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Research Article

Endpoint Estimates for a Class of Littlewood-Paley Operators with Nondoubling Measures

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Let μ be a positive Radon measure on $\mathbb{R}^d$ which may be nondoubling. The only condition that μ satisfies is $\mu(B(x,r)) \leq C_0 r^n$ for all $x \in \mathbb{R}^d$, $r > 0$, and some fixed constant C_0. In this paper, we introduce the operator $g^*_{\lambda,\mu}$ related to such a measure and assume it is bounded on $L^2(\mu)$. We then establish its boundedness, respectively, from the Lebesgue space $L^1(\mu)$ to the weak Lebesgue space $L^{1,\infty}(\mu)$, from the Hardy space $H^1(\mu)$ to $L^1(\mu)$ and from the Lesesgue space $L^\infty(\mu)$ to the space $\mathrm{RBLO}(\mu)$. As a corollary, we obtain the boundedness of $g^*_{\lambda,\mu}$ in the Lebesgue space $L^p(\mu)$ with $p \in (1,\infty)$.

1. Introduction

A positive Radon measure μ on R^d is said to be doubling if there exists some constant C such that $\mu(B(x,2r)) \leq C\mu(B(x,r))$ for all $x \in \mathrm{supp}(\mu)$, $r > 0$. It is well known that the doubling condition is an essential assumption in many results of classical Calderón-Zygmund theory. However in the recent years, it has been shown that a big part of the classical theory remains valid if the doubling assumption on μ is substituted by the growth condition as follows:

$$\mu(B(x,r)) \leq C_0 r^n \tag{1.1}$$

for all $x \in R^d$, where n is some fixed number with $0 < n \leq d$. For example, In 2001, Tolsa in [1, 2] investigated the weak (1,1) inequality for singular integrals, the Littlewood-Paley theory and the $T(1)$ theorem with nondoubling measures. In 2002, García-Cuerva and Gatto [3] investigated the boundedness properties of fractional integral operators associated to nondoubling measures. In 2005, Hu et al. [4] studied the multilinear commutators of singular integrals with nondoubling measures. Since 2007, Hu et al. [5] have proved some boundedness results of Marcinkiewicz integrals with nondoubling measures on some function spaces.

On the other hand, let ψ be a function on $\mathbb{R}^d$ such that there exist positive constants C_0, C_1, δ, and γ satisfying

(a) $\psi \in L^1(\mathbb{R}^d)$ and $\int_{R^d}\psi(x)d\mu(x) = 0$,

(b) $|\psi(x)| \le C_0(1+|x|)^{-n-\delta}$,

(c) $|\psi(x+y)-\psi(x)| \le C_1|y|^{\gamma}(1+|x|)^{-n-\gamma-\delta}$ for $2|y| \le |x|$.

For this ψ, we define the Littlewood-Paley's $g^*_{\lambda,\mu}$-function with nondoubling measures as follows:

$$g^*_{\lambda,\mu}(f)(x) = \left(\int_{\mathbb{R}^{d+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\psi_t * f(y)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2}, \quad \lambda > 1, \tag{1.2}$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ and $\psi_t * f(y) = \int_{\mathbb{R}^d}\psi_t(y-z)f(z)d\mu(z)$.

Note that if we replace $d\mu(y)$ by dy in the above definition and when $\psi_t = P_t$ is the Poisson kernel, we obtain classical g^*_λ-function defined and studied by Stein [6] and later by Fefferman [7], where the weak (1,1) with $\lambda > 2$ and weak (p,p) with $\lambda = 2/p$ boundedness of g^*_λ function were obtained. In the same paper, Fefferman [7] also established the L^p bounds of g^*_λ for $1 < p < \infty$ and $\lambda > \max\{1, 2/p\}$. For the more generalized g^*_λ-function defined by (1.1), the L^p boundedness is also well known (see, e.g., [8, pages 309–318]). On the other hand, inspired by the works of Sakamoto and Yabuta in 1999, the first author in this paper studied parametric g^*_λ-function systematically in his PhD thesis [9]. Later, in 2008, Lin and Meng [10] gave some results on parametric g^*_λ-function with nondoubling measures. But their result only valid for $\rho > n/2$, one cannot obtain the results for classical operators even for $\rho = 1$ or in the classical case studied by Stein in 1961 [6].

In this paper, we will study the properties of operator $g^*_{\lambda,\mu}$ with nondoubling measures on some function spaces under the conditions (a)–(c).

First, before stating our main results, we give some notation and definitions, let $Q \subset \mathbb{R}^n$ be a closed cube with sides parallel to the axes. Denote its side length by $l(Q)$ and its center by x_Q. Given $\alpha > 1$ and $\beta > \alpha^n$, we say Q is (α,β)-doubling if $\mu(\alpha Q) \le \beta\mu(Q)$, where αQ is the cube concentric with Q with side length $\alpha l(Q)$. If α,β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube. For any cube Q, we denote by $\widetilde{Q}$ the smallest doubling cube which contains Q and has the same center as Q.

Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$S_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}}\frac{\mu(2^kQ)}{l(2^kQ)^n}, \tag{1.3}$$

where $N_{Q,R}$ is the first integer k such that $l(2^kQ) \geq l(R)$ and

$$S^{\alpha}_{Q,R} = 1 + \sum_{k=1}^{N^{\alpha}_{Q,R}} \frac{\mu(\alpha^k Q)}{l(\alpha^k Q)^n} \tag{1.4}$$

with $\alpha > 1$, where $N^{\alpha}_{Q,R}$ is the first integer k such that $l(\alpha^k Q) \geq l(R)$.

In the article of [1, page 95], we know that $K_{Q,R} \approx K^{\alpha}_{Q,R}$ with constants that may depend on α and C_0. The following atomic Hardy space $H^{1,\infty}_{atb}$ was introduced by Tolsa in [11].

Definition 1.1. For a fixed $\rho > 1$, a function $b \in L^1_{\mathrm{loc}}(\mu)$ is called an atomic block if

(1) there exists some cube R such that $\mathrm{supp}(b) \subset R$;

(2) $\int_{\mathbb{R}^d} b\, d\mu = 0$;

(3) there are fucntions a_j with supports in cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that

$$b = \sum_j \lambda_j a_j, \qquad \|a_j\|_{L^\infty(\mu)} \leq \left[\mu(\rho Q_j) S_{Q,R}\right]^{-1}. \tag{1.5}$$

Define

$$|b|_{H^{1,\infty}_{atb}(\mu)} = \sum_j |\lambda_j|. \tag{1.6}$$

We say that $f \in H^{1,\infty}_{atb}(\mu)$ if there are atomic blocks $\{b_i\}_i$ such that $f = \sum_{i=1}^{\infty} b_i$ with $\sum_i |b_i|_{H^{1,\infty}_{atb}} < \infty$. The $H^{1,\infty}_{atb}$ norm of f is defined by

$$\|f\|_{H^{1,\infty}_{atb}(\mu)} = \inf \sum_i |b_i|_{H^{1,\infty}_{atb}(\mu)}, \tag{1.7}$$

where the infimum is taken over all the possible decompositions of f in atomic blocks.

It was shown by Tolsa that the space $H^{1,\infty}_{atb}(\mu)$ was proved to be the Hardy space $H^1(\mu)$ in [11] with equivalent norms. We will denote the space $H^{1,\infty}_{atb}(\mu)$ and the norm $\|\cdot\|_{H^{1,\infty}_{atb}(\mu)}$, respectively, by $H^1(\mu)$ and $\|\cdot\|_{H^1(\mu)}$ for convenience. He also proved that the dual space of $H^1(\mu)$ is the following space $\mathrm{RBMO}(\mu)$.

Definition 1.2. Let $\rho > 1$ be a fixed constant. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\mathrm{RBMO}(\mu)$ if there exists some constant $C > 0$ such that for any cube Q centered at some point of $\mathrm{supp}(\mu)$

$$\frac{1}{\mu(\rho Q)} \int_Q \left| f(y) - m_{\tilde{Q}}(f) \right| d\mu \leq C \tag{1.8}$$

and for any two doubling cubes $Q \subset R$

$$\left|m_Q(f) - m_R(f)\right| \leq C S_{Q,R}, \tag{1.9}$$

where $m_Q(f)$ denotes the mean value of f over cube Q. The minimal constant C above is defined to be the norm of f in the space RBMO(μ) and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

Tolsa in [11] proved that the definition of the space $H^{1,\infty}_{atb}(\mu)$ and RBMO(μ) are independent of the choice of ρ. The following space RBLO(μ) was introduced in [12]. It is easy to see that RBLO(μ) $\subset$ RBMO(μ).

Definition 1.3. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space RBLO(μ) if there exists some positive constant C such that for any $((4\sqrt{d}), (4\sqrt{d})^{n+1})$ doubling cube Q,

$$m_Q(f) - \operatorname*{ess\,inf}_{x \in Q} f(x) \leq C \tag{1.10}$$

and for any two $((4\sqrt{d}), (4\sqrt{d})^{n+1})$-doubling cubes $Q \subset R$,

$$m_Q(f) - m_R(f) \leq C S_{Q,R}. \tag{1.11}$$

The minimal constant C as above is defined to be the norm of f in the space RBLO(μ), we denote it by $\|f\|_{\mathrm{RBLO}(\mu)}$.

In this paper, we always assume that μ and n are considered as they are defined at the beginning of this paper. Our main results are as follows.

