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Research Article

General Comparison Principle for Variational-Hemivariational Inequalities

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We study quasilinear elliptic variational-hemivariational inequalities involving general Leray-Lions operators. The novelty of this paper is to provide existence and comparison results whereby only a local growth condition on Clarke's generalized gradient is required. Based on these results, in the second part the theory is extended to discontinuous variational-hemivariational inequalities.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. By $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, $1 < p < \infty$, we denote the usual Sobolev spaces with their dual spaces $(W^{1,p}(\Omega))^*$ and $W^{-1,q}(\Omega)$, respectively, where q is the Hölder conjugate satisfying $1/p + 1/q = 1$. We consider the following elliptic variational-hemivariational inequality. Find $u \in K$ such that

$$\langle Au + F(u), v - u \rangle + \int_\Omega j_1^0(\cdot, u; v - u)\,dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u)\,d\sigma \geq 0, \quad \forall v \in K, \tag{1.1}$$

where $j_k^0(x, s; r)$, $k = 1, 2$ denotes the generalized directional derivative of the locally Lipschitz functions $s \mapsto j_k(x, s)$ at s in the direction r given by

$$j_k^0(x, s; r) = \limsup_{y \to s, t \downarrow 0} \frac{j_k(x, y + tr) - j_k(x, y)}{t}, \quad k = 1, 2 \tag{1.2}$$

(cf. [1, Chapter 2]). We denote by K a closed convex subset of $W^{1,p}(\Omega)$, and A is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)). \tag{1.3}$$

The operator F stands for the Nemytskij operator associated with some Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$F(u)(x) = f(x, u(x), \nabla u(x)). \tag{1.4}$$

Furthermore, we denote the trace operator by $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is known to be linear, bounded, and even compact.

The aim of this paper is to establish the method of sub- and supersolutions for problem (1.1). We prove the existence of solutions between a given pair of sub-supersolution assuming only a local growth condition of Clarke's generalized gradient, which extends results recently obtained by Carl in [2]. To complete our findings, we also give the proof for the existence of extremal solutions of problem (1.1) for a fixed ordered pair of sub- and supersolutions in case A has the form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)). \tag{1.5}$$

In the second part we consider (1.1) with a discontinuous Nemytskij operator F involved, which extends results in [3] and partly of [4]. Let us consider next some special cases of problem (1.1), where we suppose $A = -\Delta_p$.

(1) If $K = W^{1,p}(\Omega)$ and j_k are smooth, problem (1.1) reduces to

$$\langle -\Delta_p u + F(u), v\rangle + \int_\Omega j_1'(\cdot, u) v \, dx + \int_{\partial\Omega} j_2'(\cdot, \gamma u)\gamma v \, d\sigma = 0, \quad \forall v \in W^{1,p}(\Omega), \tag{1.6}$$

which is equivalent to the weak formulation of the nonlinear boundary value problem

$$\begin{aligned} -\Delta_p u + F(u) + j_1'(u) = 0 \quad &\text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + j_2'(\gamma u) = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

where $\partial u / \partial \nu$ denotes the conormal derivative of u. The method of sub- and supersolution for this kind of problems is a special case of [5].

(2) For $f \in V_0^*$, $K \subset W_0^{1,p}(\Omega)$ and $j_2 = 0$, (1.1) corresponds to the variational-hemivariational inequality given by

$$\langle -\Delta_p u + f, v-u\rangle + \int_\Omega j_1^0(\cdot, u; v-u)dx \geq 0, \quad \forall v \in K, \tag{1.8}$$

which has been discussed in detail in [6].

(3) If $K \subset W_0^{1,p}(\Omega)$ and $j_k = 0$, then (1.1) is a classical variational inequality of the form

$$u \in K : \langle -\Delta_p u + F(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{1.9}$$

whose method of sub- and supersolution has been developed in [7, Chapter 5].

(4) Let $K = W_0^{1,p}(\Omega)$ or $K = W^{1,p}(\Omega)$ and j_k not necessarily smooth. Then problem (1.1) is a hemivariational inequality, which contains for $K = W_0^{1,p}(\Omega)$ as a special case the following Dirichlet problem for the elliptic inclusion:

$$\begin{gathered} -\Delta_p u + F(u) + \partial j_1(\cdot, u) \ni 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.10}$$

and for $K = W^{1,p}(\Omega)$ the elliptic inclusion

$$\begin{gathered} -\Delta_p u + F(u) + \partial j_1(\cdot, u) \ni 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \partial j_2(\cdot, u) \ni 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.11}$$

where the multivalued functions $s \mapsto \partial j_k(x,s)$, $k = 1,2$ stand for Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j_k(x,s)$, $k = 1,2$ given by

$$\partial j_k(x,s) = \left\{ \xi \in \mathbb{R} : j_k^0(x,s;r) \geq \xi r, \forall r \in \mathbb{R} \right\}. \tag{1.12}$$

Problems of the form (1.10) and (1.11) have been studied in [5, 8], respectively.

Existence results for variational-hemivariational inequalities with or without the method of sub- and supersolutions have been obtained under different structure and regularity conditions on the nonlinear functions by various authors. For example, we refer to [9–16]. In case that K is the whole space $W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$, respectively, problem (1.1) reduces to a hemivariational inequality which has been treated in [17–25].

Comparison principles for general elliptic operators A, including the negative p-Laplacian $-\Delta_p$, Clarke's generalized gradient $s \mapsto \partial j(x,s)$, satisfying a one-sided growth condition in the form

$$\xi_1 \leq \xi_2 + c_1(s_2 - s_1)^{p-1} \tag{1.13}$$

for all $\xi_i \in \partial j(x, s_i)$, $i = 1, 2$, for a.a. $x \in \Omega$, and for all s_1, s_2 with $s_1 < s_2$, can be found in [7]. Inspired by results recently obtained in [8, 26], we prove the existence of (extremal) solutions for the variational-hemivariational inequality (1.1) within a sector of an ordered pair of sub- and supersolutions $\underline{u}$, $\overline{u}$ without assuming a one-sided growth condition on Clarke's gradient of the form (1.13).

2. Notation of Sub- and Supersolution

For functions $u, v : \Omega \to \mathbb{R}$ we use the notation $u \wedge v = \min(u, v)$, $u \vee v = \max(u, v)$, $K \wedge K = \{u \wedge v : u, v \in K\}$, $K \vee K = \{u \vee v : u, v \in K\}$, and $u \wedge K = \{u\} \wedge K$, $u \vee K = \{u\} \vee K$ and introduce the following definitions.

Definition 2.1. A function $\underline{u} \in W^{1,p}(\Omega)$ is said to be a subsolution of (1.1) if the following holds:

(1) $F(\underline{u}) \in L^q(\Omega)$;

(2) $\langle A\underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_\Omega j_1^0(\cdot, \underline{u}; w - \underline{u})dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \geq 0$, $\forall w \in \underline{u} \wedge K$.

Definition 2.2. A function $\overline{u} \in W^{1,p}(\Omega)$ is said to be a supersolution of (1.1) if the following holds:

(1) $F(\overline{u}) \in L^q(\Omega)$;

(2) $\langle A\overline{u} + F(\overline{u}), w - \overline{u} \rangle + \int_\Omega j_1^0(\cdot, \overline{u}; w - \overline{u})dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma\overline{u}; \gamma w - \gamma\overline{u})d\sigma \geq 0$, $\forall w \in \overline{u} \vee K$.

In order to prove our main results, we additionally suppose the following assumptions:

$$\underline{u} \vee K \subset K, \qquad \overline{u} \wedge K \subset K. \tag{2.1}$$

3. Preliminaries and Hypotheses

Let $1 < p < \infty$, $1/p + 1/q = 1$, and assume for the coefficients $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, \ldots, N$ the following conditions.

(A1) Each $a_i(x, s, \xi)$ satisfies Carathéodory conditions, that is, is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in (s, ξ) for a.e. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist so that

$$|a_i(x, s, \xi)| \leq k_0(x) + c_0\left(|s|^{p-1} + |\xi|^{p-1}\right) \tag{3.1}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ.

