2009

Recommended by Ulrich Abel

The shadowing property is to find an exact solution to an iterated map that remains close to an approximate solution. In this article, using shadowing property, we show the stability of the following equation in normed group: $4{}_{n-2}C_{n/2-1}r^2 f(\sum_{j=1}^{n}(x_j/r)) + n\sum_{i_k\in\{0,1\},\, \sum_{k=1}^{n} i_k=n/2} r^2 f(\sum_{i=1}^{n}(-1)^{i_k}(x_i/r)) = 4n\cdot{}_{n-2}C_{n/2-1}\sum_{i=1}^{n} f(x_i)$, where $n \geq 2$, $r \in \mathbb{R}$ $(r^2 \neq n)$ and f is a mapping. And we prove that the even mapping which satisfies the above equation is quadratic and also the Hyers-Ulam stability of the functional equation in Banach spaces.

1. Introduction

The notion of pseudo-orbits very often appears in several areas of the dynamical systems. A pseudo-orbit is generally produced by numerical simulations of dynamical systems. One may consider a natural question which asks whether or not this predicted behavior is close to the real behavior of system. The above property is called the *shadowing property (or pseudo-orbit tracing property)*. The shadowing property is an important feature of stable dynamical systems. Moreover, a dynamical system satisfying the shadowing property is in many respects close to a (topologically, structurally) stable system. It is well known that the shadowing property is a useful notion for the study about the stability theory, and the concept of the shadowing is close to this of the stability in dynamical systems.

In this paper, we are going to investigate the stability of functional equations using the shadowing property derived from dynamical systems.

The study of stability problems for functional equations is related to the following question raised by Ulam [1] concerning the stability of group homomorphisms. *Let* G_1 *be a group, and let* G_2 *be a metric group with the* metric $d(\cdot,\cdot)$. *Given* $\varepsilon > 0$ *does there exist a* $\delta > 0$ *such that if a* mapping $h : G_1 \to G_2$*satisfies the inequality*

$$d(h(xy), h(x)h(y)) < \delta \tag{1.1}$$

for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

D. H. Hyers [2] provided the first partial solution of Ulam's question as follows. Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Hyers showed that if a function $f : X \to Y$ satisfies the following inequality:

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \tag{1.2}$$

for some $\epsilon \geq 0$ and for all $x, y \in X$, then the limit

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{1.3}$$

exists for each $x \in X$ and $a : X \to Y$ is the unique additive function such that

$$\|f(x) - a(x)\| \leq \epsilon \tag{1.4}$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

Hyers' theorem was generalized in various directions. The very first author who generalized Hyers' theorem to the case of unbounded control functions was T. Aoki [3]. In 1978 Th. M. Rassias [4] by introducing the concept of the unbounded Cauchy difference generalized Hyers's Theorem for the stability of the linear mapping between Banach spaces. Afterward Th. M. Rassias's Theorem was generalized by many authors; see [5–7].

The quadratic function $f(x) = cx^2$ $(c \in \mathbb{R})$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.5}$$

Hence this equation is called the *quadratic functional equation*, and every solution of the quadratic equation (1.5) is called a *quadratic function*.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.5) was first proved by Skof [8] for functions $f : X \to Y$, where X is a normed space, and Y is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. Several functional equations have been investigated in [10–12].

From now on, we let n be an even integer, and $r \in \mathbb{R} - \{0\}$ such that $r^2 \neq n$. We denote ${}_nC_k = n!/(n-k)!\,k!$. In this paper, we investigate that a mapping $f : X \to Y$ satisfies the following equation:

$$\begin{aligned} &4\ {}_{n-2}C_{n/2-1} r^2 f\left(\sum_{j=1}^{n} \frac{x_j}{r}\right) + n \sum_{\substack{i_k \in \{0,1\} \\ \sum_{k=1}^{n} i_k = n/2}} r^2 f\left(\sum_{i=1}^{n} (-1)^{i_k} \frac{x_i}{r}\right) \\ &\quad = 4n \cdot {}_{n-2}C_{n/2-1} \sum_{i=1}^{n} f(x_i), \end{aligned} \tag{1.6}$$

for a mapping $f : X \to Y$. We will prove the stability in normed group by using shadowing property and also the Hyers-Ulam stability of each functional equation in Banach spaces.