Theorem 1.4. *Let φ be a function on $\mathbb{R}^d$, satisfying (a)–(c), $\lambda > 2$, $0 < \gamma < \min\{(\lambda - 2)n/2, \delta\}$. If $g^*_{\lambda,\mu}$ is bounded on $L^2(\mu)$, then it is also bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.*

Theorem 1.5. *Let φ be a function on $\mathbb{R}^d$, satisfying (a)–(c), $\lambda > 2$, $0 < \gamma < \min\{(\lambda - 2)n/2, \delta\}$. If $g^*_{\lambda,\mu}$ is bounded on $L^2(\mu)$, then it is also bounded from $H^1(\mu)$ to $L^1(\mu)$.*

Theorem 1.6. *Let φ be a function on $\mathbb{R}^d$, satisfying (a)–(c), $\lambda > 2$, $0 < \gamma < \min\{(\lambda - 2)n/2, \delta\}$. If $g^*_{\lambda,\mu}$ is bounded on $L^2(\mu)$, then for $f \in L^\infty(\mu)$, $g^*_{\lambda,\mu}(f)$ is either infinite everywhere or finite almost everywhere. More precisely, if $g^*_{\lambda,\mu}(f)$ is finite at some point $x_0 \in \mathbb{R}^d$, then $g^*_{\lambda,\mu}(f)$ is μ-finite almost everywhere and*

$$\left\|g^*_{\lambda,\mu}(f)\right\|_{RBLO(\mu)} \leq C\|f\|_{L^\infty}(\mu). \tag{1.12}$$

Corollary 1.7. *Let φ be a function on $\mathbb{R}^d$, satisfying (a)–(c), $\lambda > 2$, $0 < \gamma < \min\{(\lambda - 2)n/2, \delta\}$. If $g^*_{\lambda,\mu}$ is bounded on $L^2(\mu)$, then it is also bounded on $L^p(\mu)$ for any $1 < p < \infty$.*

Remark 1.8. It is natural to consider the similar problems with more general rough kernels. However, even in the doubling measure case, if we take $\varphi(x) = (\Omega(x)/|x|^{n-1})\chi_{\{|x|<1\}}$ in (1.1) (in this case, g^*_λ is defined and studied by [13]), from the results in [8], we know that it is impossible to give similar results as above for Littlewood-Paley $g^*_{\lambda,\mu}$ function even $\Omega \in \mathrm{Lip}_\alpha$ $(0 < \alpha < 1)$ for $n > 2$. In fact, by the counter example in [13], even the L^p $(1 < p < 2n/(n+2))$ boundedness does not hold. In this sense, the condition we assumed on ψ is necessary and reasonable. On the other hand, in 2008, Lin and Meng [10] gave some results on parametric g^*_λ-function with nondoubling measures. In fact the results in [10] are only valid for $\rho > n/2$. By the same reason as above, one cannot obtain the result when $\rho = 1$ which in this case, the operator coincides with the classical operator studied by Torchinsky and Wang in [13] and it is a generalization of the classical operators studied by Stein and Fefferman.

Remark 1.9. Even in the classical case, the index $\lambda > 2$ is sharp for weak $(1,1)$ boundedness; see [6] for detail.

We arrange our paper as follows, in Section 2, we give and prove some key lemmas. The proof of our main theorems will be given in Section 3. Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables. $A \lesssim B$ will always denote that there exists a constant $C > 0$, such that $A \le CB$.

2. Main Lemmas

We need two lemmas given by Tolsa.

Lemma 2.1 (see [11]). *If $Q \subset R$ are concentric cubes such that there are no (α,β)-doubling cubes with $(\beta > \alpha^n)$ of the form $\alpha^k Q$, $k \ge 1$, with $Q \subset \alpha^k Q \subset R$, then*

$$\int_{R\setminus Q} \frac{1}{|x - x_Q|^n} d\mu(x) \le C_1, \tag{2.1}$$

where C_1 depends only on α, β, n, C_0.

Lemma 2.2 (see [11]). *For any $f \in L^1(\mu)$ and any $\lambda > 0$ $(\lambda > \alpha^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$, if $\|\mu\| < \infty)$ then we have one has the following:*

(a) *there exists a family of almost disjoint cubes $\{Q_i\}_i$ (that means $\sum_i \chi_{Q_i} \le C$, C depends only on d) such that*

  (a.1) $(1/\mu(\alpha Q_i))\int_{Q_i}|f|d\mu > \lambda/\alpha^{d+1}$;

  (a.2) $(1/\mu(\alpha\eta Q_i))\int_{\eta Q_i}|f|d\mu \le \lambda/\alpha^{d+1}$ *for any* $\eta > 2$;

  (a.3) $|f| \le \lambda$ *a.e.* μ *on* $\mathbb{R}^d \setminus \bigcup_i Q_i$,

(b) *for each i, let R_i be the smallest (β,β^{n+1})-doubling cube of the form $\beta^k Q_i$, $k \in \mathbb{N}$, with that $\beta \ge 3\alpha$, and let $w_i = \chi_{Q_i}/\sum_k \chi_{Q_k}$, then there exists a family of functions φ_i with* $\sup(\varphi_i) \subset R_i$ *and with constant sign satisfying*

  (b.1) $\int \varphi_i = \int_{Q_i} f w_i d\mu$;

  (b.2) $\sum_i |\varphi_i| \le B\lambda$, *where B is some constant;*

  (b.3) $\|\varphi_i\|_{L^\infty(\mu)}\mu(R_i) \le C\int_{Q_i}|f|d\mu$.

To prove our theorems, we prepare another two key lemmas.

For any subset $E \subset \mathbb{R}^{d+1}$, we denote

$$\begin{aligned} TE &:= \int_E \left(\frac{t}{t + |x-y|} \right)^{2n+2\epsilon} \left(\frac{t}{t + |y-z|} \right)^{2n+2\epsilon} \frac{d\mu(y)dt}{t^{3n+1}}, \\ \overline{T}E &:= \int_E \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \frac{t^{2\delta}}{(t + |y - x_i|)^{2n+2\gamma+2\delta}} \frac{d\mu(y)dt}{t^{n+1}}. \end{aligned} \tag{2.2}$$

It is easy to see that

$$\int_{\mathbb{R}^{d+1}_+} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \left(\frac{t}{t + |y-z|} \right)^{2n+2\delta} \frac{d\mu(y)dt}{t^{3n+1}} \lesssim T\mathbb{R}^{d+1}_+. \tag{2.3}$$

Lemma 2.3. *Let Q_i and R_i be the same as in Lemma 2.2, and let x_i be Q_i's center. Then for any $z \in Q_i$ and $x \in \alpha R_i \setminus \alpha Q_i$,*

$$T\mathbb{R}^{d+1}_+ \lesssim \frac{1}{|x - x_i|^{2n}}. \tag{2.4}$$

Lemma 2.4. *Let Q_i and R_i be the same as in Lemma 2.2, and let x_i be Q_i's center. For any $z \in Q_i$ and $x \in \alpha R_i \setminus \alpha Q_i$, then*

$$\overline{T}\left\{ (y,t) \in \mathbb{R}^{d+1}_+ : |y - x_i| > 2|x_i - z| \right\} \le \frac{1}{|x - x_i|^{2n}}. \tag{2.5}$$

Proof of Lemma 2.3. Denote

$$A := \left(\frac{t}{t + |x-y|} \right)^{2n+2\epsilon}, \qquad B := \left(\frac{t}{t + |y-z|} \right)^{2n+2\epsilon}. \tag{2.6}$$

Set

$$\begin{aligned} D_1 &= \left\{ (y,t) \in \mathbb{R}^{d+1}_+ : t > \max\{|y-x|, |y-z|\} \right\}, \\ D_2 &= \left\{ (y,t) \in \mathbb{R}^{d+1}_+ : |y-x| < t < |y-z| \right\}, \\ D_3 &= \left\{ (y,t) \in \mathbb{R}^{d+1}_+ : |y-z| < t < |y-x| \right\}, \\ D_4 &= \left\{ (y,t) \in \mathbb{R}^{d+1}_+ : t < \min\{|y-x|, |y-z|\} \right\}. \end{aligned} \tag{2.7}$$

It is easy to see

$$T\mathbb{R}^{d+1}_+ = TD_1 + TD_2 + TD_3 + TD_4. \tag{2.8}$$

We first estimate TD_1. Set

$$\begin{aligned}
D_{1,1} &= D_1 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : t < |y - x_i| \right\}, \\
D_{1,2} &= D_1 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : t > |y - x_i|, |x - x_i| > 2|y - x_i| \right\}, \\
D_{1,3} &= D_1 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : t > |y - x_i|, |x - x_i| \le 2|y - x_i|, |x_i - z| \le |y - x_i| \right\}, \\
D_{1,4} &= D_1 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : t > |y - x_i|, |x - x_i| \le 2|y - x_i|, |x_i - z| > |y - x_i| \right\}.
\end{aligned} \tag{2.9}$$

We have

$$TD_1 = TD_{1,1} + TD_{1,2} + TD_{1,3} + TD_{1,4}. \tag{2.10}$$

Note that $z \in Q_i$ and $x \in \alpha R_i \setminus \alpha Q_i$, then $|z - x_i| \le r_i$, $|x - x_i| \ge 16 r_i$. These two inequalities will always be used in the following proof, so we will not mention them every time.

For any $(y,t) \in D_{1,1}$, we have that $|y - x_i| \ge 8r_i$, $|y - z| \sim |y - x_i|$, $|y - x_i| > (1/2)|x - x_i|$, $t \gtrsim |x - x_i|$, $t < |y - x_i|$. Then

$$A \le \frac{t^{2n-2\epsilon}}{|x - x_i|^{2n-2\epsilon}}, \qquad B \le \frac{t^{2n+2\epsilon}}{|y - x_i|^{2n+2\epsilon}}. \tag{2.11}$$

For any $(y,t) \in D_{1,2}$, we get $|x - x_i| \le 2t$. Then

$$A \le 1, \qquad B \le \frac{t^{n-\epsilon}}{|y - z|^{n-\epsilon}}. \tag{2.12}$$

For $(y,t) \in D_{1,3}$, we obtain that $|y - z| \le 2|x - x_i|$, $t > (1/2)|x - x_i|$. Therefore

$$A \le 1, \qquad B \le \frac{t^{n-\epsilon}}{|y - z|^{n-\epsilon}}. \tag{2.13}$$

For $(y,t) \in D_{1,4}$, we have that $|x - y| \sim |x - x_i|$, $t > r_i$. Therefore

$$A \le \frac{t^{2n+2\epsilon}}{|x - x_i|^{2n+2\epsilon}}, \qquad B \le 1. \tag{2.14}$$

Thus, we get

$$
\begin{aligned}
TD_{1,1} &\le \int_{|y-x_i|>(1/2)|x-x_i|}\int_0^{|y-x_i|} \frac{t^{2n-2\epsilon}}{|x-x_i|^{2n-2\epsilon}} \frac{t^{2n+2\epsilon}}{|y-x_i|^{2n+2\epsilon}} \frac{1}{t^{3n+1}} dt\, d\mu(y) \\
&\lesssim \frac{1}{|x-x_i|^{2n}}, \\
TD_{1,2} &\le \int_{(1/2)|x-x_i|}^{+\infty}\int_{|y-z|\le t} \frac{t^{n-\epsilon}}{|y-z|^{n-\epsilon}} \frac{1}{t^{3n+1}} d\mu(y) dt \lesssim \frac{1}{|x-x_i|^{2n}}, \\
TD_{1,3} &\le \int_{(1/2)|x-x_i|}^{+\infty}\int_{|y-z|\le 2|x-x_i|} \frac{t^{n-\epsilon}}{|y-z|^{n-\epsilon}} \frac{1}{t^{3n+1}} d\mu(y) dt \lesssim \frac{1}{|x-x_i|^{2n}}, \\
TD_{1,4} &\le \int_{|y-x_i|<r_i}\int_{r_i}^{+\infty} \frac{t^{2n+2\epsilon}}{|x-x_i|^{2n+2\epsilon}} \frac{1}{t^{3n+1}} dt\, d\mu(y) \lesssim \frac{1}{|x-x_i|^{2n}}.
\end{aligned}
\tag{2.15}
$$