(A2) The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^{N}\left(a_i(x,s,\xi)-a_i(x,s,\xi')\right)(\xi_i-\xi_i')>0 \tag{3.2}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) A constant $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ exist such that

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq c_1|\xi|^p - k_1(x) \tag{3.3}$$

for a.e. $x \in \Omega$, for all $s \in R$, and for all $\xi \in \mathbb{R}^N$.

Condition (A1) implies that $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is bounded continuous and along with (A2); it holds that A is pseudomonotone. Due to (A1) the operator A generates a mapping from $W^{1,p}(\Omega)$ into its dual space defined by

$$\langle Au, \varphi\rangle = \int_{\Omega}\sum_{i=1}^{N} a_i(x,u,\nabla u)\frac{\partial\varphi}{\partial x_i}dx, \tag{3.4}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$, and assumption (A3) is a coercivity type condition.

Let $[\underline{u}, \overline{u}]$ be an ordered pair of sub- and supersolutions of problem (1.1). We impose the following hypotheses on j_k and the nonlinearity f in problem (1.1).

(j1) $x \mapsto j_1(x,s)$ and $x \mapsto j_2(x,s)$ are measurable in Ω and $\partial\Omega$, respectively, for all $s \in \mathbb{R}$.

(j2) $s \mapsto j_1(x,s)$ and $s \mapsto j_2(x,s)$ are locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.

(j3) There are functions $L_1 \in L_+^q(\Omega)$ and $L_2 \in L_+^q(\partial\Omega)$ such that for all $s \in [\underline{u}(x), \overline{u}(x)]$ the following local growth conditions hold:

$$\begin{aligned} &\eta \in \partial j_1(x,s) : |\eta| \leq L_1(x), \quad \text{for a.a. } x \in \Omega, \\ &\xi \in \partial j_2(x,s) : |\xi| \leq L_2(x), \quad \text{for a.a. } x \in \partial\Omega. \end{aligned} \tag{3.5}$$

(F1) (i) $x \mapsto f(x,s,\xi)$ is measurable in Ω for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

(ii) $(s,\xi) \mapsto f(x,s,\xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for a.a. $x \in \Omega$.

(iii) There exist a constant $c_2 > 0$ and a function $k_3 \in L_+^q(\Omega)$ such that

$$|f(x,s,\xi)| \leq k_3(x) + c_2|\xi|^{p-1} \tag{3.6}$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, and for all $s \in [\underline{u}(x), \overline{u}(x)]$.

Note that the associated Nemytskij operator F defined by $F(u)(x) = f(x, u(x), \nabla u(x))$ is continuous and bounded from $[\underline{u}, \overline{u}] \subset W^{1,p}(\Omega)$ to $L^q(\Omega)$ (cf. [27]). We recall that the normed space $L^p(\Omega)$ is equipped with the natural partial ordering of functions defined by $u \le v$ if and only if $v - u \in L^p_+(\Omega)$, where $L^p_+(\Omega)$ is the set of all nonnegative functions of $L^p(\Omega)$.

Based on an approach in [8], the main idea in our considerations is to modify the functions j_k. First we set for $k = 1, 2$

$$\alpha_k(x) := \min\{\xi : \xi \in \partial j_k(x, \underline{u}(x))\}, \qquad \beta_k(x) := \max\{\xi : \xi \in \partial j_k(x, \overline{u}(x))\}. \tag{3.7}$$

By means of (3.7) we introduce the mappings $\tilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{j}_k(x, s) = \begin{cases} j_k(x, \underline{u}(x)) + \alpha_k(x)(s - \underline{u}(x)), & \text{if } s < \underline{u}(x), \\ j_k(x, s), & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ j_k(x, \overline{u}(x)) + \beta_k(x)(s - \overline{u}(x)), & \text{if } s > \overline{u}(x). \end{cases} \tag{3.8}$$

The following lemma provides some properties of the functions $\tilde{j}_1$ and $\tilde{j}_2$.

Lemma 3.1. *Let the assumptions in (j1)–(j3) be satisfied. Then the modified functions $\tilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ have the following qualities.*

($\tilde{j}$1) *$x \mapsto \tilde{j}_1(x, s)$ and $x \mapsto \tilde{j}_2(x, s)$ are measurable in Ω and $\partial\Omega$, respectively, for all $s \in \mathbb{R}$, and $s \mapsto \tilde{j}_1(x, s)$ and $s \mapsto \tilde{j}_2(x, s)$ are locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.*

($\tilde{j}$2) *Let $\partial\tilde{j}_k(x, s)$ be Clarke's generalized gradient of $s \mapsto \tilde{j}_k(x, s)$. Then for all $s \in \mathbb{R}$ the following estimates hold true:*

$$\begin{aligned} &\eta \in \partial\tilde{j}_1(x, s) : |\eta| \le L_1(x), \quad \text{for a.a. } x \in \Omega, \\ &\xi \in \partial\tilde{j}_2(x, s) : |\xi| \le L_2(x), \quad \text{for a.a. } x \in \partial\Omega. \end{aligned} \tag{3.9}$$

($\tilde{j}$3) *Clarke's generalized gradients of $s \mapsto \tilde{j}_1(x, s)$ and $s \mapsto \tilde{j}_2(x, s)$ are given by*

$$\partial\tilde{j}_k(x, s) = \begin{cases} \alpha_k(x), & \text{if } s < \underline{u}(x), \\ \partial\tilde{j}_k(x, \underline{u}(x)), & \text{if } s = \underline{u}(x), \\ \partial j_k(x, s), & \text{if } \underline{u}(x) < s < \overline{u}(x), \\ \partial\tilde{j}_k(x, \overline{u}(x)), & \text{if } s = \overline{u}(x), \\ \beta_k(x), & \text{if } s > \overline{u}(x), \end{cases} \tag{3.10}$$

and the inclusions $\partial\tilde{j}_k(x, \underline{u}(x)) \subset \partial j_k(x, \underline{u}(x))$ and $\partial\tilde{j}_k(x, \overline{u}(x)) \subset \partial j_k(x, \overline{u}(x))$ are valid for $k = 1, 2$.

Proof. With a view to the assumptions (j1)–(j3) and the definition of $\tilde{j}_k$ in (3.8), one verifies the lemma in few steps. □

With the aid of Lemma 3.1, we introduce the integral functionals J_1 and J_2 defined on $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively, given by

$$J_1(u) = \int_\Omega \tilde{j}_1(x, u(x))dx, \quad u \in L^p(\Omega), \qquad J_2(v) = \int_{\partial\Omega} \tilde{j}_2(x, v(x))d\sigma, \quad v \in L^p(\partial\Omega). \tag{3.11}$$

Due to the properties ($\tilde{j}$1)–($\tilde{j}$2) and Lebourg's mean value theorem (see [1, Chapter 2]), the functionals $J_1 : L^p(\Omega) \to \mathbb{R}$ and $J_2 : L^p(\partial\Omega) \to \mathbb{R}$ are well defined and Lipschitz continuous on bounded sets of $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. This implies among others that Clarke's generalized gradients $\partial J_1 : L^p(\Omega) \to 2^{L^q(\Omega)}$ and $\partial J_2 : L^p(\partial\Omega) \to 2^{L^q(\partial\Omega)}$ are well defined, too. Furthermore, by means of Aubin-Clarke's theorem (see [1]), for $u \in L^p(\Omega)$ and $v \in L^p(\partial\Omega)$ we get

$$\begin{gathered}\eta \in \partial J_1(u) \Longrightarrow \eta \in L^q(\Omega) \quad \text{with } \eta(x) \in \partial\tilde{j}_1(x, u(x)) \text{ for a.a. } x \in \Omega,\\ \xi \in \partial J_2(v) \Longrightarrow \xi \in L^q(\partial\Omega) \quad \text{with } \xi(x) \in \partial\tilde{j}_2(x, v(x)) \text{ for a.a. } x \in \partial\Omega.\end{gathered} \tag{3.12}$$

An important tool in our considerations is the following surjectivity result for multivalued pseudomonotone mappings perturbed by maximal monotone operators in reflexive Banach spaces.

