Research Article

On Convergence of q-Series Involving $_{r+1}\phi_r$ Basic Hypergeometric Series

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We use inequality technique and the terminating case of the q-binomial formula to give some results on convergence of q-series involving $_{r+1}\phi_r$ basic hypergeometric series. As an application of the results, we discuss the convergence for special Thomae q-integral.

1. Introduction

q-Series, which are also called basic hypergeometric series, play a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics. Convergence of a q-series is an important problem in the study of q-series. There are some results about it in [1–3]. For example, Ito used inequality technique to give a sufficient condition for convergence of a special q-series called Jackson integral. In this paper, by using inequality technique, we derive the following two theorems on convergence of q-series involving $_{r+1}\phi_r$ basic hypergeometric series, which can be used for convergence of special Thomae q-integral.

2. Notations and Known Results

We recall some definitions, notations, and known results which will be used in the proofs. Throughout this paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1 - aq^k\right), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}\left(1 - aq^k\right). \qquad (2.1)$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \tag{2.2}$$

where n is an integer or ∞.

The q-binomial theorem [4, 5] is

$$\sum_{k=0}^{\infty} \frac{(a; q)_k z^k}{(q; q)_k} = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1, \ |q| < 1. \tag{2.3}$$

When $a = q^{-n}$, where n denotes a nonnegative integer

$$\sum_{k=0}^{n} \frac{(q^{-n}; q)_k z^k}{(q; q)_k} = (zq^{-n}; q)_n. \tag{2.4}$$

Heine introduced the $_{r+1}\phi_r$ basic hypergeometric series, which is defined by [4, 5]

$$_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{array} ; \ q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n z^n}{(q, b_1, b_2, \ldots, b_r; q)_n}. \tag{2.5}$$

3. Main Results

The main purpose of the present paper is to establish the following two theorems on convergence of q-series involving $_{r+1}\phi_r$ basic hypergeometric series.

Theorem 3.1. *Suppose a_i, b_i, t are any real numbers such that $t > 0$ and $b_i < 1$ with $i = 1, 2, \ldots, r$. Let $\{c_n\}$ be any sequence of numbers. If*

$$\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = p < 1, \tag{3.1}$$

then the q-series

$$\sum_{n=0}^{\infty} c_n \cdot {}_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array} ; \ q, tq^n \right) \tag{3.2}$$

converges absolutely.

Proof. Let $b < 1$ and

$$f(t) = \frac{1 - at}{1 - bt}, \quad 0 \le t \le 1, \tag{3.3}$$

It is easy to see that $f(t)$ is a monotone function with respect to $0 \le t \le 1$.

Consequently, one has

$$\left|\frac{1-at}{1-bt}\right| \le \max\left\{1, \frac{|1-a|}{1-b}\right\}. \tag{3.4}$$

From (3.4), one knows

$$\left|\frac{(a_i;q)_k}{(b_i;q)_k}\right| = \left|\frac{1-a_i}{1-b_i}\right| \cdot \left|\frac{1-a_iq}{1-b_iq}\right| \cdots \left|\frac{1-a_iq^{k-1}}{1-b_iq^{k-1}}\right| \le M_i^k, \tag{3.5}$$

where $M_i = \max\{1, |1-a_i|/(1-b_i)\}$ for $i = 1, 2, \ldots, r$.

So, one has

$$\left|\frac{(a_1, a_2, \ldots, a_r; q)_k(-1)^k}{(b_1, b_2, \ldots, b_r; q)_k}\right| = \left|\frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_r; q)_k}\right| \le \left(\prod_{i=1}^{r} M_i\right)^k. \tag{3.6}$$

It is obvious that

$$\frac{(q^{-n};q)_k(-tq^n)^k}{(q;q)_k} > 0, \quad t > 0, \ k = 1, 2, \ldots, n. \tag{3.7}$$

Multiplying both sides of (3.6) by

$$\frac{(q^{-n};q)_k(-tq^n)^k}{(q;q)_k} \tag{3.8}$$

gives

$$\left|\frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k}\right| \le \frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n\prod_{i=1}^{r} M_i\right)^k. \tag{3.9}$$

Hence,

$$\begin{aligned}
\left|_{r+1}\phi_r\left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array}; q, tq^n\right)\right| &= \left|\sum_{k=0}^{n}\frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k}\right| \\
&\le \sum_{k=0}^{n}\left|\frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k}\right| \\
&\le \sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n\prod_{i=1}^{r} M_i\right)^k.
\end{aligned} \tag{3.10}$$