Next we estimate TD_2. Set

$$
\begin{aligned}
D_{2,1} &= D_2 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |y-x_i| > 2|x-x_i| \right\}, \\
D_{2,2} &= D_2 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |y-x_i| \le 2|x-x_i| \right\}.
\end{aligned}
\tag{2.16}
$$

For any $(y,t) \in D_{2,1}$, there exist two constants C_1, C_2 such that $C_1|y-x_i| < t < C_2|y-x_i|$. Since $A \le 1$, $B \le 1$, we then have that

$$
TD_{2,1} \le \int_{|y-x_i|>2|x-x_i|}\int_{C_1|y-x_i|}^{C_2|y-x_i|} \frac{1}{t^{3n+1}} dt\, d\mu(y) \lesssim \frac{1}{|x-x_i|^{2n}}.
\tag{2.17}
$$

For any $(y,t) \in D_{2,2}$, the following inequalities hold: $t < 3|x-x_i|$, $|x-z| \sim |x-x_i|$, and $t + |y-z| > |x-z|$. It follows that

$$
\begin{aligned}
&A \le 1, \qquad B \lesssim \frac{t^{2n+2\epsilon}}{|x-x_i|^{2n+2\epsilon}}, \\
&TD_{2,2} \lesssim \int_0^{3|x-x_i|}\int_{|y-x|<t} \frac{t^{2n+2\epsilon}}{|x-x_i|^{2n+2\epsilon}} \frac{1}{t^{3n+1}} d\mu(y) dt \lesssim \frac{1}{|x-x_i|^{2n}}.
\end{aligned}
\tag{2.18}
$$

Next we estimate TD_3. Set

$$
\begin{aligned}
D_{3,1} &= D_3 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |x-x_i| \le 2|y-x_i| \right\}, \\
D_{3,2} &= D_3 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |x-x_i| > 2|y-x_i| \right\}.
\end{aligned}
\tag{2.19}
$$

Then

$$TD_3 = TD_{3,1} + TD_{3,2}. \tag{2.20}$$

For any $(y,t) \in D_{3,1}$, we can get that $|y-z| \sim |y-x_i|$ and there exist two constants C_1 and C_2 such that $C_1|y-x_i| < t < C_2|y-x_i|$. Then

$$\begin{gathered} A \le 1, \qquad B \lesssim \frac{t^{2n+2\epsilon}}{\left|y-x_i\right|^{2n+2\epsilon}}, \\ TD_{3,1} \lesssim \int_{|y-x_i| \ge (1/2)|x-x_i|} \int_{C_1|y-x_i|}^{C_2|y-x_i|} \frac{t^{2n+2\epsilon}}{\left|y-x_i\right|^{2n+2\epsilon}} \frac{1}{t^{3n+1}} dt\, d\mu(y) \lesssim \frac{1}{|x-x_i|^{2n}}. \end{gathered} \tag{2.21}$$

For any $(y,t) \in D_{3,2}$, we can get $|x-y| \ge (1/2)|x-x_i|$ and $t < (3/2)|x-x_i|$. Then

$$A \lesssim \frac{t^{2n+2\epsilon}}{|x-x_i|^{2n+2\epsilon}}, \qquad B \le 1, \qquad TD_{3,2} \lesssim \int_0^{(3/2)|x-x_i|} \int_{|y-z|<t} \frac{t^{2n+2\epsilon}}{|x-x_i|^{2n+2\epsilon}} \lesssim d\mu(y)\, dt. \tag{2.22}$$

Next we estimate TD_4. Set

$$\begin{aligned} D_{4,1} &= D_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : \left|x-y\right| > 2|x-z| \right\}, \\ D_{4,2} &= D_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : \frac{1}{2}|x-z| < \left|x-y\right| \le 2|x-z| \right\}, \\ D_{4,3} &= D_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : \left|x-y\right| \le \frac{1}{2}|x-z| \right\}. \end{aligned} \tag{2.23}$$

If $(y,t) \in D_{4,1}$, we obtain $|y-z| > |x-z|$, $|x-z| \sim |x-x_i|$ and

$$A \lesssim \frac{t^{2n-2\epsilon}}{|x-z|^{2n-2\epsilon}}, \qquad B \le \frac{t^{2n+2\epsilon}}{\left|y-z\right|^{2n+2\epsilon}}. \tag{2.24}$$

Therefore

$$TD_{4,1} \lesssim \int_{|y-z|>|x-z|} \int_0^{|y-z|} \frac{t^{2n-2\epsilon}}{|x-z|^{2n-2\epsilon}} \frac{t^{2n+2\epsilon}}{\left|y-z\right|^{2n+2\epsilon}} \frac{1}{t^{3n+1}} dt\, d\mu(y) \lesssim \frac{1}{|x-x_i|^{2n}}. \tag{2.25}$$

If $(y,t) \in D_{4,2}$, we have $t < 2|x-z|$ and

$$A \lesssim \frac{t^{2n+2\epsilon}}{|x-z|^{2n+2\epsilon}}, \qquad B \le \frac{t^{2n-2\epsilon}}{\left|y-z\right|^{2n-2\epsilon}}. \tag{2.26}$$

Therefore

$$TD_{4,2} \lesssim \int_0^{2|x-z|} \int_{|y-z|>t} \frac{t^{2n+2\epsilon}}{|x-z|^{2n+2\epsilon}} \frac{t^{2n-2\epsilon}}{\left|y-z\right|^{2n-2\epsilon}} \frac{1}{t^{3n+1}} d\mu(y)\,dt. \tag{2.27}$$

If $(y,t) \in D_{4,3}$, we have $|y-z| \sim |x-z|$ and $t < (1/2)|x-z|$. Then

$$\begin{gathered} A \le \frac{t^{2n-2\epsilon}}{\left|x-y\right|^{2n-2\epsilon}}, \qquad B \lesssim \frac{t^{2n+2\epsilon}}{|x-z|^{2n+2\epsilon}}, \\ TD_{4,3} \lesssim \int_0^{(1/2)|x-z|} \int_{|x-y|>t} \frac{t^{2n-2\epsilon}}{\left|x-y\right|^{2n-2\epsilon}} \frac{t^{2n+2\epsilon}}{|x-z|^{2n+2\epsilon}} \frac{1}{t^{3n+1}} d\mu(y)\,dt \lesssim \frac{1}{|x-x_i|^{2n}}. \end{gathered} \tag{2.28}$$

The proof of Lemma 2.3 is finished. □

Proof of Lemma 2.4. Denote

$$\overline{A} := \left(\frac{t}{t+|x-y|}\right)^{\lambda n}, \qquad \overline{B} := \frac{t^{2\delta}}{\left(t+\left|y-x_i\right|\right)^{2n+2\gamma+2\delta}}. \tag{2.29}$$

Let $K = \{(y,t) \in \mathbb{R}^{d+1}_+ : |y-x_i| > 2|x_i - z|\}$ and divide K into four parts

$$\begin{aligned} K_1 &= K \cap \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \max\left\{\left|y-x\right|, \left|y-z\right|\right\} \le t\right\}, \\ K_2 &= K \cap \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \left|y-x\right| < t < \left|y-z\right|\right\}, \\ K_3 &= K \cap \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \left|y-x\right| \le t,\ \left|y-x_i\right| < 4r_i\right\}, \\ K_4 &= K \cap \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \left|y-x\right| \le t,\ \left|y-x_i\right| \ge 4r_i\right\}. \end{aligned} \tag{2.30}$$

Then $\overline{T}K = \overline{T}K_1 + \overline{T}K_2 + \overline{T}K_3 + \overline{T}K_4$.

Since $x \in \mathbb{R}^n \setminus \alpha R_i$ and $z \in R_i$, we have that $|z-x_i| \le r_i$, $|x-x_i| \ge 16 r_i$.

We first estimate $\overline{T}K_1$. Set

$$\begin{aligned} K_{1,1} &= K_1 \cap \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \left|y-x_i\right| \le \frac{3}{2} r_i\right\}, \\ K_{1,2} &= K_1 \cap \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \left|y-x_i\right| > \frac{3}{2} r_i,\ t < \left|y-x_i\right| + 2r_i\right\}, \\ K_{1,3} &= K_1 \cap \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \left|y-x_i\right| > \frac{3}{2} r_i,\ t \ge \left|y-x_i\right| + 2r_i, |x-x_i| \le 2\left|y-x_i\right|\right\}, \\ K_{1,4} &= K_1 \cap \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \left|y-x_i\right| > \frac{3}{2} r_i,\ t \ge \left|y-x_i\right| + 2r_i, |x-x_i| > 2\left|y-x_i\right|\right\}. \end{aligned} \tag{2.31}$$

Then

$$\overline{T}K_1 = \overline{T}K_{1,1} + \overline{T}K_{1,2} + \overline{T}K_{1,3} + \overline{T}K_{1,4}. \tag{2.32}$$

For any $(y,t) \in K_{1,1}$, we have that $|x-x_i| \sim |x-y|$ and $t \geq (1/2)|x-x_i|$. Then we get

$$\overline{A} \leq 1, \qquad \overline{B} \leq \frac{t^{2\delta}}{t^{2n+2\gamma+2\delta}}, \tag{2.33}$$

and therefore

$$\overline{T}K_{1,1} \leq \int_{(1/2)|x-x_i|}^{\infty} \int_{|y-z|\leq t} \frac{t^{2\delta}}{t^{2n+2\gamma+2\delta}} \frac{d\mu(y)\,dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}. \tag{2.34}$$