Theorem 3.2. *Let X be a real reflexive Banach space with the dual space X^*, $\Phi : X \to 2^{X^*}$ a maximal monotone operator, and $u_0 \in \mathrm{dom}(\Phi)$. Let $A : X \to 2^{X^*}$ be a pseudomonotone operator, and assume that either A_{u_0} is quasibounded or Φ_{u_0} is strongly quasibounded. Assume further that $A : X \to 2^{X^*}$ is u_0-coercive, that is, there exists a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all $(u, u^*) \in \mathrm{graph}(A)$ one has $\langle u^*, u - u_0\rangle \geq c(\|u\|_X)\|u\|_X$. Then $A + \Phi$ is surjective, that is,* $\mathrm{range}(A + \Phi) = X^*$.

The proof of the theorem can be found, for example, in [28, Theorem 2.12]. The notation A_{u_0} and Φ_{u_0} stand for $A_{u_0}(u) := A(u_0 + u)$ and $\Phi_{u_0}(u) := \Phi(u_0 + u)$, respectively. Note that any bounded operator is, in particular, also quasibounded and strongly quasibounded. For more details we refer to [28]. The next proposition provides a sufficient condition to prove the pseudomonotonicity of multivalued operators and plays an important part in our argumentations. The proof is presented, for example, in [28, Chapter 2].

Proposition 3.3. *Let X be a reflexive Banach space, and assume that $A : X \to 2^{X^*}$ satisfies the following conditions:*

(i) *for each $u \in X$ one has that $A(u)$ is a nonempty, closed, and convex subset of X^*;*

(ii) *$A : X \to 2^{X^*}$ is bounded;*

(iii) *if $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* with $u_n^* \in A(u_n)$ and if $\limsup\langle u_n^*, u_n - u\rangle \leq 0$, then $u^* \in A(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.*

Then the operator $A : X \to 2^{X^*}$ is pseudomonotone.

We denote by $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ and $\gamma^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ the adjoint operators of the imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ and the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, respectively, given by

$$\langle i^*\eta, \varphi\rangle = \int_\Omega \eta\varphi\, dx, \quad \forall \varphi \in W^{1,p}(\Omega), \qquad \langle \gamma^*\xi, \varphi\rangle = \int_{\partial\Omega} \xi\gamma\varphi\, d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \tag{3.13}$$

Next, we introduce the following multivalued operators:

$$\Phi_1(u) := (i^* \circ \partial J_1 \circ i)(u), \qquad \Phi_2(u) := \left(\gamma^* \circ \partial J_2 \circ \gamma\right)(u), \tag{3.14}$$

where i, i^*, γ, γ^* are defined as mentioned above. The operators Φ_k, $k = 1,2$, have the following properties (see, e.g., [5, Lemmas 3.1 and 3.2]).

Lemma 3.4. *The multivalued operators $\Phi_1 : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ and $\Phi_2 : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ are bounded and pseudomonotone.*

Let $b : \Omega \times \mathbb{R} \to \mathbb{R}$ be the cutoff function related to the given ordered pair $\underline{u}$, $\overline{u}$ of sub- and supersolutions defined by

$$b(x,s) = \begin{cases} (s - \overline{u}(x))^{p-1}, & \text{if } s > \overline{u}(x), \\ 0, & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ -\left(\underline{u}(x) - s\right)^{p-1}, & \text{if } s < \underline{u}(x). \end{cases} \tag{3.15}$$

Clearly, the mapping b is a Carathéodory function satisfying the growth condition

$$|b(x,s)| \le k_4(x) + c_3|s|^{p-1} \tag{3.16}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, where $k_4 \in L^q_+(\Omega)$ and $c_3 > 0$. Furthermore, elementary calculations show the following estimate:

$$\int_\Omega b(x,u(x))u(x)dx \ge c_4\|u\|^p_{L^p(\Omega)} - c_5, \quad \forall u \in L^p(\Omega), \tag{3.17}$$

where c_4 and c_5 are some positive constants. Due to (3.16) the associated Nemytskij operator $B : L^p(\Omega) \to L^q(\Omega)$ defined by

$$Bu(x) = b(x,u(x)) \tag{3.18}$$

is bounded and continuous. Since the embedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, the composed operator $\widehat{B} := i^* \circ B \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is completely continuous.

For $u \in W^{1,p}(\Omega)$, we define the truncation operator T with respect to the functions $\underline{u}$ and $\overline{u}$ given by

$$Tu(x) = \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x), \\ u(x), & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \underline{u}(x), & \text{if } u(x) < \underline{u}(x). \end{cases} \tag{3.19}$$

The mapping T is continuous and bounded from $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)$ which follows from the fact that the functions $\min(\cdot,\cdot)$ and $\max(\cdot,\cdot)$ are continuous from $W^{1,p}(\Omega)$ to itself and that T can be represented as $Tu = \max(u,\underline{u}) + \min(u,\overline{u}) - u$ (cf. [29]). Let $F \circ T$ be the composition of the Nemytskij operator F and T given by

$$(F \circ T)(u)(x) = f(x, Tu(x), \nabla Tu(x)). \tag{3.20}$$

Due to hypothesis (F1)(iii), the mapping $F \circ T : W^{1,p}(\Omega) \to L^q(\Omega)$ is bounded and continuous. We set $\widehat{F} : i^* \circ (F \circ T) : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$, and consider the multivalued operator

$$\widetilde{A} = A_T u + \widehat{F} + \lambda \widehat{B} + \Phi_1 + \Phi_2 : W^{1,p}(\Omega) \longrightarrow 2^{(W^{1,p}(\Omega))^*}, \tag{3.21}$$

where λ is a constant specified later, and the operator A_T is given by

$$\langle A_T u, \varphi \rangle = -\sum_{i=1}^{N} \int_{\Omega} a_i(x, Tu, \nabla u) \frac{\partial \varphi}{\partial x_i} dx. \tag{3.22}$$

We are going to prove the following properties for the operator $\widetilde{A}$.

Lemma 3.5. *The operator $\widetilde{A} : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is bounded, pseudomonotone, and coercive for λ sufficiently large.*

Proof. The boundedness of $\widetilde{A}$ follows directly from the boundedness of the specific operators A_T, $\widehat{F}$, $\widehat{B}$, Φ_1, and Φ_2. As seen above, the operator $\widehat{B}$ is completely continuous and thus pseudomonotone. The elliptic operator $A_T + \widehat{F}$ is pseudomonotone because of hypotheses (A1), (A2), and (F1), and in view of Lemma 3.4 the operators Φ_1 and Φ_2 are bounded and pseudomonotone as well. Since pseudomonotonicity is invariant under addition, we conclude that $\widetilde{A} : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is bounded and pseudomonotone. To prove the coercivity of $\widetilde{A}$, we have to find the existence of a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\lim_{s \to +\infty} c(s) = +\infty, \tag{3.23}$$

such that for all $u \in W^{1,p}(\Omega)$ and $u^* \in \widetilde{A}(u)$ the following holds

$$\langle u^*, u - u_0 \rangle \geq c\Big(\|u\|_{W^{1,p}(\Omega)}\Big) \|u\|_{W^{1,p}(\Omega)} \tag{3.24}$$

for some $u_0 \in K$. Let $u^* \in \widetilde{A}(u)$; that is, u^* is of the form

$$u^* = \left(A_T + \widehat{F} + \lambda \widehat{B}\right)(u) + i^*\eta + \gamma^*\xi, \tag{3.25}$$

where $\eta \in L^q(\Omega)$ with $\eta(x) \in \partial\widetilde{j}_1(x, u(x))$ for a.a. $x \in \Omega$ and $\xi \in L^q(\partial\Omega)$ with $\xi(x) \in \partial\widetilde{j}_2(x, u(x))$ for a.a. $x \in \partial\Omega$. Applying (A1), (A3), (F1)(iii), (3.17), and ($\widetilde{j}$2), the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ and Young's inequality yield