2. A Generalized Quadratic Functional Equation in Several Variables

Lemma 2.1. *Let $n \geq 2$ be an even integer number, $r \in \mathbb{R} - \{0\}$ with $r^2 \neq n$, and X, Y vector spaces. If an even mapping $f : X \to Y$ which satisfies*

$$\begin{aligned} &4\ {}_{n-2}C_{n/2-1} r^2 f\left(\sum_{j=1}^{n} \frac{x_j}{r}\right) + n \sum_{\substack{i_k \in \{0,1\} \\ \sum_{k=1}^{n} i_k = n/2}} r^2 f\left(\sum_{i=1}^{n} (-1)^{i_k} \frac{x_i}{r}\right) \\ &\quad = 4n \cdot {}_{n-2}C_{n/2-1} \sum_{i=1}^{n} f(x_i), \end{aligned} \tag{2.1}$$

then f is quadratic, for all $x_1, \ldots, x_n \in X$.

Proof. By letting $x_1 = \cdots = x_n = 0$ in the equation (2.1), we have

$$4\ {}_{n-2}C_{n/2-1} r^2 f(0) + n\ {}_nC_{n/2} r^2 f(0) = 4n^2 \cdot {}_{n-2}C_{n/2-1} f(0). \tag{2.2}$$

Since ${}_nC_{n/2} = 4(n-1)/n\ {}_{n-2}C_{n/2-1}$, we have

$$(4 + 4(n-1)) r^2 f(0) = 4n^2 f(0), \tag{2.3}$$

that is, $(r^2 - n) f(0) = 0$. By the assumption $r^2 \neq n$, we have $f(0) = 0$. Now, by letting $x_1 = x,\ x_2 = y$, and $x_3 = \cdots = x_n = 0$, we get

$$\begin{aligned} 4\ {}_{n-2}C_{n/2-1} r^2 f\left(\frac{x+y}{r}\right) &+ n\ {}_{n-2}C_{n/2} r^2 f\left(\frac{x+y}{r}\right) + n\ {}_{n-2}C_{n/2-1} r^2 f\left(\frac{-x+y}{r}\right) \\ &+ n_{n-2}C_{n/2-1} r^2 f\left(\frac{x-y}{r}\right) + n_{n-2}C_{n/2-2} r^2 f\left(\frac{-x-y}{r}\right) \\ = 4n \cdot {}_{n-2}C_{n/2-1} \big(f(x) + f(y)\big) & \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Since f is even, we have

$$4\ {}_{n-2}C_{n/2-1} r^2 f\left(\frac{x+y}{r}\right) + 2n\ {}_{n-2}C_{n/2} r^2 f\left(\frac{x+y}{r}\right) + 2n\ {}_{n-2}C_{n/2-1} r^2 f\left(\frac{-x+y}{r}\right) = 4n\cdot {}_{n-2}C_{n/2-1}\left(f(x)+f(y)\right) \tag{2.5}$$

for all $x, y \in X$. From the following equation:

$$4\ {}_{n-2}C_{n/2-1} + 2n\ {}_{n-2}C_{n/2} = 4\frac{(n-2)!}{(n/2-1)!(n/2-1)!} + \frac{2\cdot n(n-2)!}{(n/2)!(n/2-2)!} = 2n\cdot {}_{n-2}C_{n/2-1}, \tag{2.6}$$

we have

$$2n\cdot {}_{n-2}C_{n/2-1} r^2 f\left(\frac{x+y}{r}\right) + 2n\cdot {}_{n-2}C_{n/2-1} r^2 f\left(\frac{x-y}{r}\right) = 4n\cdot {}_{n-2}C_{n/2-1}\left(f(x)+f(y)\right). \tag{2.7}$$