For any $(y,t) \in K_{1,2}$, we have $|x-x_i| \leq 3|y-x_i|$, $(1/2)|y-x_i| < t < 2|y-x_i|$. It follows that

$$\begin{gathered}\overline{A} \leq 1, \qquad \overline{B} \leq \frac{t^{2\delta}}{\left|x_i-y\right|^{2n+2\gamma+2\delta}}, \\ \overline{T}K_{1,2} \leq \int_{\left|y-x_i\right|>(1/3)|x-x_i|} \int_{(1/2)|y-x_i|}^{2|y-x_i|} \frac{t^{2\delta}}{\left|x_i-y\right|^{2n+2\gamma+2\delta}} \frac{dt\,d\mu(y)}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}.\end{gathered} \tag{2.35}$$

If $(y,t) \in K_{1,3}$, we will get $|x-x_i| \leq |y-x_i| \leq 2t$. It follows that

$$\overline{A} \leq 1, \qquad \overline{B} \leq \frac{1}{t^{2n+2\gamma}}, \qquad \overline{T}K_{1,3} \leq \int_{(1/2)|x-x_i|}^{\infty} \int_{|y-x_i|<t} \frac{1}{t^{2n+2\gamma}} \frac{dt\,d\mu(y)}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}. \tag{2.36}$$

If $(y,t) \in K_{1,4}$, we obtain that $t > (1/2)|x-x_i|$. Thus we can get

$$\begin{gathered}\overline{A} \leq 1, \qquad \overline{B} \leq \frac{1}{t^{2n+2\gamma}}, \\ \overline{T}K_{1,4} \leq \int_{(1/2)|x-x_i|}^{+\infty} \frac{1}{t^{2n+2\gamma}} \frac{d\mu(y)\,dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}.\end{gathered} \tag{2.37}$$

Next we estimate $\overline{T}K_2$. Set

$$\begin{aligned}K_{2,1} &= K_2 \cap \left\{ (y,t) \in \mathbb{R}_+^{n+1} : \left|y-x_i\right| > 2|x-x_i| \right\}, \\ K_{2,2} &= K_2 \cap \left\{ (y,t) \in \mathbb{R}_+^{n+1} : \left|y-x_i\right| \leq 2|x-x_i| \right\}.\end{aligned} \tag{2.38}$$

Then

$$\overline{T}K_2 = \overline{T}K_{2,1} + \overline{T}K_{2,2}. \tag{2.39}$$

For any $(y,t) \in K_{2,1}$, the inequalities $t > (1/2)|y-x_i|$ and $t \le 2|y-x_i|$ hold, and we have

$$\overline{A} \le 1, \qquad \overline{B} \le \frac{1}{t^{2n+2\gamma}},$$
$$\overline{T}K_{2,1} \le \int_{|y-x_i|>2|x-x_i|} \int_{(1/2)|y-x_i|}^{2|y-x_i|} \frac{1}{t^{2n+2\gamma}} \frac{dt\, d\mu(y)}{t^{n+1}}. \tag{2.40}$$

For any $(y,t) \in K_{2,2}$, the inequalities $t < 3|x-x_i|$ and $t + |y-x_i| > |x-x_i|$ hold, and we can get

$$\overline{A} \le 1, \qquad \overline{B} \le \frac{t^{2\delta}}{|x-x_i|^{2n+2\gamma+2\delta}},$$
$$\overline{T}K_{2,2} \le \int_0^{2|x-x_i|} \int_{|y-x|<t} \frac{t^{2\delta}}{|x-x_i|^{2n+2\gamma+2\delta}} \frac{d\mu(y)\,dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}. \tag{2.41}$$

Next we estimate $\overline{T}K_3$. Let

$$K_{3,1} = K_3 \cap \left\{ (y,t) \in \mathbb{R}^d : t > \left|y-x_i\right| \right\},$$
$$K_{3,2} = K_3 \cap \left\{ (y,t) \in \mathbb{R}^d : t \le \left|y-x_i\right| \right\}. \tag{2.42}$$

Then

$$\overline{T}K_3 = \overline{T}K_{3,1} + \overline{T}K_{3,2}. \tag{2.43}$$

Since $2n + 2\gamma < \lambda n$, we can choose $\epsilon > 0$ small enough such that $2n + 2\gamma + 2\epsilon < \lambda n$ and $\epsilon < (1/2)n$.

Now denote

$$\widetilde{A} := \left(\frac{t}{t + |x-y|} \right)^{2n+2\gamma+2\epsilon}, \tag{2.44}$$

and for a subset $E \subset \mathbb{R}^d$, we denote

$$\widetilde{T}E := \int_E \left(\frac{t}{t+|x-y|} \right)^{2n+2\gamma+2\epsilon} \frac{t^{2\delta}}{\left(t+|y-x_i|\right)^{2n+2\gamma+2\delta}} \frac{d\mu(y)\,dt}{t^{n+1}}. \tag{2.45}$$

It is easy to see that $\overline{T}E \le \widetilde{T}E$.

For any $(y,t) \in K_{3,1}$, we have $|x-y| \sim |x-x_i|$, $t \le 2|x-x_i|$. It follows that

$$\widetilde{A} \lesssim \frac{t^{2n+2\epsilon+2\gamma}}{|x-x_i|^{2n+2\epsilon+2\gamma}},$$
$$\widetilde{A}\overline{B} \le \frac{t^{2n+2\epsilon}}{|x-x_i|^{2n+2\epsilon}} \frac{t^{2\delta+2\gamma}}{\left(t+|y-x_i|\right)^{2n+2\gamma+2\delta}} \le \frac{t^{2n+2\epsilon}}{|x-x_i|^{2n+2\epsilon}} \frac{1}{|y-x_i|^{n-\epsilon}} \frac{1}{t^{n+\epsilon}}. \tag{2.46}$$

The last inequality holds because $t > |y-x_i|$ and we choose $\epsilon > 0$ small enough such that $n-\epsilon > 0$, and then we can get $(t/|y-x_i|)^{n-\epsilon} > 1$, which leads to the above inequality. Using these inequalities, we obtain

$$\widetilde{T}K_{3,1} \lesssim \int_{|y-x_i|<4r_i} \int_0^{2|x-x_i|} \frac{t^{2n+2\epsilon}}{|x-x_i|^{2n+2\epsilon}} \frac{1}{|y-x_i|^{n-\epsilon}} \frac{1}{t^{n+\epsilon}} \frac{dt\, d\mu(y)}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}. \tag{2.47}$$

If $(y,t) \in K_{3,2}$, we get $t \le |x-x_i|$. It follows that

$$\widetilde{A} \le \frac{t^{2n+2\gamma+2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}},$$
$$\widetilde{A}\overline{B} \le \frac{t^{2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}} \frac{t^{2n+2\gamma+2\delta}}{\left(t+|y-x_i|\right)^{2n+2\gamma+2\delta}} \le \frac{t^{2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}} \frac{t^{n+\gamma+\delta}}{|y-x_i|^{n+\gamma+\delta}}, \tag{2.48}$$
$$\widetilde{T}K_{3,2} \le \int_0^{|x-x_i|} \int_{|y-x_i|\ge t} \frac{t^{2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}} \times \frac{t^{n+\gamma+\delta}}{|y-x_i|^{n+\gamma+\delta}} \frac{d\mu(y)dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}.$$

Next we estimate $\overline{T}K_4$. Set

$$\begin{aligned}
K_{4,1} &= K_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |y-x_i| \le t < |y-x_i| + 2r_i,\ 2|y-x_i| > |x-x_i| \right\},\\
K_{4,2} &= K_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |y-x_i| \le t < |y-x_i| + 2r_i,\ 2|y-x_i| \le |x-x_i| \right\},\\
K_{4,3} &= K_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : \max\{|y-x_i|, |y-x_i| + 2r_i\} \le t,\ 2|y-x_i| > |x-x_i| \right\},\\
K_{4,4} &= K_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : \max\{|y-x_i|, |y-x_i| + 2r_i\} \le t,\ 2|y-x_i| \le |x-x_i| \right\},\\
K_{4,5} &= K_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |y-x_i| > t,\ |x-y| > 2|x-x_i| \right\},\\
K_{4,6} &= K_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |y-x_i| > t,\ \frac{1}{2}|x-x_i| < |x-y| \le 2|x-x_i| \right\},\\
K_{4,7} &= K_4 \cap \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |y-x_i| > t,\ |x-y| < \frac{1}{2}|x-x_i| \right\}.
\end{aligned} \tag{2.49}$$

Then

$$\overline{T}K_4 = \overline{T}K_{4,1} + \overline{T}K_{4,2} + \overline{T}K_{4,3} + \overline{T}K_{4,4} + \overline{T}K_{4,5} + \overline{T}K_{4,6} + \overline{T}K_{4,7}. \tag{2.50}$$

For any $(y,t) \in K_{4,1}$, we have $(3/4)|y-x_i| < t < (3/2)|y-x_i|$. Then we get

$$\begin{gathered}\overline{A} \leq 1, \qquad \overline{B} \leq \frac{t^{2\delta}}{t^{2n+2\gamma+2\delta}},\\ \overline{T}K_{4,1} \leq \int_{|y-x_i|\geq(1/2)|x-x_i|}\int_{(3/4)|y-x_i|}^{(3/2)|y-x_i|} \frac{t^{2\delta}}{t^{2n+2\gamma+2\delta}} \frac{dt\,d\mu(y)}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}.\end{gathered} \tag{2.51}$$

If $(y,t) \in K_{4,2}$, we get $|x-y| > (1/2)|x-x_i|$ and $t < 2|x-x_i|$. Then we can obtain

$$\begin{gathered}\widetilde{A} < \frac{t^{2n+2\gamma+2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}}, \qquad \overline{B} < \frac{t^{2\delta}}{t^{2n+2\gamma+2\delta}},\\ \widetilde{T}K_{4,2} \leq \int_0^{2|x-x_i|}\int_{|y-x_i|<t} \frac{t^{2n+2\gamma+2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}} \frac{t^{2\delta}}{t^{2n+2\gamma+2\delta}} \frac{d\mu(y)\,dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}.\end{gathered} \tag{2.52}$$

If $(y,t) \in K_{4,3}$, we get $t > (1/2)|x-x_i|$. Then

$$\begin{gathered}\overline{A} \leq 1, \qquad \overline{B} = \frac{t^{2\delta}}{\left(t+\left|y-x_i\right|\right)^{2\delta}} \frac{1}{\left(t+\left|y-x_i\right|\right)^{2n+2\gamma}} \leq \frac{1}{\left|y-x_i\right|^{2n+2\gamma}},\\ \overline{T}K_{4,3} \lesssim \int_{|y-x_i|>(1/2)|x-x_i|}\int_{(1/2)|x-x_i|}^{\infty} \frac{1}{\left|y-x_i\right|^{2n+2\gamma}} \frac{dt\,d\mu(y)}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}.\end{gathered} \tag{2.53}$$