$$\begin{aligned}
&\langle u^*, u - u_0\rangle \\
&\quad = \left\langle \left(A_T + \widehat{F} + \lambda \widehat{B}\right)(u) + i^*\eta + \gamma^*\xi, u - u_0 \right\rangle \\
&\quad = \int_\Omega \sum_{i=1}^N a_i(x, Tu, \nabla u) \frac{\partial u - \partial u_0}{\partial x_i} dx + \int_\Omega \left(f(\cdot, Tu, \nabla Tu)(u - u_0) + \lambda b(x, u)(u - u_0)\right) dx \\
&\qquad + \int_\Omega (\eta(u - u_0)) dx + \int_{\partial\Omega} \xi\gamma(u - u_0) d\sigma \\
&\quad \geq c_1 \|\nabla u\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - d_1\|u\|^{p-1}_{L^p(\Omega)} - d_2\|\nabla u\|^{p-1}_{L^p(\Omega)} - d_3 - \varepsilon\|\nabla u\|^p_{L^p(\Omega)} - c(\varepsilon)\|u\|^p_{L^p(\Omega)} \\
&\qquad - d_5\|u\|_{L^p(\Omega)} - d_6\|\nabla u\|^{p-1}_{L^p(\Omega)} - d_7 + \lambda c_4\|u\|^p_{L^p(\Omega)} - \lambda c_5 - d_8 - d_9\|u\|^{p-1}_{L^p(\Omega)} \\
&\qquad - d_{10}\|u\|_{L^p(\Omega)} - d_{11} - d_{12}\|u\|_{L^p(\partial\Omega)} - d_{13} \\
&\quad = (c_1 - \varepsilon)\|\nabla u\|^p_{L^p(\Omega)} + (\lambda c_4 - c(\varepsilon))\|u\|^p_{L^p(\Omega)} - d_{14}\|\nabla u\|^{p-1}_{L^p(\Omega)} - d_{15}\|u\|^{p-1}_{L^p(\Omega)} \\
&\qquad - d_{16}\|u\|_{L^p(\Omega)} - d_{17},
\end{aligned} \tag{3.26}$$

where d_j are some positive constants. Choosing $\varepsilon < c_1$ and λ such that $\lambda > c(\varepsilon)/c_4$ yields the estimate

$$\langle u^*, u - u_0\rangle \geq d_{18}\|u\|^p_{W^{1,p}(\Omega)} - d_{19}\|u\|^{p-1}_{W^{1,p}(\Omega)} - d_{20}\|u\|_{W^{1,p}(\Omega)} - d_{21}. \tag{3.27}$$

Setting $c(s) = d_{18}s^{p-1} - d_{19}s^{p-2} - d_{20} - d_{21}/s$ for $s > 0$ and $c(0) = 0$ provides the estimate in (3.24) satisfying (3.23). This proves the coercivity of A and completes the proof of the lemma. □

4. Main Results

Theorem 4.1. *Let hypotheses (A1)–(A3), (j1)–(j3), and (F1) be satisfied, and assume the existence of sub- and supersolutions* $\underline{u}$ *and* $\overline{u}$*, respectively, satisfying* $\underline{u} \le \overline{u}$ *and* (2.1)*. Then, there exists a solution of* (1.1) *in the order interval* $[\underline{u}, \overline{u}]$*.*

Proof. Let $I_K : W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be the indicator function corresponding to the closed convex set $K \neq \emptyset$ given by

$$I_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{if } u \notin K, \end{cases} \tag{4.1}$$

which is known to be proper, convex, and lower semicontinuous. The variational-hemivariational inequality (1.1) can be rewritten as follows. Find $u \in K$ such that

$$\langle Au + F(u), v - u \rangle + I_K(v) - I_K(u) + \int_\Omega j_1^0(\cdot, u; v - u)\,dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u)\,d\sigma \ge 0 \tag{4.2}$$

for all $v \in W^{1,p}(\Omega)$. By using the operators A_T, $\widehat{F}$, $\widehat{B}$ and the functions $\widetilde{j}_1$, $\widetilde{j}_2$ introduced in Section 3, we consider the following auxiliary problem. Find $u \in K$ such that

$$\begin{aligned} &\left\langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u), v - u \right\rangle + I_K(v) - I_K(u) + \int_\Omega \widetilde{j}_1^0(\cdot, u; v - u)\,dx \\ &\quad + \int_{\partial\Omega} \widetilde{j}_2^0(\cdot, \gamma u; \gamma v - \gamma u)\,d\sigma \ge 0 \end{aligned} \tag{4.3}$$

for all $v \in W^{1,p}(\Omega)$. Consider now the multivalued operator

$$\widetilde{A} + \partial I_K : W^{1,p}(\Omega) \longrightarrow 2^{(W^{1,p}(\Omega))^*}, \tag{4.4}$$

where $\widetilde{A}$ is as in (3.21), and $\partial I_K : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is the subdifferential of the indicator function I_K which is known to be a maximal monotone operator (cf. [28, page 20]). Lemma 3.5 provides that $\widetilde{A}$ is bounded, pseudomonotone, and coercive. Applying Theorem 3.2 proves the surjectivity of $\widetilde{A} + \partial I_K$ meaning that range$(\widetilde{A} + \partial I_K) = (W^{1,p}(\Omega))^*$. Since $0 \in (W^{1,p}(\Omega))^*$, there exists a solution $u \in K$ of the inclusion

$$\widetilde{A}(u) + \partial I_K(u) \ni 0. \tag{4.5}$$

This implies the existence of $\eta^* \in \Phi_1(u)$, $\xi^* \in \Phi_2(u)$, and $\theta^* \in \partial I_K(u)$ such that

$$A_T u + \widehat{F}(u) + \lambda \widehat{B}(u) + \eta^* + \xi^* + \theta^* = 0, \quad \text{in } \left(W^{1,p}(\Omega)\right)^*, \tag{4.6}$$

where it holds in view of (3.12) and (3.14) that

$$\eta^* = i^*\eta, \qquad \xi^* = \gamma^*\xi \tag{4.7}$$

with

$$\eta \in L^q(\Omega), \quad \eta(x) \in \partial\tilde{j}_1(x, u(x)) \quad \text{as well as} \quad \xi \in L^q(\partial\Omega), \quad \xi(x) \in \partial\tilde{j}_2(x, \gamma u(x)). \tag{4.8}$$

Due to the Definition of Clarke's generalized gradient $\partial\tilde{j}_k(\cdot, u)$, $k = 1, 2$, one gets

$$\begin{aligned} \langle \eta^*, \varphi \rangle &= \int_\Omega \eta(x)\varphi(x)dx \le \int_\Omega \tilde{j}_1^0(x, u(x); \varphi(x))dx, \quad \forall \varphi \in W^{1,p}(\Omega), \\ \langle \xi^*, \varphi \rangle &= \int_{\partial\Omega} \xi(x)\gamma\varphi(x)d\sigma \le \int_{\partial\Omega} \tilde{j}_2^0(x, \gamma u(x); \gamma\varphi(x))d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{4.9}$$

Moreover, we have the following estimate:

$$\langle \theta^*, v - u \rangle \le I_K(v) - I_K(u), \quad \forall v \in W^{1,p}(\Omega). \tag{4.10}$$

From (4.6) we conclude

$$\left\langle A_T u + \hat{F}(u) + \lambda\hat{B}(u) + \eta^* + \xi^* + \theta^*, \varphi \right\rangle = 0, \quad \forall \varphi \in W^{1,p}(\Omega). \tag{4.11}$$

Using the estimates in (4.9) and (4.10) to the equation above where φ is replaced by $v - u$, yields for all $v \in W^{1,p}(\Omega)$

$$\begin{aligned} 0 &= \left\langle A_T + \hat{F}(u) + \lambda\hat{B}(u) + \eta^* + \xi^* + \theta^*, v - u \right\rangle \\ &\le \left\langle A_T u + \hat{F}(u) + \lambda\hat{B}(u), v - u \right\rangle + I_K(v) - I_K(u) \\ &\quad + \int_\Omega \tilde{j}_1^0(\cdot, u; v - u)dx + \int_{\partial\Omega} \tilde{j}_2^0(\cdot, \gamma u; \gamma v - \gamma u)d\sigma. \end{aligned} \tag{4.12}$$