Now letting $x = x$ and $y = 0$, we have

$$r^2 f\left(\frac{x}{r}\right) = f(x). \tag{2.8}$$

Hence

$$f(x+y) + f(x-y) = r^2 f\left(\frac{x+y}{r}\right) + r^2 f\left(\frac{x-y}{r}\right) = 2\cdot\left(f(x)+f(y)\right) \tag{2.9}$$

for all $x, y \in X$. Then it is easily obtained that f is quadratic. This completes the proof. □

We call this quadratic mapping a *generalized quadratic mapping of r-type.*

3. Stability Using Shadowing Property

In this section, we will take $r = 1$, that is, we will investigate the generalized mappings of 1-type, and hence we will study the stability of the following functional equation by using shadowing property:

$$Df(x_1, \ldots, x_n) := 4\ {}_{n-2}C_{n/2-1} f\left(\sum_{j=1}^{n} x_j\right) + n \sum_{\substack{i_k\in\{0,1\}\\ \sum_{k=1}^{n} i_k = n/2}} f\left(\sum_{i=1}^{n} (-1)^{i_k} x_i\right) - 4n\cdot {}_{n-2}C_{n/2-1} \sum_{i=1}^{n} f(x_i) \tag{3.1}$$

for all $x_1, \ldots, x_n \in G$, where G is a commutative semigroup.

Before we proceed, we would like to introduce some basic definitions concerning shadowing and concepts to establish the stability; see [13]. After then we will investigate the stability of the given functional equation based on ideas from dynamical systems.

Let us introduce some notations which will be used throughout this section. We denote $\mathbb{N}$ the set of all nonnegative integers, X a complete normed space, $B(x,s)$ the closed ball centered at x with radius s, and let $\phi : X \to X$ be given.

Definition 3.1. Let $\delta \geq 0$. A sequence $(x_k)_{k\in\mathbb{N}}$ in X is a δ-pseudo-orbit for ϕ if

$$d(x_{k+1}, \phi(x_k)) \leq \delta \quad \text{for } k \in \mathbb{N}. \tag{3.2}$$

A 0-pseudo-orbit is called an orbit.

Definition 3.2. Let $s, R > 0$ be given. A function $\phi : X \to X$ is locally (s,R)-invertible at $x_0 \in X$ if for any point y in $B(\phi(x_0), R)$, there exists a unique element x in $B(x_0, s)$ such that $\phi(x) = y$. If ϕ is locally (s,R)-invertible at each $x \in X$, then we say that ϕ is locally(s,R)-invertible.

For a locally (s,R)-invertible function ϕ, we define a function $\phi_{x_0}^{-1} : B(\phi(x_0), R) \to B(x_0, s)$ in such a way that $\phi_{x_0}^{-1}(y)$ denote the unique x from the above definition which satisfies $\phi(x) = y$. Moreover, we put

$$\operatorname{lip}_R \phi^{-1} := \sup_{x_0 \in X} \operatorname{lip}\left(\phi_{x_0}^{-1}\right). \tag{3.3}$$

Theorem 3.3 (see [14]). *Let $l \in (0,1), R \in (0,\infty)$ be fixed, and let $\phi : X \to X$ be locally (lR, R)-invertible. We assume additionally that $lip_R(\phi^{-1}) \leq l$. Let $\delta \leq (1-l)R$, and let $(x_k)_{k\in\mathbb{N}}$ be an arbitrary δ-pseudo-orbit. Then there exists a unique $y \in X$ such that*

$$d\left(x_k, \phi^k(y)\right) \leq lR \quad \textit{for } k \in \mathbb{N}. \tag{3.4}$$

Moreover,

$$d\left(x_k, \phi^k(y)\right) \leq \frac{l\delta}{1-l} \quad \textit{for } k \in \mathbb{N}. \tag{3.5}$$

Let X be a semigroup. Then the mapping $\|\cdot\| : X \to \mathbb{R}$ is called a (*semigroup*) *norm* if it satisfies the following properties:

(1) for all $x \in X$, $\|x\| \geq 0$;

(2) for all $x \in X, k \in \mathbb{N}$, $\|kx\| = k\|x\|$;

(3) for all $x, y \in X$, $\|x\| + \|y\| \geq \|x * y\|$ and also the equality holds when $x = y$, where $*$ is the binary operation on X.