For any $(y,t) \in K_{4,4}$, the inequalities $|x-y| \geq (1/2)|x-x_i|$, $0 < t < 2|x-x_i|$, and $t > |y-x_i|$ hold, from which we obtain

$$\begin{gathered}\widetilde{A} \lesssim \frac{t^{2n+2\gamma+2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}}, \qquad \overline{B} \leq \frac{1}{t^{2n+2\gamma}},\\ \widetilde{T}K_{4,4} \lesssim \int_0^{2|x-x_i|}\int_{|y-x_i|\leq t} \frac{t^{2n+2\gamma+2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}} \frac{1}{t^{2n+2\gamma}} \frac{d\mu(y)\,dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}.\end{gathered} \tag{2.54}$$

If $(y,t) \in K_{4,5}$, we have $|y-x_i| \geq |x-x_i|$. Then

$$\begin{gathered}\overline{A} \leq \left(\frac{t}{t+\left|x-y\right|}\right)^{2n+2\gamma-2\epsilon} \lesssim \frac{t^{2n+2\gamma-2\epsilon}}{|x-x_i|^{2n+2\gamma-2\epsilon}}, \qquad \overline{B} \leq \frac{t^{2\delta}}{\left|y-x_i\right|^{2n+2\gamma+2\delta}},\\ \overline{T}K_{4,5} \lesssim \int_{|y-x_i|>|x-x_i|}\int_0^{|y-x_i|} \frac{t^{2n+2\gamma-2\epsilon}}{|x-x_i|^{2n+2\gamma-2\epsilon}} \frac{t^{2\delta}}{\left|y-x_i\right|^{2n+2\gamma+2\delta}} \frac{dt\,d\mu(y)}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}.\end{gathered} \tag{2.55}$$

If $(y,t) \in K_{4,6}$, we have $t < 3|x-x_i|$ and it follows that

$$\widetilde{A} \lesssim \frac{t^{2n+2\gamma+2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}}, \qquad \overline{B} \le \frac{t^{2\delta}}{\left|y-x_i\right|^{2n+2\gamma+2\delta}},$$
$$\widetilde{T}K_{4,6} \lesssim \int_0^{3|x-x_i|}\int_{|y-x_i|>t} \frac{t^{2n+2\gamma+2\epsilon}}{|x-x_i|^{2n+2\gamma+2\epsilon}} \frac{t^{2\delta}}{\left|y-x_i\right|^{2n+2\gamma+2\delta}} \frac{d\mu(y)dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}. \tag{2.56}$$

If $(y,t) \in K_{4,7}$, we get $|y-x_i| \sim |x-x_i|$ and $t \lesssim |x-x_i|$. Then we have

$$\overline{A} \le \left(\frac{t}{\left|x-y\right|}\right)^{2n+2\gamma}, \qquad \overline{B} \lesssim \frac{t^{2\delta}}{|x-x_i|^{2n+2\gamma+2\delta}},$$
$$\overline{T}K_{4,7} \lesssim \int_0^{(3/2)|x-x_i|}\int_{|y-x|>t} \left(\frac{t}{\left|x-y\right|}\right)^{2n+2\gamma} \frac{t^{2\delta}}{|x-x_i|^{2n+2\gamma+2\delta}} \frac{d\mu(y)dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}. \tag{2.57}$$

□

3. Proof of Theorems

Proof of Theorem 1.4. To prove Theorem 1.4, we will choose $\alpha = 16\sqrt{d}$ and $\beta = 3\alpha$.

Let $f \in L^1(\mu)$ and $\lambda > \alpha^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$. Applying Lemma 2.2 to f and λ, we obtain a family of almost disjoint cubes $\{Q_i\}_i$. With the notation w_i, φ_i, R_i the same as in Lemma 2.2, we can decompose $f = g + b$, with that $g = f\chi_{\mathbb{R}^d \setminus \bigcup_i Q_i} + \sum_j \varphi_i$ and $b = \sum_j (w_j f - \varphi_j) = \sum_i b_i$. And $\mathbb{R}^d$ can be decomposed as $\mathbb{R}^d = (\beta R_i)^c \cup (\beta R_i \setminus \alpha Q_i) \cup \alpha Q_i$. By (a.1) of Lemma 2.2, we have $\mu(\bigcup_{i=1}^{\infty} \alpha Q_i) \le (C/\lambda)\sum_i^{\infty}\int_{Q_i}|f|d\mu \le (C/\lambda)\int|f|d\mu$.

Thus, to prove that $g^*_{\lambda,\mu}$ is of weak type $(1,1)$, we only need to prove

$$\mu\left\{x \in \mathbb{R}^d \setminus \bigcup_{i=1}^{\infty}\alpha Q_i : g^*_{\lambda,\mu}(f)(x) > \lambda\right\} \le \frac{C}{\lambda}\int|f|d\mu. \tag{3.1}$$

Since $f = g + b$ and $g^*_{\lambda,\mu}(f) \le g^*_{\lambda,\mu}(g) + g^*_{\lambda,\mu}(b)$, we only need to show that both g and b satisfy the inequality (2.4).

For g, it follows from (b.1) of Lemma 2.2 that $\|\sum\varphi_i\|_{L^1(\mu)} \le \sum\int_{Q_i}|f|d\mu \lesssim \|f\|_{L^1(\mu)}$, and we have $\|g\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)} < \infty$. Using L^2-boundedness of $g^*_{\lambda,\mu}$ and (a.3) and (b.2) from Lemma 2.2, we obtain

$$\mu\left\{x \in \mathbb{R}^d \setminus \bigcup_{i=1}^{\infty}\alpha Q_i : g^*_{\lambda,\mu}(g)(x) > \lambda\right\} \le \frac{C}{\lambda^2}\int|g|^2 d\mu \le \frac{C}{\lambda}\int|g|d\mu \lesssim \frac{\|f\|_{L^1(\mu)}}{\lambda}. \tag{3.2}$$

To prove that b satisfies inequality (3.1), it suffices to show that

$$\int_{\mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} \alpha Q_i} g^*_{\lambda,\mu}(b)(x)\,d\mu \lesssim \|b\|_{L^1(\mu)} \lesssim \|f\|_{L^1((\mu)}. \tag{3.3}$$

Since $b = \sum_j (w_j f - \varphi_j) = \sum_i b_i$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} \alpha Q_i} g^*_{\lambda,\mu}(b)(x)\,d\mu &\le \sum_i \int_{\mathbb{R}^d \setminus \alpha Q_i} g^*_{\lambda,\mu}(b_i)(x)\,d\mu \\
&\le \sum_i \left[\int_{\mathbb{R}^d \setminus \alpha R_i} g^*_{\lambda,\mu}(b_i)(x)\,d\mu + \int_{\alpha R_i \setminus \alpha Q_i} g^*_{\lambda,\mu}(b_i)(x)\,d\mu\right] \\
&:= \sum_i (A_i + B_i).
\end{aligned} \tag{3.4}$$

If we can prove

$$A_i \lesssim \int_{Q_i} |f|\,d\mu, \qquad B_i \lesssim \int_{Q_i} |f|\,d\mu, \tag{3.5}$$

then we finish the proof of Theorem 1.4.

We first estimate B_i and divide it into two parts

$$\begin{aligned}
B_i &\le \int_{\alpha R_i \setminus \alpha Q_i} g^*_{\lambda,\mu}(w_i f)(x)\,d\mu(x) + \int_{\alpha R_i \setminus \alpha Q_i} g^*_{\lambda,\mu}(\varphi_i)(x)\,d\mu(x) \\
&:= B_{i,1} + B_{i,2}.
\end{aligned} \tag{3.6}$$

By the assumption of L^2 boundedness of $g^*_{\lambda,\mu}$, and the fact that R_i is the (β, β^{n+1}) doubling cube, it is easy to see that

$$\begin{aligned}
B_{i,2} &\le \int_{\alpha R_i} g^*_{\lambda,\mu}(\varphi_i)(x)\,d\mu(x) \le \left[\int_{\alpha R_i} \left|g^*_{\lambda,\mu}(\varphi_i)(x)\right|^2 d\mu(x)\right]^{1/2} \mu(\alpha R_i)^{1/2} \\
&\lesssim \left[\int_{\alpha R_i} |\varphi_i|^2 d\mu(x)\right]^{1/2} \mu(R_i)^{1/2} \lesssim \int_{Q_i} |f|\,d\mu.
\end{aligned} \tag{3.7}$$

Next we estimate $B_{i,1}$. By the Minkowski's inequality, Fubini's theorem and condition (b) of φ that $|\varphi(x)| \le C_0(1+|x|)^{-n-\delta}$, we have the following estimate:

$$
\begin{aligned}
B_{i,1} &= \int_{\alpha R_i \backslash \alpha Q_i} \left[\int_{\mathbb{R}^{d+1}_+} \left| \left(\frac{t}{t+|x-y|} \right)^{\lambda n/2} \int \varphi_t(y-z) w_i(z) f(z) d\mu(z) \right|^2 \frac{d\mu(y)dt}{t^{n+1}} \right]^{1/2} d\mu(x) \\
&\lesssim \int_{\alpha R_i \backslash \alpha Q_i} \int_{Q_i} |f(z)| \left[\int_{\mathbb{R}^{d+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \varphi_t(y-z)^2 \frac{d\mu(y)dt}{t^{n+1}} \right]^{1/2} d\mu(z) d\mu(x) \\
&\lesssim \int_{Q_i} |f(z)| \int_{\alpha R_i \backslash \alpha Q_i} \left[\int_{\mathbb{R}^{d+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left(\frac{t}{t+|y-z|} \right)^{2n+2\delta} \frac{d\mu(y)dt}{t^{3n+1}} \right]^{1/2} d\mu(x) d\mu(z).
\end{aligned}
\tag{3.8}
$$

Choose $\epsilon > 0$ small enough such that $\epsilon < \delta$, $2n + 2\epsilon < \lambda n$ and $2\epsilon < n$.