By using (2.4) one obtains

$$\sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} \left(-tq^n \prod_{i=1}^{r} M_i \right)^k = \left(-t \prod_{i=1}^{r} M_i; q \right)_n. \tag{3.11}$$

Substituting (3.11) into (3.10), one has

$$\left| {}_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array} ; q, tq^n \right) \right| \leq \left(-t \prod_{i=1}^{r} M_i; q \right)_n. \tag{3.12}$$

Multiplying both sides of (3.12) by $|c_n|$, one has

$$\left| c_n \cdot {}_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array} ; q, tq^n \right) \right| \leq |c_n| \left(-t \prod_{i=1}^{r} M_i; q \right)_n. \tag{3.13}$$

The ratio test shows that the series

$$\sum_{n=0}^{\infty} c_n \left(-t \prod_{i=1}^{r} M_i; q \right)_n \tag{3.14}$$

is absolutely convergent. From (3.13), it is sufficient to establish that (3.2) is absolutely convergent. □

Theorem 3.2. *Suppose a_i, b_i, t are any real numbers such that $t > 0$ and $a_i < 1$, $b_i < 1$ with $i = 1, 2, \ldots, r$. Let $\{c_n\}$ be any sequence of numbers. If*

$$\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = p > 1, \quad or \quad \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = +\infty, \tag{3.15}$$

then the q-series

$$\sum_{n=0}^{\infty} c_n \cdot {}_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array} ; q, -tq^n \right) \tag{3.16}$$

diverges.

Proof. Let $a < 1$, $b < 1$ and

$$f(t) = \frac{1 - at}{1 - bt}, \quad 0 \leq t \leq 1, \tag{3.17}$$

It is easy to see that $f(t)$ is a monotone function with respect to $0 \leq t \leq 1$.

Consequently, one has

$$\frac{1-at}{1-bt} \geq \min\left\{1, \frac{1-a}{1-b}\right\}. \tag{3.18}$$

From (3.18), one knows

$$\frac{(a_i;q)_k}{(b_i;q)_k} = \frac{1-a_i}{1-b_i} \cdot \frac{1-a_iq}{1-b_iq} \cdots \frac{1-a_iq^{k-1}}{1-b_iq^{k-1}} \geq m_i^k, \tag{3.19}$$

where $m_i = \min\{1, (1-a_i)/(1-b_i)\}$ for $i = 1, 2, \ldots, r$.

So, one has

$$\frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_r; q)_k} \geq \left(\prod_{i=1}^{r} m_i\right)^k. \tag{3.20}$$

It is obvious that

$$\frac{(q^{-n};q)_k(-tq^n)^k}{(q;q)_k} > 0, \quad t > 0, \ k = 1, 2, \ldots, n. \tag{3.21}$$

Multiplying both sides of (3.20) by

$$\frac{(q^{-n};q)_k(-tq^n)^k}{(q;q)_k} \tag{3.22}$$

gives

$$\frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(-tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k} \geq \frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n\prod_{i=1}^{r} m_i\right)^k. \tag{3.23}$$

Hence,

$$\begin{aligned}
{}_{r+1}\phi_r\left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array}; \ q, -tq^n\right) &= \sum_{k=0}^{n} \frac{(q^{-n}, a_1, a_2, \ldots, a_r; q)_k(-tq^n)^k}{(q, b_1, b_2, \ldots, b_r; q)_k} \\
&\geq \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n\prod_{i=1}^{r} m_i\right)^k.
\end{aligned} \tag{3.24}$$

By using (2.4) one obtains

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}\left(-tq^n\prod_{i=1}^{r} m_i\right)^k = \left(-t\prod_{i=1}^{r} m_i; q\right)_n. \tag{3.25}$$

Substituting (3.25) into (3.24), one has

$$
{}_{r+1}\phi_r\left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array} ; \ q, -tq^n\right) \geq \left(-t\prod_{i=1}^{r} m_i; q\right)_n. \tag{3.26}
$$

Multiplying both sides of (3.26) by $|c_n|$, one has

$$
|c_n| \cdot {}_{r+1}\phi_r\left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array} ; \ q, -tq^n\right) \geq |c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n. \tag{3.27}
$$

Since

$$
\lim_{n \to \infty} \frac{|c_{n+1}|\left(-t\prod_{i=1}^{r} m_i; q\right)_{n+1}}{|c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n} = \lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right|. \tag{3.28}
$$