Note that $\|\cdot\|$ is called a *group norm* if X is a group with an identity 0_X, and it additionally satisfies that $\|x\| = 0$ if and only if $x = 0_X$.

From now on, we will simply denote the identity 0_G of G and the identity 0_X of X by 0. We say that $(X, *, \|\cdot\|)$ is a *normed (semi)group* if X is a (semi)group with the (semi)group norm $\|\cdot\|$. Now, given an Abelian group X and $n \in \mathbb{Z}$, we define the mapping $[n_X] : X \to X$ by the formula

$$[n_X](x) := nx \quad \text{for } x \in \mathrm{X}. \tag{3.6}$$

Since X is a normed group, it is clear that $[n_X]$ is locally $(R/n, R)$-invertible at 0, and $\mathrm{lip}_R[n_X]^{-1} = 1/n$.

Also, we are going to need the following result. Tabor et al. proved the next lemma by using Theorem 3.3.

Lemma 3.4. *Let $l \in (0,1), R \in (0,\infty), \delta \in (0,(1-l)R), \varepsilon > 0, m \in \mathbb{N}, n \in \mathbb{Z}$. Let G be a commutative semigroup, and X a complete Abelian metric group. We assume that the mapping $[n_X]$ is locally (lR, R)-invertible and that $lip_R([n_X]^{-1}) \le l$. Let $f : G \to X$ satisfy the following two inequalities:*

$$\begin{gathered}\left\|\sum_{i=1}^{N} a_i f(b_{i_1}x_1 + \cdots + b_{i_n}x_n)\right\| \le \varepsilon \quad \textit{for } x_1, \ldots, x_n \in G, \\ \|f(mx) - nf(x)\| \le \delta \quad \textit{for } x \in G,\end{gathered} \tag{3.7}$$

where a_i are endomorphisms in X, and b_{i_j}, are endomorphisms in G. We assume additionally that there exists $K \in \{1, \ldots, N\}$ such that

$$\sum_{i=1}^{K} \mathrm{lip(a_i)ffi} \le \mathrm{(1-l)R}, \qquad {''} + \sum_{\mathrm{i=K+1}}^{\mathrm{N}} \mathrm{lip(a_i)}\frac{\mathrm{lffi}}{1-\mathrm{l}} \le \mathrm{lR}. \tag{3.8}$$

Then there exists a unique function $F : G \to X$ such that

$$\begin{gathered}F(mx) = nF(x) \quad \textit{for} \quad x \in G, \\ \|f(x) - F(x)\| \le \frac{l\delta}{1-l} \quad \textit{for} \quad x \in G.\end{gathered} \tag{3.9}$$

Moreover, then F satisfies

$$\sum_{i=1}^{N} a_i F(b_{i_1}x_1 + \cdots + b_{i_n}x_n) = 0 \quad \textit{for } x_1, \ldots, x_n \in G. \tag{3.10}$$

Proof. Using the proof of [13, Theorem 2], one can easily show this lemma. □

Let $R > 0$, $n \ge 2$ an even integer, G an Abelian group, and X a complete normed Abelian group.

Theorem 3.5. *Let $\varepsilon \leq 3n/4(n^3+n^2+4n+1)R$ be arbitrary, and let $f : G \to X$ be a function such that*

$$\left\| Df(x_1,\ldots,x_n) \right\| \leq \varepsilon \tag{3.11}$$

for all $x_1,\ldots,x_n \in G$. Then there exists a unique function $F : G \to X$ such that

$$\begin{gathered} F(nx) = n^2 F(x), \\ DF(x_1,\ldots,x_n) = 0, \\ \left\| F(x) - f(x) \right\| \leq \frac{n+1}{12n\ {}_{n-2}C_{n/2-1}}\varepsilon \end{gathered} \tag{3.12}$$

for all $x_1,\ldots,x_n, x \in G$.