For any subset $E \subset \mathbb{R}^{d+1}$, we denote

$$
TE := \int_E \left(\frac{t}{t+|x-y|} \right)^{2n+2\epsilon} \left(\frac{t}{t+|y-z|} \right)^{2n+2\epsilon} \frac{d\mu(y)dt}{t^{3n+1}}. \tag{3.9}
$$

Let x_i be $Q_i's$ center. Using the above notations, it is easy to see that

$$
\int_{\mathbb{R}^{d+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left(\frac{t}{t+|y-z|} \right)^{2n+2\delta} \frac{d\mu(y)dt}{t^{3n+1}} \lesssim T\mathbb{R}^{d+1}_+. \tag{3.10}
$$

By Lemma 2.3, we will have the following inequalities:

$$
g^*_{\lambda,\mu}(fw_i)(x) \lesssim \frac{1}{|x-x_i|^n} \int_{Q_i} |fw_i| d\mu(z), \tag{3.11}
$$

$$
B_{i,1} \lesssim \int_{Q_i} |f(z)| d\mu(z) \int_{\alpha R_i \backslash \alpha Q_i} \frac{1}{|x-x_i|^n} d\mu(x) \lesssim \int_{Q_i} |f(z)| d\mu(z), \tag{3.12}
$$

where we used Lemma 2.1 to estimate

$$
\begin{aligned}
\int_{\alpha R_i \backslash \alpha Q_i} \frac{1}{|x-x_i|^n} d\mu(x) &\le \int_{\alpha R_i \backslash R_i} \frac{1}{|x-x_i|^n} d\mu(x) + \int_{R_i \backslash \beta Q_i} \frac{1}{|x-x_i|^n} d\mu(x) \\
&\quad + \int_{\beta Q_i \backslash \alpha Q_i} \frac{1}{|x-x_i|^n} d\mu(x) \le C_{\alpha,\beta,n,C_0}.
\end{aligned}
\tag{3.13}
$$

We now estimate A_i. Let $r_i = (\sqrt{d}/2)l(R_i)$. Recall that $A_i = \int_{\mathbb{R}^d \setminus \alpha R_i} g^*_{\lambda,\mu}(b_i)(x)d\mu(x)$, and

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus \alpha R_i} g^*_{\lambda,\mu}(b_i)(x)d\mu(x) \\
&\quad \leq \int_{\mathbb{R}^d \setminus \alpha R_i} \left[\int\!\!\int_{|y-x_i| \leq 2|x_i - z|} \left| \left(\frac{t}{t + |x-y|} \right)^{\lambda n/2} \int \varphi_t(y-z) b_i(z) d\mu(z) \right|^2 \frac{d\mu(y)dt}{t^{n+1}} \right]^{1/2} d\mu(x) \\
&\qquad + \int_{\mathbb{R}^d \setminus \alpha R_i} \left[\int\!\!\int_{|y-x_i| > 2|x_i - z|} \left| \left(\frac{t}{t + |x-y|} \right)^{\lambda n/2} \right. \right. \\
&\qquad\qquad \left. \left. \times \int (\varphi_t(y-z) - \varphi_t(y-x_i)) b_i(z) d\mu(z) \right|^2 \frac{d\mu(y)dt}{t^{n+1}} \right]^{1/2} d\mu(x) \\
&:= A_i^1 + A_i^2.
\end{aligned} \tag{3.14}$$

We first estimate A_i^1. By the Minkowski's inequality, Fubini's theorem and property (b) of φ, we can obtain that

$$\begin{aligned}
A_i^1 &\leq C \int_{R_i} |b_i(z)| \\
&\quad \times \int_{\mathbb{R}^d \setminus \alpha R_i} \left[\int\!\!\int_{|y-x_i| \leq 2|x_i - z|} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \left(\frac{t}{t + |y-z|} \right)^{2n+2\delta} \frac{d\mu(y)dt}{t^{3n+1}} \right]^{1/2} d\mu(x) d\mu(z).
\end{aligned} \tag{3.15}$$

If we prove

$$\int\!\!\int_{|y-x_i| \leq 2|x_i - z|} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \left(\frac{t}{t + |y-z|} \right)^{2n+2\delta} \frac{d\mu(y)dt}{t^{3n+1}} \lesssim \frac{r_i^{2\epsilon}}{|x - x_i|^{2n+2\epsilon}}, \tag{3.16}$$

for any $z \in R_i$ and $x \in (\alpha R_i)^c$, then

$$\int_{\mathbb{R}^d \setminus \alpha R_i} \left[\int\!\!\int_{|y-x_i| \leq 2|x_i - z|} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} \left(\frac{t}{t + |y-z|} \right)^{2n+2\delta} \frac{d\mu(y)dt}{t^{3n+1}} \right]^{1/2} d\mu(x) \lesssim 1, \tag{3.17}$$

and by Lemma 2.2, we conclude that $A_i^1 \lesssim \int_{R_i} |b_i(z)| d\mu(z) \leq \int_{Q_i} |f(z)| d\mu(z)$.

Now choose $\epsilon > 0$ small enough such that $\epsilon < \delta$, $2n + 2\epsilon < \lambda n$ and $2\epsilon < n$. Then for any $z \in R_i$ and $x \in (\alpha R_i)^c$, note that $|x - y| \sim |x - x_i|$

$$\begin{aligned}
&\int_{|y-x_i|\le 2|x_i-z|}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left(\frac{t}{t+|y-z|}\right)^{2n+2\delta}\frac{d\mu(y)dt}{t^{3n+1}} \\
&\quad\le \int_{|y-x_i|\le 2|x_i-z|}\left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon}\left(\frac{t}{t+|y-z|}\right)^{2n+2\epsilon}\frac{d\mu(y)dt}{t^{3n+1}} \\
&\quad\le \int_{|y-z|\le 3r_i}\int_0^{r_i}\left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon}\left(\frac{t}{t+|y-z|}\right)^{2n+2\epsilon}\frac{d\mu(y)dt}{t^{3n+1}} \\
&\qquad+ \int_{|y-z|\le 3r_i}\int_{r_i}^{\infty}\left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon}\left(\frac{t}{t+|y-z|}\right)^{2n+2\epsilon}\frac{d\mu(y)dt}{t^{3n+1}} \\
&\quad\le \int_{|y-z|\le 3r_i}\int_0^{r_i}\left(\frac{t}{|x-y|}\right)^{2n+2\epsilon}\left(\frac{t}{|y-z|}\right)^{n-\epsilon}\frac{dt\,d\mu(y)}{t^{3n+1}} \\
&\qquad+ \int_{|y-z|\le 3r_i}\int_{r_i}^{\infty}\left(\frac{t}{|x-y|}\right)^{2n+2\epsilon}\frac{dt\,d\mu(y)}{t^{3n+1}} \lesssim \frac{r_i^{2\epsilon}}{|x-x_i|^{2n+2\epsilon}}.
\end{aligned} \tag{3.18}$$

Next we estimate A_i^2. By property (c) of φ and $2|x_i - z| \le |y - x_i|$, we have

$$\left|\varphi_t(y-z) - \varphi_t(y-x_i)\right| \le \frac{1}{t^n}\left(\frac{|x_i-z|}{t}\right)^{\gamma}\left(\frac{t}{t+|y-x_i|}\right)^{n+\gamma+\delta} \le \frac{r_i^{\gamma}t^{\delta}}{\left(t+|y-x_i|\right)^{n+\gamma+\delta}}, \tag{3.19}$$

since $|x_i - z| \le r_i$. Using the above inequality, Minkowski's inequality and Fubini's theorem, we get

$$\begin{aligned}
A_i^2 &\le C\int_{R_i}|b_i(z)| \\
&\quad\times \int_{\mathbb{R}^d\setminus\alpha R_i}\left[\int_{|y-x_i|>2|x_i-z|}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{r_i^{2\gamma}t^{2\delta}}{\left(t+|y-x_i|\right)^{2n+2\gamma+2\delta}}\frac{d\mu(y)dt}{t^{n+1}}\right]^{1/2} d\mu(x)d\mu(z).
\end{aligned} \tag{3.20}$$

So by Lemma 2.4, for $x \in \mathbb{R}^d \setminus \alpha R_i$ and any $z \in R_i$

$$\int_{|y-x_i|>2|x_i-z|}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{t^{2\delta}}{\left(t+|y-x_i|\right)^{2n+2\gamma+2\delta}}\frac{d\mu(y)dt}{t^{n+1}} \lesssim \frac{1}{|x-x_i|^{2n+2\gamma}}. \tag{3.21}$$

Then we get

$$\int_{\mathbb{R}^d\setminus\alpha R_i}\left[\int\int_{|y-x_i|>2|x_i-z|}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{r_i^{2\gamma}t^{2\delta}}{(t+|y-x_i|)^{2n+2\gamma+2\delta}}\frac{d\mu(y)dt}{t^{n+1}}\right]^{1/2}d\mu(x)\lesssim 1, \quad (3.22)$$

and we will have $A_i^2 \lesssim \int_{R_i}|b_i(z)|d\mu(z) \le \int_{Q_i}|f(z)|d\mu(z)$. With the estimates of

$$\begin{aligned} A_i^1 &\lesssim \int_{R_i}|b_i(z)|d\mu(z) \le \int_{Q_i}|f(z)|d\mu(z), \\ A_i^2 &\lesssim \int_{R_i}|b_i(z)|d\mu(z) \le \int_{Q_i}|f(z)|d\mu(z), \end{aligned} \quad (3.23)$$

we obtain that

$$A_i \lesssim \int_{R_i}|b_i(z)|d\mu(z) \le \int_{Q_i}|f(z)|d\mu(z). \quad (3.24)$$

Thus,

$$\int_{\mathbb{R}^d\setminus\alpha R_i} g^*_{\lambda,\mu}(b_i)(x)d\mu(x) \lesssim \int_{R_i}|b_i(z)|d\mu(z). \quad (3.25)$$

Therefore we finish the proof of Theorem 1.4. □

Proof of Theorem 1.5. Note that the definition of $H^1(\mu)$ is independent of the choice of the constant ρ, we can assume that $\rho = \alpha$, still with $\alpha = 16\sqrt{d}$. By Theorem 1.4, the operator $g^*_{\lambda,\mu}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. By a standard argument, we only need to prove that $\|g^*_{\lambda,\mu}(b)\|_{L^1(\mu)} \le |b|_{H^1(\mu)}$ for any atomic block b with $\operatorname{supp}(b)\subset R$. Write

$$\begin{aligned} \int_{\mathbb{R}} g^*_{\lambda,\mu}(b)(x)d\mu(x) &= \int_{\mathbb{R}\setminus\alpha R} g^*_{\lambda,\mu}(b)(x)d\mu(x) + \int_{\alpha R} g^*_{\lambda,\mu}(b)(x)d\mu(x) \\ &\equiv J_1 + J_2. \end{aligned} \quad (3.26)$$