By hypothesis

$$
\lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right| = p > 1, \quad \text{or} \quad \lim_{n \to \infty} \left|\frac{c_{n+1}}{c_n}\right| = +\infty, \tag{3.29}
$$

therefore, in both cases there exists a integer $N_0 > 0$ such that $\forall n > N_0$

$$
\frac{|c_{n+1}|\left(-t\prod_{i=1}^{r} m_i; q\right)_{n+1}}{|c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n} > 1. \tag{3.30}
$$

So, one can conclude that

$$
|c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n > |c_{N_0}|\left(-t\prod_{i=1}^{r} m_i; q\right)_{N_0}, \quad \forall n > N_0. \tag{3.31}
$$

Now, from (3.27) and (3.31)

$$
|c_n| \cdot {}_{r+1}\phi_r\left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array} ; \ q, -tq^n\right) \geq |c_n|\left(-t\prod_{i=1}^{r} m_i; q\right)_n
$$

$$
> |c_{N_0}|\left(-t\prod_{i=1}^{r} m_i; q\right)_{N_0} \tag{3.32}
$$

$$
> 0.
$$

Thereby, (3.16) diverges. $\square$

We want to point out that some q-integral can be written as (3.2) or (3.16). So, the results obtained here can be used to discuss the convergence of q-integrals.

4. Some Applications

In [6, 7], Thomae defined the q-integral on the interval $[0, 1]$ by

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n. \tag{4.1}$$

The right side of (4.1) corresponds to use a Riemann sum with partition points $t_n = q^n$, $n = 0, 1, 2, \ldots$. Jackson [8] extended Thomae q-integral via

$$\int_0^d f(t)d_qt = d(1-q)\sum_0^{\infty} f(dq^n)q^n,$$

$$\int_c^d f(t)d_qt = \int_0^d f(t)d_qt - \int_0^c f(t)d_qt. \tag{4.2}$$

In this section, we use the theorems derived in this paper to discuss two examples of the convergence for Thomae q-integral. We have the following theorems.

Theorem 4.1. *Let a_i, b_i, t be any real numbers such that $t > 0$ and $b_i < 1$ with $i = 1, 2, \ldots, r$. If $\alpha > -1$, then the Thomae q-integral*

$$\int_0^1 t^\alpha \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, t^{-1} \\ b_1, b_2, \ldots, b_r \end{matrix} \; ; \; q, t \right)d_qt \tag{4.3}$$

converges absolutely.

Proof. By the definition of Thomae q-integral (4.1), one has

$$\int_0^1 t^\alpha \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, t^{-1} \\ b_1, b_2, \ldots, b_r \end{matrix} \; ; \; q, t \right)d_qt$$

$$= (1-q)\sum_{n=0}^{\infty} q^{n(1+\alpha)} \, {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{matrix} \; ; \; q, q^n \right). \tag{4.4}$$

Using Theorem 3.1 and noticing,

$$\lim_{n \to \infty} \frac{q^{(n+1)(1+\alpha)}}{q^{n(1+\alpha)}} = q^{1+\alpha} < 1, \tag{4.5}$$

one knows that (4.3) converges absolutely. □

Theorem 4.2. *Let a_i, b_i, t be any real numbers such that $t > 0$ and $a_i < 1$, $b_i < 1$ with $i = 1, 2, \ldots, r$. If $\alpha > 1$, then the Thomae q-integral*

$$\int_0^1 t^{-\alpha} \cdot {}_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \ldots, a_r, t^{-1} \\ b_1, b_2, \ldots, b_r \end{array} \; ; \; q, -t \right) d_q t \tag{4.6}$$

diverges.

Proof. By the definition of Thomae q-integral (4.1), one has

$$\int_0^1 t^{-\alpha} \cdot {}_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \ldots, a_r, t^{-1} \\ b_1, b_2, \ldots, b_r \end{array} \; ; \; q, -t \right) d_q t$$
$$= (1 - q) \sum_{n=0}^{\infty} q^{(1-\alpha)n} {}_{r+1}\phi_r \left(\begin{array}{c} a_1, a_2, \ldots, a_r, q^{-n} \\ b_1, b_2, \ldots, b_r \end{array} \; ; \; q, -q^n \right). \tag{4.7}$$

Using Theorem 3.2 and noticing,

$$\lim_{n \to \infty} \frac{q^{(1-\alpha)(n+1)}}{q^{(1-\alpha)n}} = q^{1-\alpha} > 1, \tag{4.8}$$

one knows that (4.6) diverges. $\qquad\square$

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