Proof. By letting $x_1 = \cdots = x_n = 0$ in (3.11), we have

$$\left\| 4\ {}_{n-2}C_{n/2-1} f(0) + n\ {}_nC_{n/2} f(0) - 4n^2 \cdot {}_{n-2}C_{n/2-1} f(0) \right\| = \left\| 4n(n-1)_{n-2}C_{n/2-1} f(0) \right\| \leq \varepsilon, \tag{3.13}$$

Thus $\|f(0)\| \leq \varepsilon/4n(n-1)_{n-2}C_{n/2-1}$. Now, let $x_k = x$ $(k = 1,\ldots,n)$ in (3.11). From the inequality $\|f(0)\| \leq \varepsilon/4n(n-1)_{n-2}C_{n/2-1}$, we have

$$\left\| 4\ {}_{n-2}C_{n/2-1} f(nx) - 4n^2 \cdot {}_{n-2}C_{n/2-1} f(x) \right\| \leq \frac{n+1}{n}\varepsilon. \tag{3.14}$$

Thus we obtain

$$\left\| f(nx) - n^2 f(x) \right\| \leq \frac{n+1}{4n\ {}_{n-2}C_{n/2-1}}\varepsilon \tag{3.15}$$

for all $x \in G$. To apply Lemma 3.4 for the function f, we may let

$$\begin{gathered} l = \frac{1}{4}, \quad \delta = \frac{n+1}{4n\ {}_{n-2}C_{n/2-1}}\varepsilon, \quad K = {}_nC_{n/2}, \\ a_1 = \cdots = a_K = id_X, \quad a_{K+1} = 4\ {}_{n-2}C_{n/2-1} id_X, \\ a_{K+2} = \cdots = a_{K+(n+1)} = -4n\ {}_{n-2}C_{n/2-1} id_X, \quad \text{where } N = K+n+1. \end{gathered} \tag{3.16}$$

Then we have

$$\delta = \frac{n+1}{4n\ {}_{n-2}C_{n/2-1}}\varepsilon \le \frac{n+1}{4(n^3+n^2+4n+1)\ {}_{n-2}C_{n/2-1}} \cdot \frac{3}{4}R < \frac{3}{4}R = (1-l)R,$$
$$\sum_{i=1}^{K}\mathrm{lip}(a_i)\delta = K \cdot \frac{n+1}{4n\ {}_{n-2}C_{n/2-1}}\varepsilon \le \frac{n^2-1}{n^3+n^2+4n+1} \cdot \frac{3}{4}R \le \frac{3}{4}R = (1-l)R, \tag{3.17}$$
$$\varepsilon + \sum_{i=K+1}^{N}\mathrm{lip}(a_i)\frac{l\delta}{1-l} = \varepsilon + \left(4_{n-2}C_{n/2-1} + 4n^2\ {}_{n-2}C_{n/2-1}\right)\frac{\delta}{3} = \varepsilon \cdot \frac{n^3+n^2+4n+1}{3n} \le \frac{1}{4}R = lR.$$

Thus we also obtain $\mathrm{lip}_R([n_X]^{-1}) \le l$, and so all conditions of Lemma 3.4 are satisfied. Hence we conclude that there exists a unique function $F : G \to X$ such that

$$F(nx) = n^2F(x),$$
$$4_{n-2}C_{n/2-1}F\left(\sum_{j=1}^{n}x_j\right) + n\sum_{\substack{i_k\in\{0,1\}\\ \sum_{k=1}^{n} i_k=n/2}} F\left(\sum_{i=1}^{n}(-1)^{i_k}x_i\right) = 4n\cdot_{n-2}C_{n/2-1}\sum_{i=1}^{n}F(x_i), \tag{3.18}$$

and also we have

$$\|f(x) - F(x)\| \le \frac{n+1}{12n\ {}_{n-2}C_{n/2-1}}\varepsilon \quad \text{for all } x_1,\ldots,x_n,x \in G. \tag{3.19}$$