Hence, we obtain a solution u of the auxiliary problem (4.3) which is equivalent to the problem. Find $u \in K$ such that

$$\left\langle A_T u + \hat{F}(u) + \lambda\hat{B}(u), v - u \right\rangle + \int_\Omega \tilde{j}_1^0(\cdot, u; v - u)dx + \int_{\partial\Omega} \tilde{j}_2^0(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K. \tag{4.13}$$

In the next step we have to show that any solution u of (4.13) belongs to $[\underline{u}, \overline{u}]$. By Definition 2.2 and by choosing $w = \overline{u} \vee u = \overline{u} + (u - \overline{u})^+ \in \overline{u} \vee K$, we obtain

$$\left\langle A\overline{u} + F(\overline{u}), (u - \overline{u})^+ \right\rangle + \int_\Omega j_1^0(\cdot, \overline{u}; (u - \overline{u})^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma\overline{u}; \gamma(u - \overline{u})^+) d\sigma \geq 0, \tag{4.14}$$

and selecting $v = \overline{u} \wedge u = u - (u - \overline{u})^+ \in K$ in (4.13) provides

$$\left\langle A_T u + \widehat{F}(u) + \lambda\widehat{B}(u), -(u - \overline{u})^+ \right\rangle + \int_\Omega \widetilde{j}_1^0(\cdot, u; -(u - \overline{u})^+) dx + \int_{\partial\Omega} \widetilde{j}_2^0(\cdot, \gamma u; -\gamma(u - \overline{u})^+) d\sigma \geq 0. \tag{4.15}$$

Adding these inequalities yields

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega (a_i(x, \overline{u}, \nabla\overline{u}) - a_i(x, Tu, \nabla u)) \frac{\partial(u - \overline{u})^+}{\partial x_i} dx + \int_\Omega (F(\overline{u}) - (F \circ T)(u))(u - \overline{u})^+ dx \\ &\quad + \int_\Omega \left(j_1^0(\cdot, \overline{u}; 1) + \widetilde{j}_1^0(\cdot, u; -1) \right) (u - \overline{u})^+ dx + \int_{\partial\Omega} \left(j_2^0(\cdot, \gamma\overline{u}; 1) + \widetilde{j}_2^0(\cdot, \gamma u; -1) \right) \gamma(u - \overline{u})^+ d\sigma \\ &\quad \geq \lambda \int_\Omega B(u)(u - \overline{u})^+ dx. \end{aligned} \tag{4.16}$$

Let us analyze the specific integrals in (4.16). By using (A2) and the definition of the truncation operator, we obtain

$$\begin{gathered} \int_\Omega (a_i(x, \overline{u}, \nabla\overline{u}) - a_i(x, Tu, \nabla u)) \frac{\partial(u - \overline{u})^+}{\partial x_i} dx \leq 0, \\ \int_\Omega (F(\overline{u}) - (F \circ T)(u))(u - \overline{u})^+ dx = 0. \end{gathered} \tag{4.17}$$

Furthermore, we consider the third integral of (4.16) in case $u > \overline{u}$; otherwise it would be zero. Applying (1.12) and (3.8) proves

$$\begin{aligned} &\widetilde{j}_1^0(x, u(x); -1) \\ &\quad = \limsup_{s \to u(x), t \downarrow 0} \frac{\widetilde{j}_1(x, s - t) - \widetilde{j}_1(x, s)}{t} \\ &\quad = \limsup_{s \to u(x), t \downarrow 0} \frac{j_1(x, \overline{u}(x)) + \beta_1(x)(s - t - \overline{u}(x)) - j_1(x, \overline{u}(x)) - \beta_1(x)(s - \overline{u}(x))}{t} \\ &\quad = \limsup_{s \to u(x), t \downarrow 0} \frac{-\beta_1(x)t}{t} \\ &\quad = -\beta_1(x). \end{aligned} \tag{4.18}$$

Proposition 2.1.2 in [1] along with (3.7) shows

$$j_1^0(x,\overline{u}(x);1) = \max\{\xi : \xi \in \partial j_1(x,\overline{u}(x))\} = \beta_1(x). \tag{4.19}$$

In view of (4.18) and (4.19) we obtain

$$\int_\Omega \Big(j_1^0(\cdot,\overline{u};1) + \widetilde{j}_1^0(\cdot,u;-1)\Big)(u-\overline{u})^+ dx = \int_\Omega (\beta_1(x) - \beta_1(x))(u-\overline{u})^+ dx = 0, \tag{4.20}$$

and analog to this calculation

$$\int_{\partial\Omega} \Big(j_2^0(\cdot,\gamma\overline{u};1) + \widetilde{j}_2^0(\cdot,\gamma u;-1)\Big)\gamma(u-\overline{u})^+ d\sigma = 0. \tag{4.21}$$

Due to (4.17), (4.20), and (4.21), we immediately realize that the left-hand side in (4.16) is nonpositive. Thus, we have

$$\begin{aligned} 0 &\geq \lambda \int_\Omega B(u)(u-\overline{u})^+ dx \\ &= \lambda \int_\Omega b(\cdot,u)(u-\overline{u})^+ dx \\ &= \lambda \int_{\{x:u(x)>\overline{u}(x)\}} (u-\overline{u})^p dx \\ &= \lambda \int_\Omega \left((u-\overline{u})^+\right)^p dx \\ &\geq 0, \end{aligned} \tag{4.22}$$

which implies $(u-\overline{u})^+ = 0$ and hence, $u \leq \overline{u}$. The proof for $\underline{u} \leq u$ is done in a similar way. So far we have shown that any solution of the inclusion (4.5) (which is a solution of (4.3) as well) belongs to the interval $[\underline{u},\overline{u}]$. The latter implies $A_T u = Au$, $B(u) = 0$ and $(F \circ T)(u) = F(u)$, and thus from (4.5) it follows

$$\langle Au + F(u) + i^*\eta + \gamma^*\xi, v - u\rangle \geq 0, \quad \forall v \in K, \tag{4.23}$$

where $\eta(x) \in \partial\widetilde{j}_1(x,u(x)) \subset \partial j_1(x,u(x))$ and $\xi(x) \in \partial\widetilde{j}_2(x,\gamma u(x)) \subset \partial j_2(x,\gamma u(x))$, which proves that $u \in [\underline{u},\overline{u}]$ is also a solution of our original problem (1.1). This completes the proof of the theorem. □

Let $\mathcal{S}$ denote the set of all solutions of (1.1) within the order interval $[\underline{u},\overline{u}]$. In addition, we will assume that K has lattice structure, that is, K fulfills

$$K \vee K \subset K, \qquad K \wedge K \subset K. \tag{4.24}$$

We are going to show that $\mathcal{S}$ possesses the smallest and the greatest element with respect to the given partial ordering.

Theorem 4.2. *Let the hypothesis of Theorem 4.1 be satisfied. Then the solution set $\mathcal{S}$ is compact.*

Proof. First, we are going to show that $\mathcal{S}$ is bounded in $W^{1,p}(\Omega)$. Let $u \in \mathcal{S}$ be a solution of (4.2), and notice that $\mathcal{S}$ is $L^p(\Omega)$-bounded because of $\underline{u} \le u \le \overline{u}$. This implies $\gamma\underline{u} \le \gamma u \le \gamma\overline{u}$, and thus, u is also bounded in $L^p(\partial\Omega)$. Choosing a fixed $v = u_0 \in K$ in (4.2) delivers

$$\langle Au + F(u), u_0 - u\rangle + \int_\Omega j_1^0(\cdot, u; u_0 - u)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma u_0 - \gamma u)d\sigma \ge 0. \tag{4.25}$$