By definition of b, we have $\|b\|_{L^1(\mu)} \le |b|_{H^1(\mu)}$. By (3.25) in the proof of Theorem 1.4, we can get

$$J_1 \lesssim \int_{\mathbb{R}^d\setminus\alpha R} g^*_{\lambda,\mu}(b)(x)d\mu(x) \lesssim \int_R |b(z)|d\mu(z) \lesssim |b|_{H^1(\mu)}. \quad (3.27)$$

To estimate J_2, let $b = \sum_j \lambda_j a_j$ be as in Definition 1.1 and write

$$
\begin{aligned}
J_2 &= \int_{\alpha R \setminus \cup \alpha Q_j} g^*_{\lambda,\mu}(b)(x)d\mu(x) + \int_{\cup \alpha Q_j} g^*_{\lambda,\mu}(b)(x)d\mu(x) \\
&\leq \sum_j |\lambda_j| \int_{\alpha R \setminus \alpha Q_j} g^*_{\lambda,\mu}(a_j)(x)d\mu(x) + \sum_j |\lambda_j| \int_{\alpha Q_j} g^*_{\lambda,\mu}(a_j)(x)d\mu(x).
\end{aligned}
\tag{3.28}
$$

By the assumption of L^2 boundedness of $g^*_{\lambda,\mu}$ and the Holder inequality,

$$
\begin{aligned}
&\int_{\alpha Q_j} g^*_{\lambda,\mu}(a_j)(x)d\mu(x) \\
&\quad \leq \left\| g^*_{\lambda,\mu} \right\|_{L^2(\mu)} \cdot [\mu(\alpha Q_j)]^{1/2} \leq C \left\| g^*_{\lambda,\mu} \right\| \cdot \|a_j\|_{L^2(\mu)} \cdot [\mu(\alpha Q_j)]^{1/2} \\
&\quad \leq C \|a_j\|_{L^\infty} \cdot \mu(\alpha Q_j) \leq C \left[\mu(\alpha Q_j) S_{Q_j,R}\right]^{-1} \cdot \mu(\alpha Q_j) \leq C.
\end{aligned}
\tag{3.29}
$$

By (3.11) in the proof of Theorem 1.4,

$$
\begin{aligned}
\int_{\alpha R \setminus \alpha Q_j} g^*_{\lambda,\mu}(a_j)(x)d\mu(x) &\lesssim \int_{\alpha R \setminus \alpha Q_j} \frac{1}{|x - x_j|^n} \int_{Q_j} |a_j(y)| d\mu(y) d\mu(x) \\
&\lesssim \|a_j\|_{L^1(\mu)} \int_{\alpha R \setminus \alpha Q_j} \frac{1}{|x - x_j|^n} d\mu(x)
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
&\lesssim \|a_j\|_{L^1(\mu)} \int_{\alpha R_j \setminus \alpha Q_j} \frac{1}{|x - x_j|^n} d\mu(x) \\
&\lesssim \|a_j\|_{L^\infty} \cdot \mu(Q_j) \cdot S_{Q_j,R},
\end{aligned}
\tag{3.31}
$$

where R_j is a cube and has the same center as Q_j and $l(R_j) = l(R)$, x_j is the center of Q_j. From (3.29) to (3.30), we use the conclusion

$$
\int_{\alpha R_j \setminus \alpha Q_j} \frac{1}{|x - x_j|^n} d\mu(x) \lesssim S_{Q_j,R}.
\tag{3.32}
$$

As a matter of fact, let $N = N_{Q_j,R}$ and then by the definition of N, we get that

$$\alpha R_j \setminus \alpha Q_j \subset \bigcup_{k=1}^{N}\left(\alpha^{k+1}Q_j \setminus \alpha^k Q_j\right),$$

$$\begin{aligned}\int_{\alpha R_j\setminus\alpha Q_j}\frac{1}{|x-x_j|^n}d\mu(x) &\le \sum_{k=1}^{N}\int_{\alpha^{k+1}Q_j\setminus\alpha^k Q_j}\frac{1}{|x-x_j|^n}d\mu(x)\\ &\le \sum_{k=1}^{N}\frac{\mu(\alpha^{k+1}Q_j)}{((1/2)l(\alpha^k Q_j))^n} \lesssim \sum_{k=1}^{N}\frac{\mu(\alpha^{k+1}Q_j)}{l(\alpha^{k+1}Q_j)^n}\\ &\lesssim S^{\alpha}_{Q_j,R} \approx S_{Q_j,R}.\end{aligned} \tag{3.33}$$

Using (3.30) and the fact that

$$\|a_j\|_{L^\infty(\mu)} \le [\mu(\rho Q_j)S_{Q,R}]^{-1}, \tag{3.34}$$

we have

$$\begin{aligned}\int_{\alpha R\setminus\alpha Q_j} g^*_{\lambda,\mu}(a_j)(x)d\mu(x) &\lesssim \|a_j\|_{L^\infty}\cdot\mu(Q_j)\cdot S_{Q_j,R}\\ &\lesssim \left[\mu(\alpha Q_j)S_{Q_j,R}\right]^{-1}\cdot\mu(Q_j)\cdot S_{Q_j,R} \lesssim 1.\end{aligned} \tag{3.35}$$

From (3.29) and (3.35), we can obtain that

$$\int_{\alpha R} g^*_{\lambda,\mu}(b)(x)d\mu(x) \lesssim \sum_j|\lambda_j|. \tag{3.36}$$

We have

$$J_2 \lesssim |b|_{H^1(\mu)}. \tag{3.37}$$

Combining (3.27) and (3.37), we finish the proof of Theorem 1.5. □

Proof of Theorem 1.6. Now we begin to prove Theorem 1.6. First we claim that there is a positive constant C such that for any $f \in L^\infty(\mu)$ and $(4\sqrt{d}, (4\sqrt{d})^{n+1})$-doubling cube Q, the following inequality holds:

$$\frac{1}{\mu(Q)}\int_Q g^*_{\lambda,\mu}(f)(y)d\mu(y) \le C\|f\|_{L^\infty(\mu)} + \inf_{y\in Q} g^*_{\lambda,\mu}(f)(y). \tag{3.38}$$

To prove (3.38), for each fixed cube Q, let B be the smallest ball which contains Q and has the same center as Q. Then $2B \subset 4\sqrt{d}Q$. We decompose f as

$$f(x) = f(x)\chi_{2B} + f(x)\chi_{\mathbb{R}^d\setminus 2B} := f_1(x) + f_2(x). \tag{3.39}$$

By the Hölder's inequality and $L^2(\mu)$ boundedness of $g^*_{\lambda,\mu}$, we have

$$\begin{aligned}
\frac{1}{\mu(Q)}\int_Q g^*_{\lambda,\mu}(f_1)(x)d\mu(x) &\le \frac{1}{\mu(Q)}\left[\int_Q \left|g^*_{\lambda,\mu}(f_1)(x)\right|^2 d\mu(x)\right]^{1/2}\mu(Q)^{1/2} \lesssim \frac{\|f_1\|_{L^2(\mu)}}{\mu(Q)^{1/2}} \\
&\lesssim \|f\|_{L^\infty(\mu)}\left(\frac{\mu(2B)}{\mu(Q)}\right)^{1/2} \lesssim \|f\|_{L^\infty(\mu)}\left(\frac{\mu\left(4\sqrt{d}Q\right)}{\mu(Q)}\right)^{1/2} \\
&\lesssim \|f\|_{L^\infty(\mu)}.
\end{aligned} \tag{3.40}$$

We denote by r the radius of B. Note that $|x-z| \ge r$ for any $x \in Q$ and for any $z \in \mathbb{R}^d \setminus 2B$, then by Minkowski's inequality, we have

$$\begin{aligned}
&g^*_{\lambda,\mu}(f_2)(x) \\
&\quad= \left(\iint_{\mathbb{R}^{d+1}_+}\left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n/2}\int_{|x-z|\ge r}\varphi_t(y-z)f_2(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2} \\
&\quad\le \left(\iint_{\mathbb{R}^{d+1}_+}\left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n/2}\int_{|x-z|\ge r}\varphi_t(y-z)f(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2} \\
&\qquad+ \left(\iint_{\mathbb{R}^{d+1}_+}\left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n/2}\int_{|x-z|\le r}\varphi_t(y-z)f_1(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2} \\
&\quad\le g^*_{\lambda,\mu}(f)(x) + \left(\iint_{\mathbb{R}^{d+1}_+}\left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n/2}\int_{|x-z|\le r}\varphi_t(y-z)f_1(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2}.
\end{aligned} \tag{3.41}$$

Since we have the following estimate:

$$\left(\iint_{\mathbb{R}_+^{d+1}}\left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n/2}\int_{|x-z|\le r}\varphi_t(y-z)f_1(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2}$$
$$\lesssim \|f\|_{L^\infty(\mu)}\int_{r\le|x-z|\le 3r}\left(\int_{\mathbb{R}_+^{d+1}}\left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon}\left(\frac{t}{t+|y-z|}\right)^{2n+2\epsilon}\frac{d\mu(y)dt}{t^{3n+1}}\right)^{1/2}d\mu(z). \tag{3.42}$$

We claim

$$\int_{\mathbb{R}_+^{d+1}}\left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon}\left(\frac{t}{t+|y-z|}\right)^{2n+2\epsilon}\frac{d\mu(y)dt}{t^{3n+1}}\lesssim\frac{1}{|z-x|^{2n}}. \tag{3.43}$$

Then

$$\int_{r\le|x-z|\le 3r}\left(\int_{\mathbb{R}_+^{d+1}}\left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon}\left(\frac{t}{t+|y-z|}\right)^{2n+2\epsilon}\frac{d\mu(y)dt}{t^{3n+1}}\right)^{1/2}d\mu(z)\lesssim 1. \tag{3.44}$$

By (3.42), (3.44), and (3.45), we can obtain

$$g^*_{\lambda,\mu}(f_2)(x)\lesssim g^*_{\lambda,\mu}(f)(x)+\|f\|_{L^\infty(\mu)}. \tag{3.45}$$

The method to prove (3.43) is quite similar as to prove (2.4) in the proof of Theorem 1.4 and we omit it.