□

Theorem 3.6. *Suppose that $[(2n\ {}_{n-2}C_{n/2-1})_X]$ is locally $(R/2n\ {}_{n-2}C_{n/2-1}, R)$-invertible, $[(n\ {}_{n-1}C_{n/2-1})_X]$ is locally $(R/n\ {}_{n-1}C_{n/2-1}, R)$-invertible, and $[(4n(n-1)_{n-2}C_{n/2-1})_X]$ is locally $(R/4n(n-1)\ {}_{n-1}C_{n/2-1}, R)$-invertible. If a function $f : G \to X$ satisfies the following equation:*

$$Df(x_1,\ldots,x_n) = 0 \tag{3.20}$$

for all $x_1,\ldots,x_n \in G$ then f is a quadratic function.

Proof. By letting $x_k = 0$ $(k = 1,\ldots,n)$ in (3.20), we have

$$4n(n-1)_{n-2}C_{n/2-1}f(0) = 0. \tag{3.21}$$

By the uniqueness of the local division by $4n(n-1)\ {}_{n-2}C_{n/2-1}$, we get $f(0) = 0$. Also, setting $x_1 = x, x_k = 0$ $(k = 2,\ldots,n)$ in (3.20), $f(0) = 0$ implies that

$$4\ {}_{n-2}C_{n/2-1}f(x) + n\ {}_{n-1}C_{n/2}f(x) + n\ {}_{n-1}C_{n/2-1}f(-x) = 4n\cdot_{n-2}C_{n/2-1}f(x), \tag{3.22}$$

that is, we have

$$n\ {}_{n-1}C_{n/2}f(x) = n\ {}_{n-1}C_{n/2-1}f(-x) \tag{3.23}$$

for all $x \in G$. By the uniqueness of the local division by $n\ {}_{n-1}C_{n/2-1}$, we get $f(x) = f(-x)$ for all $x \in G$. Now, by letting $x_1 = x, x_2 = y$, and $x_3 = \ldots = x_n = 0$ in (3.20), we get

$$\begin{aligned} &4\ {}_{n-2}C_{n/2-1}f(x+y) + 2n\ {}_{n-2}C_{n/2}f(x+y) + 2n\ {}_{n-2}C_{n/2-1}f(x-y) \\ &\quad = 4n\cdot{}_{n-2}C_{n/2-1}(f(x)+f(y)) \end{aligned} \tag{3.24}$$

for all $x, y \in G$. By the uniqueness of the local division by $2n\ {}_{n-2}C_{n/2-1}$, we have

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{3.25}$$

for all $x, y \in G$. Hence f is a quadratic mapping which completes the proof. □

Theorems 3.5 and 3.6 yield the following corollary.

Corollary 3.7. *Let $f : G \to X$ be a function satisfying (3.11), and let $\varepsilon \leq 3n/4(n^3 + n^2 + 4n + 1)R$ be arbitrary. Suppose that $[(4n(n-1)\ {}_{n-2}C_{n/2-1})_X]$ is locally $(R/4n(n-1)\ {}_{n-1}C_{n/2-1}, R)$-invertible, $[(n\ {}_{n-1}C_{n/2-1})_X]$ is locally $(R/n\ {}_{n-1}C_{n/2-1}, R)$-invertible, and $[(2n\ {}_{n-2}C_{n/2-1})_X]$ is locally $(R/2n\ {}_{n-2}C_{n/2-1}, R)$-invertible. Then there exists a quadratic function $F : G \to X$ such that*

$$\|F(x) - f(x)\| \leq \frac{n+1}{12n\ {}_{n-2}C_{n/2-1}}\varepsilon \tag{3.26}$$

for all $x \in G$.