Using (A1), (j3), (F1)(iii), Proposition 2.1.2 in [1], and Young's inequality yields

$$\begin{aligned}
\langle Au, u\rangle &\le \int_\Omega \sum_{i=1}^N |a_i(x,u,\nabla u)|\left|\frac{\partial u_0}{\partial x_i}\right| dx + \int_\Omega \bigl|f(x,u,\nabla u)\bigr||u_0 - u|dx \\
&\quad + \int_\Omega \max\{\eta(u_0 - u) : \eta \in \partial j_1(x,u)\}dx + \int_{\partial\Omega} \max\{\xi(u_0 - \mathrm{u}) : \xi \in \partial j_2(x,u)\}d\sigma \\
&\le \int_\Omega \sum_{i=1}^N \Bigl(k_0 + c_0|u|^{p-1} + c_0|\nabla u|^{p-1}\Bigr)|\nabla u_0|dx + \int_\Omega \Bigl(k_3 + c_2|\nabla u|^{p-1}\Bigr)|u_0 - u|dx \\
&\quad + \int_\Omega L_1|u_0 - u|dx + \int_{\partial\Omega} L_2\bigl|\gamma u_0 - \gamma u\bigr|d\sigma \\
&\le e_1 + e_2\|u\|_{L^p(\Omega)}^{p-1} + e_3\|\nabla u\|_{L^p(\Omega)}^{p-1} + e_4 + e_5\|u\|_{\mathrm{L}^p(\Omega)} + e_6\|\nabla u\|_{L^p(\Omega)}^{p-1} + \varepsilon\|\nabla u\|_{L^p(\Omega)}^{p} \\
&\quad + c(\varepsilon)\|u\|_{L^p(\Omega)}^{p} + e_7 + e_8\|u\|_{L^p(\Omega)} + e_9 + e_{10}\|u\|_{L^p(\partial\Omega)} \\
&\le \varepsilon\|\nabla u\|_{L^p(\Omega)}^{p} + e_{11}\|\nabla u\|_{L^p(\Omega)}^{p-1} + e_{12}\|\nabla u\|_{L^p(\Omega)} + e_{13},
\end{aligned} \tag{4.26}$$

where the left-hand side fulfills the estimate

$$\langle Au, u\rangle \ge c_1\|\nabla u\|_{L^p(\Omega)}^{p} - k_1. \tag{4.27}$$

Thus, one has

$$(c_1 - \varepsilon)\|\nabla u\|_{L^p(\Omega)}^{p} \le e_{11}\|\nabla u\|_{L^p(\Omega)}^{p-1} + e_{13}, \tag{4.28}$$

where the choice $\varepsilon < c_1$ proves that $\|\nabla u\|_{L^p(\Omega)}$ is bounded. Hence, we obtain the boundedness of u in $W^{1,p}(\Omega)$. Let $(u_n) \subset \mathcal{S}$. Since $W^{1,p}(\Omega)$, $1 < p < \infty$, is reflexive, there exists a weak convergent subsequence, not relabelled, which yields along with the compact imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ and the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W^{1,p}(\Omega), \\ u_n &\longrightarrow u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega, \\ \gamma u_n &\longrightarrow \gamma u \quad \text{in } L^p(\partial\Omega) \text{ and a.e. pointwise in } \partial\Omega. \end{aligned} \tag{4.29}$$

As u_n solves (4.2), in particular, for $v = u \in K$, we obtain

$$\langle Au_n, u_n - u\rangle \le \langle F(u_n), u - u_n\rangle + \int_\Omega j_1^0(\cdot, u_n; u - u_n)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n)d\sigma. \tag{4.30}$$

Since $(s,r) \mapsto j_k^0(x,s;r)$, $k = 1,2$, is upper semicontinuous and due to Fatou's Lemma, we get from (4.30)

$$\begin{aligned} \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle &\le \underbrace{\limsup_{n\to\infty}\langle F(u_n), u - u_n\rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j_1^0(\cdot, u_n; u - u_n)}_{\le j_1^0(\cdot,u,0)=0}dx \\ &\quad + \int_{\partial\Omega} \underbrace{\limsup_{n\to\infty} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n)}_{\le j_2^0(\cdot,\gamma u,\gamma 0)=0}d\sigma \le 0. \end{aligned} \tag{4.31}$$

The elliptic operator A satisfies the (S_+)-property, which due to (4.31) and (4.29) implies

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\Omega). \tag{4.32}$$

Replacing u by u_n in (1.1) yields the following inequality:

$$\langle Au_n + F(u_n), v - u_n\rangle + \int_\Omega j_1^0(\cdot, u_n;\ v - u_n)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n;\ \gamma v - \gamma u_n)d\sigma \ge 0, \quad \forall v \in K. \tag{4.33}$$

Passing to the limes superior in (4.33) and using Fatou's Lemma, the strong convergence of (u_n) in $W^{1,p}(\Omega)$, and the upper semicontinuity of $(s,r) \to j_k^0(x,s;r)$, $k = 1,2$, we have

$$\langle Au + F(u), v - u\rangle + \int_\Omega j_1^0(\cdot, u;\ v - u)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u;\ \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K. \tag{4.34}$$

Hence, $u \in \mathcal{S}$. This shows the compactness of the solution set $\mathcal{S}$. □

In order to prove the existence of extremal elements of the solution set $\mathcal{S}$, we drop the u-dependence of the operator A. Then, our assumptions read as follows.

(A1′) Each $a_i(x,\xi)$ satisfies Carathéodory conditions, that is, is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in ξ for a.e. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist so that

$$|a_i(x,\xi)| \le k_0(x) + |\xi|^{p-1} \tag{4.35}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ.

(A2′) The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^{N}\left(a_i(x,\xi) - a_i(x,\xi')\right)\left(\xi_i - \xi_i'\right) > 0 \tag{4.36}$$

for a.e. $x \in \Omega$, and for all $\xi,\xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3′) A constant $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ exist such that

$$\sum_{i=1}^{N} a_i(x,\xi)\xi_i \ge c_1|\xi|^p - k_1(x) \tag{4.37}$$

for a.e. $x \in \Omega$, and for all $\xi \in \mathbb{R}^N$.

Then the operator $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ acts in the following way:

$$\langle Au, \varphi\rangle = \int_\Omega \sum_{i=1}^{N} a_i(x,\nabla u)\frac{\partial \varphi}{\partial x_i}dx. \tag{4.38}$$

Let us recall the definition of a directed set.

Definition 4.3. Let $(\mathcal{P},\le)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be upward directed if for each pair $x,y \in \mathcal{C}$ there is a $z \in \mathcal{C}$ such that $x \le z$ and $y \le z$. Similarly, $\mathcal{C}$ is downward directed if for each pair $x,y \in \mathcal{C}$ there is a $w \in \mathcal{C}$ such that $w \le x$ and $w \le y$. If $\mathcal{C}$ is both upward and downward directed, it is called directed.

Theorem 4.4. *Let hypotheses (A1′)–(A3′) and (j1)–(j3) be fulfilled, and assume that (F1) and* (4.24) *are valid. Then the solution set $\mathcal{S}$ of problem* (1.1) *is a directed set.*

Proof. By Theorem 4.1, we have $\mathcal{S} \neq \emptyset$. Let $u_1, u_2 \in \mathcal{S}$ be given solutions of (1.1), and let $u_0 = \max\{u_1,u_2\}$. We have to show that there is a $u \in \mathcal{S}$ such that $u_0 \le u$. Our proof is mainly based on an approach developed recently in [26] which relies on a properly constructed auxiliary problem. Let the operator $\widehat{B}$ be given basically as in (3.15)–(3.18) with the following slight change:

$$b(x,s) = \begin{cases} (s-\overline{u}(x))^{p-1}, & \text{if } s > \overline{u}(x), \\ 0, & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ -(u_0(x)-s)^{p-1}, & \text{if } s < u_0(x). \end{cases} \tag{4.39}$$

We introduce truncation operators T_j related to u_j and modify the truncation operator T as follows. For $j = 1,2$, we define

$$T_j u(x) = \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x), \\ u(x), & \text{if } u_j(x) \le u(x) \le \overline{u}(x), \\ u_j(x), & \text{if } u(x) < u_j(x), \end{cases}$$
$$Tu(x) = \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x), \\ u(x), & \text{if } u_0(x) \le u(x) \le \overline{u}(x), \\ u_0(x), & \text{if } u(x) < u_0(x), \end{cases} \tag{4.40}$$

and we set

$$Gu(x) = f(x,Tu(x),\nabla Tu(x)) - \sum_{j=1}^{2} \left| f(x,Tu(x),\nabla Tu(x)) - f\left(x,T_j u(x),\nabla T_j u(x)\right) \right| \tag{4.41}$$

as well as

$$\widehat{F} : i^* \circ G : W^{1,p}(\Omega) \longrightarrow \left(W^{1,p}(\Omega)\right)^*. \tag{4.42}$$