Thus, to prove (3.38), we only need to prove for any $x, y \in Q$, the following inequality holds:

$$\left|g^*_{\lambda,\mu}(f_2)(x)-g^*_{\lambda,\mu}(f_2)(y)\right|\lesssim\|f\|_{L^\infty(\mu)}. \tag{3.46}$$

We note that $g^*_{\lambda,\mu}$ can be looked as a vector valued Calderón-Zygmund singular integral operator in the following Hilbert space H:

$$H=\left\{h:\|h\|_H=\left(\int_0^\infty\int_{\mathbb{R}^d}|h(t,y)|^2\frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2}<\infty\right\}. \tag{3.47}$$

In fact, $g^*_{\lambda,\mu}$ can be written by

$$g^*_{\lambda,\mu}f(x) = \left(\int_{\mathbb{R}^{d+1}_+}\left|\left(\frac{t}{t+|x-y|}\right)^{\lambda/2}\int \varphi_t(y-z)f(z)d\mu(z)\right|^2\frac{d\mu(y)dt}{t^{n+1}}\right)^{1/2} := \|\varphi_{t,y}(f)(x)\|_H, \tag{3.48}$$

where $\varphi_{t,y}(f)(x) = (t/(t+|x-y|))^{\lambda n/2}\int\varphi_t(y-z)f(z)d\mu(z)$. Also note that for $u,v\in Q$,

$$\begin{aligned}\left|g^*_{\lambda,\mu}(f_2)(u) - g^*_{\lambda,\mu}(f_2)(v)\right| &= \left|\|\varphi_{t,y}(f_2)(u)\|_H - \|\varphi_{t,y}(f_2)(v)\|_H\right| \\ &\le \left\|\left|\varphi_{t,y}(f_2)(u) - \varphi_{t,y}(f_2)(v)\right|\right\|_H \\ &:= I,\end{aligned} \tag{3.49}$$

where $\varphi_{t,y}(f)(x) = (t/(t+|x-y|))^{\lambda n/2}\int\varphi_t(y-z)f(z)dz$. We will divide I into four parts, namely,

$$\begin{aligned} I \le{}& \left\|\left|\varphi_{t,y}(f_2)(u) - \varphi_{t,y}(f_2)(v)\right|\chi_{\{|u-y|\le t,|v-y|\le t\}}\right\|_H \\ &+ \left\|\left|\varphi_{t,y}(f_2)(u) - \varphi_{t,y}(f_2)(v)\right|\chi_{\{|u-y|> t,|v-y|\le t\}}\right\|_H \\ &+ \left\|\left|\varphi_{t,y}(f_2)(u) - \varphi_{t,y}(f_2)(v)\right|\chi_{\{|u-y|\le t,|v-y|> t\}}\right\|_H \\ &+ \left\|\left|\varphi_{t,y}(f_2)(u) - \varphi_{t,y}(f_2)(v)\right|\chi_{\{|u-y|> t,|v-y|> t\}}\right\|_H \\ :={}& I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.50}$$

Then

$$I_i \le CM(f)(x), \quad \text{for } i = 1,\dots,4. \tag{3.51}$$

This can be obtained from the same idea used before, see also the main step in [14], here we omit the proof of it.

From (3.38), we get that for $f\in L^\infty(\mu)$, if $g^*_{\lambda,\mu}(f)(x_0)<\infty$ for some point $x_0\in\mathbb{R}^d$, then $g^*_{\lambda,\mu}(f)$ is μ-finite almost everywhere and in this case we have

$$m_Q\left(g^*_{\lambda,\mu}(f)\right) - \operatorname*{ess\,inf}_{x\in Q} g^*_{\lambda,\mu}(f)(x) \lesssim \|f\|_{L^\infty(\mu)}, \tag{3.52}$$

provided that Q is a $(4\sqrt{d},(4\sqrt{d})^{n+1})$-doubling cube.

To prove $g_{\lambda,\mu}^{*}(f) \in \mathrm{RBLO}(\mu)$, we still need to prove that $g_{\lambda,\mu}^{*}(f)$ satisfies

$$m_Q\left(g_{\lambda,\mu}^{*}(f)\right) - m_R\left(g_{\lambda,\mu}^{*}(f)\right) \leq CS_{Q,R} \tag{3.53}$$

for any two $((4\sqrt{d}), (4\sqrt{d})^{n+1})$-doubling cubes $Q \subset R$.

Let $\alpha_1 = 4\sqrt{d}$ and set $N = N_{Q,R} + 1$, then

$$\begin{aligned}
g_{\lambda,\mu}^{*}(f)(x) &\leq C g_{\lambda,\mu}^{*}(f\chi_{\alpha_1 Q}) + g_{\lambda,\mu}^{*}\left(f\chi_{(\alpha_Q)^c}\right)\\
&\leq C g_{\lambda,\mu}^{*}(f\chi_{\alpha_1 Q}) + g_{\lambda,\mu}^{*}\left(f\chi_{\alpha_1^N Q\setminus \alpha_1 Q}\right) + g_{\lambda,\mu}^{*}\left(f\chi_{(\alpha_1^N Q)^c}\right)\\
&\leq g_{\lambda,\mu}^{*}(f\chi_{\alpha_1 Q}) + \Sigma_{k=1}^{N_{Q,R}} g_{\lambda,\mu}^{*}\left(f\chi_{\alpha_1^{k+1} Q\setminus \alpha_1^k Q}\right)(x)\\
&\quad + \left[g_{\lambda,\mu}^{*}\left(f\chi_{(\alpha_1^N Q)^c}\right)(x) - g_{\lambda,\mu}^{*}\left(f\chi_{(\alpha_1^N Q)^c}\right)(y)\right]\\
&\quad + g_{\lambda,\mu}^{*}\left(f\chi_{(\alpha_1^N Q)^c}\right)(y).
\end{aligned} \tag{3.54}$$

Again, we can get the conclusion under the nondoubling condition which is similar to the proof of (2.2) in [14] that

$$\left|g_{\lambda,\mu}^{*}\left(f\chi_{(\alpha_1^N Q)^c}\right)(x) - g_{\lambda,\mu}^{*}\left(f\chi_{(\alpha_1^N Q)^c}\right)(y)\right| \lesssim \|f\|_{L^\infty(\mu)}. \tag{3.55}$$

For any $x \in Q$ and each fixed $k \in \mathbb{N}$, similar to the proof of (2.4) from Lemma 2.3, we have that

$$\begin{aligned}
&g_{\lambda,\mu}^{*}\left(f\chi_{\alpha_1^{k+1} Q\setminus \alpha_1^k Q}\right)(x)\\
&\quad\leq C\int_{\alpha_1^{k+1} Q\setminus \alpha_1^k Q} |f(z)| \left[\int_{\mathbb{R}_+^{d+1}} \left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon} \left(\frac{t}{t+|y-z|}\right)^{2n+2\epsilon} \frac{d\mu(y)dt}{t^{3n+1}}\right]^{1/2} d\mu(z)\\
&\quad\leq C\|f\|_{L^\infty(\mu)}\int_{\alpha_1^{k+1} Q\setminus \alpha_1^k Q} \left[\int_{\mathbb{R}_+^{d+1}} \left(\frac{t}{t+|x-y|}\right)^{2n+2\epsilon} \left(\frac{t}{t+|y-z|}\right)^{2n+2\epsilon} \frac{d\mu(y)dt}{t^{3n+1}}\right]^{1/2} d\mu(z)\\
&\quad\leq C\|f\|_{L^\infty(\mu)}\int \frac{1}{|x-x_i|^n} d\mu(x).
\end{aligned} \tag{3.56}$$

Therefore we have

$$\sum_{k=1}^{N_{Q,R}} g^{*}_{\lambda,\mu}\left(f\chi_{\alpha_1^{k+1}Q\setminus\alpha_1^k Q}\right)(x) \leq C\|f\|_{L^\infty(\mu)} S_{Q,R}. \tag{3.57}$$

Next we estimate $g^{*}_{\lambda,\mu}(f\chi_{(\alpha_1^N Q)^c})(y)$. By an estimate similar to (3.45), we have

$$g^{*}_{\lambda,\mu}\left(f\chi_{(\alpha_1^N Q)^c}\right)(y) \lesssim g^{*}_{\lambda,\mu}(f)(y) + \|f\|_{L^\infty(\mu)}. \tag{3.58}$$

By (3.54), (3.55), (3.57), and (3.58), we can get that

$$g^{*}_{\lambda,\mu}(f)(x) \lesssim g^{*}_{\lambda,\mu}(f\chi_{\alpha_1 Q})(x) + \|f\|_{L^\infty(\mu)} S_{Q,R} + \|f\|_{L^\infty(\mu)} + g^{*}_{\lambda,\mu}(f)(y). \tag{3.59}$$

Take mean value over Q for x and over R for y, we get

$$m_Q\left(g^{*}_{\lambda,\mu}(f)\right) \lesssim m_Q\left(g^{*}_{\lambda,\mu}(f\chi_{\alpha_1 Q})\right) + \|f\|_{L^\infty(\mu)} S_{Q,R} + m_R\left(g^{*}_{\lambda,\mu}(f)\right). \tag{3.60}$$

Therefore

$$\begin{aligned} m_Q&\left(g^{*}_{\lambda,\mu}(f)\right) - m_R\left(g^{*}_{\lambda,\mu}(f)\right) \\ &\leq C\|f\|_{L^\infty(\mu)} S_{Q,R} + m_Q\left(g^{*}_{\lambda,\mu}(f\chi_{\alpha_1 Q})\right) + m_R\left(g^{*}_{\lambda,\mu}\left(f\chi_{\alpha_1^N Q}\right)\right). \end{aligned} \tag{3.61}$$

Since by Hölder's inequality, the $L^2(\mu)$ boundedness assumption of $g^{*}_{\lambda,\mu}$ and the fact that Q is $(\alpha_1, \alpha_1^{n+1})$-doubling, we have that

$$\begin{aligned} m_Q\left(g^{*}_{\lambda,\mu}(f\chi_{\alpha_1 Q})\right) &\leq \frac{C}{\mu(Q)^{1/2}}\left[\int_Q g^{*}_{\lambda,\mu}(f\chi_{\alpha_1 Q})(x)^2 d\mu(x)\right]^{1/2} \\ &\leq \frac{C\|f\chi_{\alpha_1 Q}\|_{L^2(\mu)}}{\mu(Q)^{1/2}} \leq \frac{C\|f\|_{L^\infty(\mu)}\mu(\alpha_1 Q)^{1/2}}{\mu(Q)^{1/2}} \\ &\leq C\|f\|_{L^\infty(\mu)}, \end{aligned} \tag{3.62}$$

which completes the proof of Theorem 1.6. □

Using Theorems 1.4 and 1.5 and [4, Theorem 3.1], Corollary 1.7 is obvious.

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