4. On Hyers-Ulam-Rassias Stabilities

In this section, let X be a normed vector space with norm $\|\cdot\|$, Y a Banach space with norm $\|\cdot\|$ and $n \geq 2$ an even integer. For the given mapping $f : X \to Y$, we define

$$\begin{aligned} &Df(x_1, \ldots, x_n) \\ &\quad := 4\ {}_{n-2}C_{n/2-1}r^2 f\left(\sum_{j=1}^{n}\frac{x_j}{r}\right) + n \sum_{\substack{i_k\in\{0,1\} \\ \sum_{k=1}^{n} i_k = n/2}} r^2 f\left(\sum_{i=1}^{n}(-1)^{i_k}\frac{x_i}{r}\right) - 4n\cdot{}_{n-2}C_{n/2-1}\sum_{i=1}^{n} f(x_i), \end{aligned} \tag{4.1}$$

for all $x_1, \ldots, x_n \in X$.

Theorem 4.1. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$. Assume that there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\tilde{\phi}(x_1, \ldots, x_n) := \sum_{j=0}^{\infty}\left(\frac{r}{2}\right)^{2j}\phi\left(\left(\frac{2}{r}\right)^{j}x_1, \ldots, \left(\frac{2}{r}\right)^{j}x_n\right) < \infty, \tag{4.2}$$

$$\|Df(x_1,\ldots,x_n)\| \le \phi(x_1,\ldots,x_n) \tag{4.3}$$

for all $x_1,\ldots,x_n \in X$. *Then there exists a unique generalized quadratic mapping of* r*-type* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \le \frac{1}{8n\cdot{}_{n-2}C_{n/2-1}}\widetilde{\phi}(x,x,0,\ldots,0) \tag{4.4}$$

for all $x \in X$.

Proof. By letting $x_1 = x_2 = x$ and $x_j = 0\ (j = 3,\ldots,n)$ in (4.3), since f is an even mapping and ${}_{n-2}C_{n/2} = n-2/n\cdot{}_{n-2}C_{n/2-1}$, we have

$$\begin{aligned}&\left\|4\ {}_{n-2}C_{n/2-1}r^2 f\left(\frac{2}{r}x\right) + 2n\ {}_{n-2}C_{n/2}r^2 f\left(\frac{2}{r}x\right) - 8n\cdot{}_{n-2}C_{n/2-1}f(x)\right\|\\ &\quad= 8n\ {}_{n-2}C_{n/2-1}\left\|\left(\frac{r}{2}\right)^2 f\left(\frac{2}{r}x\right) - f(x)\right\|\\ &\quad\le \phi(x,x,0,\ldots,0)\end{aligned} \tag{4.5}$$

for all $x \in X$. Then we obtain that

$$\left\|f(x) - \left(\frac{r}{2}\right)^2 f\left(\frac{2}{r}x\right)\right\| \le \frac{1}{8n\cdot{}_{n-2}C_{n/2-1}}\phi(x,x,0,\ldots,0), \tag{4.6}$$

for all $x \in X$.

Using (4.6), we have

$$\begin{aligned}&\left\|\left(\frac{r}{2}\right)^{2d} f\left(\left(\frac{2}{r}\right)^d x\right) - \left(\frac{r}{2}\right)^{2(d+1)} f\left(\left(\frac{2}{r}\right)^{d+1} x\right)\right\|\\ &\quad\le \left(\frac{r}{2}\right)^{2d}\cdot\frac{1}{8n}\cdot\frac{1}{{}_{n-2}C_{n/2-1}}\phi\left(\left(\frac{2}{r}\right)^d x, \left(\frac{2}{r}\right)^d x, 0,\ldots,0\right)\end{aligned} \tag{4.7}$$

for all $x \in X$ and all positive integer d. Hence we get

$$\begin{aligned}&\left\|\left(\frac{r}{2}\right)^{2s} f\left(\left(\frac{2}{r}\right)^s x\right) - \left(\frac{r}{2}\right)^{2d} f\left(\left(\frac{2}{r}\right)^d x\right)\right\|\\ &\quad\le \sum_{j=s}^{d-1}\left(\frac{r}{2}\right)^{2j}\cdot\frac{1}{8n}\cdot\frac{1}{{}_{n-2}C_{n/2-1}}\phi\left(\left(\frac{2}{r}\right)^j x, \left(\frac{2}{r}\right)^j x, 0,\ldots,0\right)\end{aligned} \tag{4.8}$$