Moreover, we define

$$\alpha_{k,j}(x) := \min\{\xi : \xi \in \partial j_k(x,u_j(x))\}, \qquad \beta_k(x) := \max\{\xi : \xi \in \partial j_k(x,\overline{u}(x))\},$$
$$\alpha_{k,0}(x) := \begin{cases} \alpha_{k,1}(x), & \text{if } x \in \{u_1 \ge u_2\}, \\ \alpha_{k,2}(x), & \text{if } x \in \{u_2 > u_1\} \end{cases} \tag{4.43}$$

for $k,j = 1,2$, and introduce the functions $\widetilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{j}_k(x,s) = \begin{cases} j_k(x,u_0(x)) + \alpha_{k,0}(x)(s-u_0(x)), & \text{if } s < u_0(x), \\ j_k(x,s), & \text{if } u_0(x) \le s \le \overline{u}(x), \\ j_k(x,\overline{u}(x)) + \beta_k(x)(s-\overline{u}(x)), & \text{if } s > \overline{u}(x). \end{cases} \tag{4.44}$$

Furthermore, we define the functions $h_{1,j} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h_{2,j} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ for $j = 0, 1, 2$ as follows:

$$h_{k,0}(x,s) = \begin{cases} \alpha_{k,0}(x), & \text{if } s \leq u_0(x), \\ \alpha_{k,0}(x) + \dfrac{\beta_k(x) - \alpha_{k,0}(x)}{\overline{u}(x) - u_0(x)}(s - u_0(x)), & \text{if } u_0(x) < s < \overline{u}(x), \\ \beta_k(x), & \text{if } s \geq \overline{u}(x), \end{cases} \tag{4.45}$$

and for $j = 1, 2$

$$h_{k,j}(x,s) = \begin{cases} \alpha_{k,j}(x,) & \text{if } s \leq u_j(x), \\ \alpha_{k,j}(x) + \dfrac{\alpha_{k,0}(x) - \alpha_{k,j}(x)}{u_0(x) - u_j(x)}(s - u_j(x)), & \text{if } u_j(x) < s < u_0(x), \\ h_{k,0}(x,s), & \text{if } s \geq u_0(x), \end{cases} \tag{4.46}$$

where $k = 1, 2$. (Note that for $k = 2$ we understand the functions above being defined on $\partial\Omega$.) Apparently, the mappings $(x,s) \mapsto h_{k,j}(x,s)$ are Carathéodory functions which are piecewise linear with respect to s. Let us introduce the Nemytskij operators $H_1 : L^p(\Omega) \to L^q(\Omega)$ and $H_2 : L^p(\partial\Omega) \to L^q(\partial\Omega)$ defined by

$$\begin{aligned} H_1u(x) &= \sum_{j=1}^{2} \left| h_{1,j}(x,u(x)) - h_{1,0}(x,u(x)) \right|, \\ H_2u(x) &= \sum_{j=1}^{2} \left| h_{2,j}(x,\gamma(u(x))) - h_{2,0}(x,\gamma(u(x))) \right|. \end{aligned} \tag{4.47}$$

Due to the compact imbedding $W^{1,p}(\Omega) \to L^p(\Omega)$ and the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, the operators $\widetilde{H}_1 = i^* \circ H_1 \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ and $\widetilde{H}_2 = \gamma^* \circ H_2 \circ \gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ are bounded and completely continuous and thus pseudomonotone. Now, we consider the following auxiliary variational-hemivariational inequality. Find $u \in K$ such that

$$\begin{aligned} &\left\langle Au + \widehat{F}(u) + \lambda\widehat{B}(u), v - u \right\rangle + \int_\Omega \widetilde{j}_1^0(\cdot, u; v - u)\,dx - \left\langle \widetilde{H}_1 u, v - u \right\rangle \\ &\quad + \int_{\partial\Omega} \widetilde{j}_2^0(\cdot, \gamma u; \gamma v - \gamma u)\,d\sigma - \left\langle \widetilde{H}_2 \gamma u, \gamma v - \gamma u \right\rangle \geq 0 \end{aligned} \tag{4.48}$$

for all $v \in K$. The construction of the auxiliary problem (4.48) including the functions H_k and G is inspired by a very recent approach introduced by Carl and Motreanu in [26]. The first part of the proof of Theorem 4.1 delivers the existence of a solution u of (4.48), since all calculations in Section 3 are still valid. In order to show that the solution set $\mathcal{S}$ of (1.1) is upward directed, we have to verify that a solution u of (4.48) satisfies $u_l \le u \le \overline{u}$, $l = 1, 2$. By assumption $u_l \in \mathcal{S}$, that is, u_l solves

$$u_l \in K : \langle Au_l + F(u_l), v - u_l \rangle + \int_\Omega j_1^0(\cdot, u_l;\ v - u_l)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_l;\ \gamma v - \gamma u_l)d\sigma \ge 0 \tag{4.49}$$

for all $v \in K$. Selecting $v = u \wedge u_l = u_l - (u_l - u)^+ \in K$ in the inequality above yields

$$\langle Au_l + F(u_l), -(u_l - u)^+ \rangle + \int_\Omega j_1^0(\cdot, u_l;\ -(u_l - u)^+)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_l;\ -\gamma(u_l - u)^+)d\sigma \ge 0. \tag{4.50}$$

Taking the special test function $v = u \vee u_l = u + (u_l - u)^+ \in K$ in (4.48), we get

$$\begin{aligned} &\langle Au + \widehat{F}(u) + \lambda\widehat{B}(u), (u_l - u)^+ \rangle + \int_\Omega \widetilde{j}_1^0(\cdot, u;\ (u_l - u)^+)dx - \left\langle \widetilde{H}_1, (u_l - u)^+ \right\rangle \\ &\quad + \int_{\partial\Omega} \widetilde{j}_2^0(\cdot, \gamma u;\ \gamma(u_l - u)^+)d\sigma - \left\langle \widetilde{H}_2\gamma u, \gamma(u_l - u)^+ \right\rangle \ge 0. \end{aligned} \tag{4.51}$$

Adding (4.50) and (4.51) yields

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N (a_i(x, \nabla u) - a_i(x, \nabla u_l)) \frac{\partial(u_l - u)^+}{\partial x_i} dx \\ &\quad + \int_\Omega \left(f(x, Tu, \nabla Tu) - f(x, u_l, \nabla u_l) - \sum_{j=1}^2 \left| f(x, Tu, \nabla Tu) - f(x, T_j u, \nabla T_j u) \right| \right) (u_l - u)^+ dx \\ &\quad + \int_\Omega \left(\widetilde{j}_1^0(\cdot, u; 1) + j_1^0(\cdot, u_l; -1) - \sum_{j=1}^2 \left| h_{1,j}(x, u) - h_{1,0}(x, u) \right| \right) (u_l - u)^+ dx \\ &\quad + \int_{\partial\Omega} \left(\widetilde{j}_2^0(\cdot, \gamma u; 1) + j_2^0(\cdot, \gamma u_l; -1) - \sum_{j=1}^2 \left| h_{2,j}(x, \gamma u) - h_{2,0}(x, \gamma u) \right| \right) \gamma(u_l - u)^+ d\sigma \\ &\ge -\lambda \int_\Omega B(u)(u_l - u)^+ dx. \end{aligned} \tag{4.52}$$

The condition (A2$'$) implies directly

$$\int_\Omega \sum_{i=1}^N (a_i(x, \nabla u) - a_i(x, \nabla u_l)) \frac{\partial(u_l - u)^+}{\partial x_i} dx \le 0, \tag{4.53}$$

and the second integral can be estimated to obtain

$$
\begin{aligned}
&\int_{\Omega}\left(f(x,Tu,\nabla Tu)-f(x,u_l,\nabla u_l)-\sum_{j=1}^{2}\left|f(x,Tu,\nabla Tu)-f\left(x,T_ju,\nabla T_ju\right)\right|\right)(u_l-u)^{+}dx\\
&\quad\le\int_{\Omega}\left(f(x,Tu,\nabla Tu)-f(x,u_l,\nabla u_l)-\left|f(x,Tu,\nabla Tu)-f(x,T_lu,\nabla T_lu)\right|\right)(u_l-u)^{+}dx\\
&\quad=\int_{\{x\in\Omega:u_l(x)>u(x)\}}\left(f(x,Tu,\nabla Tu)-f(x,u_l,\nabla u_l)-\left|f(x,Tu,\nabla Tu)-f(x,u_l,\nabla u_l)\right|\right)(u_l-u)dx\\
&\quad\le 0.
\end{aligned}
\tag{4.54}
$$