for all $x \in X$ and all positive integers s and d with $s < d$. Hence the sequence $\{(r/2)^{2s} f((2/r)^s x)\}$ is a Cauchy sequence. From the completeness of Y, we may define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{s\to\infty} \left(\frac{r}{2}\right)^{2s} f\left(\left(\frac{2}{r}\right)^s x\right) \tag{4.9}$$

for all $x \in X$. Since f is even, so is Q. By the definition of $DQ(x_1, \ldots, x_n)$ and (4.3), we have that

$$\begin{aligned} \|DQ(x_1, \ldots, x_n)\| &= \lim_{s\to\infty} \left(\frac{r}{2}\right)^{2s} \left\| Df\left(\left(\frac{2}{r}\right)^s x_1, \ldots, \left(\frac{2}{r}\right)^s x_n\right)\right\| \\ &\le \lim_{s\to\infty} \left(\frac{r}{2}\right)^{2s} \phi\left(\left(\frac{2}{r}\right)^s x_1, \ldots, \left(\frac{2}{r}\right)^s x_n\right) = 0 \end{aligned} \tag{4.10}$$

for all $x_1, \ldots, x_n \in X$. Since $DQ(x_1, \ldots, x_n) = 0$, the mapping $Q : X \to Y$ is a generalized quadratic mapping of r-type by Lemma 2.1. Also, letting $s = 0$ and passing the limit $d \to \infty$ in (4.8), we get (4.4).

To prove the uniqueness, suppose that $Q' : X \to Y$ is another generalized quadratic mapping of r-type satisfying (4.4). Then we have

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \left(\frac{r}{2}\right)^{2s} \left\| Q\left(\left(\frac{2}{r}\right)^s x\right) - Q'\left(\left(\frac{2}{r}\right)^s x\right)\right\| \\ &\le \left(\frac{r}{2}\right)^{2s} \left(\left\| Q\left(\left(\frac{2}{r}\right)^s x\right) - f\left(\left(\frac{2}{r}\right)^s x\right)\right\| + \left\| Q'\left(\left(\frac{2}{r}\right)^s x\right) - f\left(\left(\frac{2}{r}\right)^s x\right)\right\| \right) \\ &\le 2 \cdot \frac{1}{8n \cdot {}_{n-2}C_{\mathrm{n}/2-1}} \cdot \left(\frac{r}{2}\right)^{2s} \tilde{\phi}\left(\left(\frac{2}{r}\right)^s x, \left(\frac{2}{r}\right)^s x, 0, \ldots, 0\right) \\ &= \frac{1}{4n} \cdot \frac{1}{{}_{n-2}C_{n/2-1}} \sum_{j=s}^{\infty} \left(\frac{r}{2}\right)^{2j} \phi\left(\left(\frac{2}{r}\right)^j x, \left(\frac{2}{r}\right)^j x, 0, \ldots, 0\right) \longrightarrow 0 \end{aligned} \tag{4.11}$$

for all $x \in X$ as $s \to \infty$. Thus the generalized quadratic mapping Q is unique. □

Theorem 4.2. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$. Assume that there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\tilde{\phi}(x_1, \ldots, x_n) := \sum_{j=1}^{\infty} \left(\frac{2}{r}\right)^{2j} \phi\left(\left(\frac{r}{2}\right)^j x_1, \ldots, \left(\frac{r}{2}\right)^j x_n\right) < \infty, \tag{4.12}$$

$$\|Df(x_1, \ldots, x_n)\| \le \phi(x_1, \ldots, x_n) \tag{4.13}$$

for all $x_1, \ldots, x_n \in X$. *Then there exists a unique generalized quadratic mapping of* r*-type* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \le \frac{1}{8n \cdot {}_{n-2}C_{n/2-1}} \widetilde{\phi}(x, x, 0, \ldots, 0) \tag{4.14}$$

for all $x \in X$.

Proof. If x is replaced by $(r/2)x$ in the inequality (4.6), then the proof follows from the proof of Theorem 4.1. □

Acknowledgment

The authors would like to thank the referee for his (or her) constructive comments and suggestions.

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