In order to investigate the third integral, we make use of some auxiliary calculation. In view of (4.44) we have for $u_l(x)>u(x)$

$$
\begin{aligned}
\tilde{j}_1^{0}(x,u(x);1)&=\limsup_{s\to u(x),t\downarrow 0}\frac{\tilde{j}_1(x,s+t)-\tilde{j}_1(x,s)}{t}\\
&=\limsup_{s\to u(x),t\downarrow 0}\frac{j_1(x,u_0(x))+\alpha_{1,0}(x)(s+t-u_0(x))-j_1(x,u_0(x))-\alpha_{1,0}(x)(s-u_0(x))}{t}\\
&=\limsup_{s\to u(x),t\downarrow 0}\frac{\alpha_{1,0}(x)t}{t}\\
&=\alpha_{1,0}(x).
\end{aligned}
\tag{4.55}
$$

Applying Proposition 2.1.2 in [1] and (3.7) results in

$$
\begin{aligned}
j_1^{0}(x,u_l(x);-1)&=\max\{-\xi:\xi\in\partial j_1(x,u_l(x))\}\\
&=-\min\{\xi:\xi\in\partial j_1(x,u_l(x))\}\\
&=-\alpha_{1,l}(x).
\end{aligned}
\tag{4.56}
$$

Furthermore, we have in case $u_l(x)>u(x)$

$$
\begin{aligned}
h_{1,l}(x,u(x))&=\alpha_{1,l}(x),\\
h_{1,0}(x,u(x))&=\alpha_{1,0}(x).
\end{aligned}
\tag{4.57}
$$

Thus, we get

$$\begin{aligned}
&\int_\Omega \left(\tilde{j}_1^0(\cdot, u; 1) + j_1^0(\cdot, u_l; -1) - \sum_{j=1}^{2} \left| h_{1,j}(x,u) - h_{1,0}(x,u) \right| \right) (u_l - u)^+ dx \\
&\quad \le \int_\Omega \left(\tilde{j}_1^0(\cdot, u; 1) + j_1^0(\cdot, u_l; -1) - \left| h_{1,l}(x,u) - h_{1,0}(x,u) \right| \right) (u_l - u)^+ dx \\
&\quad = \int_{\{x\in\Omega : u_l(x) > u(x)\}} (\alpha_{1,0}(x) - \alpha_{1,l}(x) - |\alpha_{1,l}(x) - \alpha_{1,0}(x)|)(u_l - u)^+ dx \\
&\quad \le 0.
\end{aligned} \tag{4.58}$$

The same result can be proven for the boundary integral meaning

$$\int_{\partial\Omega} \left(\tilde{j}_2^0(\cdot, \gamma u; 1) + j_2^0(\cdot, \gamma u_l; -1) - \sum_{j=1}^{2} \left| h_{2,j}(x,\gamma u) - h_{2,0}(x,\gamma u) \right| \right) \gamma(u_l - u)^+ d\sigma \le 0. \tag{4.59}$$

Applying (4.53)–(4.59) to (4.52) yields

$$\begin{aligned}
0 &\ge -\lambda \int_\Omega B(u)(u_l - u)^+ dx \\
&= -\lambda \int_{\{x\in\Omega : u_l(x) > u(x)\}} -(u_0 - u)^{p-1}(u_l - u) dx \\
&\ge \lambda \int_\Omega ((u_l - u)^+)^p dx \\
&\ge 0,
\end{aligned} \tag{4.60}$$

and hence, $(u_l - u)^+ = 0$ meaning that $u_l \le u$ for $l = 1, 2$. This proves $u_0 = \max\{u_1, u_2\} \le u$. The proof for $u \le \overline{u}$ can be shown in a similar way. More precisely, we obtain a solution $u \in K$ of (4.48) satisfying $\underline{u} \le u_0 \le u \le \overline{u}$ which implies $\widehat{F}(u) = f(\cdot, u, \nabla u)$, $\widehat{B}(u) = 0$ and $H_1(u) = H_2(\gamma u) = 0$. The same arguments as at the end of the proof of Theorem 4.1 apply, which shows that u is in fact a solution of problem (1.1) belonging to the interval $[u_0, \overline{u}]$. Thus, the solution set $\mathcal{S}$ is upward directed. Analogously, one proves that $\mathcal{S}$ is downward directed. □

Theorems 4.2 and 4.4 allow us to formulate the next theorem about the existence of extremal solutions.

Theorem 4.5. *Let the hypotheses of Theorem 4.4 be satisfied. Then the solution set $\mathcal{S}$ possesses extremal elements.*

Proof. Since $\mathcal{S} \subset W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ are separable, $\mathcal{S}$ is also separable; that is, there exists a countable, dense subset $Z = \{z_n : n \in \mathbb{N}\}$ of $\mathcal{S}$. We construct an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$ and select $u_{n+1} \in \mathcal{S}$ such that

$$\max(z_n, u_n) \le u_{n+1} \le \overline{u}. \tag{4.61}$$

By Theorem 4.4, the element u_{n+1} exists because $\mathcal{S}$ is upward directed. Moreover, we can choose by Theorem 4.2 a convergent subsequence (denoted again by u_n) with $u_n \to u$ in $W^{1,p}(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω. Since (u_n) is increasing, the entire sequence converges in $W^{1,p}(\Omega)$ and further, $u = \sup u_n$. One sees at once that $Z \subset [\underline{u}, u]$ which follows from

$$\max(z_1, \dots, z_n) \le u_{n+1} \le u, \quad \forall n, \tag{4.62}$$

and the fact that $[\underline{u}, u]$ is closed in $W^{1,p}(\Omega)$ implies

$$\mathcal{S} \subset \overline{Z} \subset \overline{[\underline{u}, u]} = [\underline{u}, u]. \tag{4.63}$$

Therefore, as $u \in \mathcal{S}$, we conclude that u is the greatest element in $\mathcal{S}$. The existence of the smallest solution of (1.1) in $[\underline{u}, \overline{u}]$ can be proven in a similar way. □

Remark 4.6. If A depends on s, we have to require additional assumptions. For example, if A satisfies in s a monotonicity condition, the existence of extremal solutions can be shown, too. In case $K = W^{1,p}(\Omega)$, a Lipschitz condition with respect to s is sufficient for proving extremal solutions. For more details we refer to [7].

5. Generalization to Discontinuous Nemytskij Operators

In this section, we will extend our problem in (1.1) to include discontinuous nonlinearities f of the form $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. We consider again the elliptic variational-hemivariational inequality

$$\langle Au + F(u), v - u\rangle + \int_\Omega j_1^0(\cdot, u; v - u)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K, \tag{5.1}$$

where all denotations of Section 1 are valid. Here, F denotes the Nemytskij operator given by

$$F(u)(x) = f(x, u(x), u(x), \nabla u(x)), \tag{5.2}$$

where we will allow f to depend discontinuously on its third argument. The aim of this section is to deal with discontinuous Nemytskij operators $F : [\underline{u}, \overline{u}] \subset W^{1,p}(\Omega) \to L^q(\Omega)$ by combining the results of Section 4 with an abstract fixed point result for not necessarily continuous operators, cf. [30, Theorem 1.1.1]. This will extend recent results obtained in [3]. Let us recall the Definitions of sub- and supersolutions.

Definition 5.1. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (5.1) if the following holds:

(1) $F(\underline{u}) \in L^q(\Omega)$;

(2) $\langle A\underline{u} + F(\underline{u}), w - \underline{u}\rangle + \int_\Omega j_1^0(\cdot, \underline{u}; w - \underline{u})dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \ge 0$, $\forall w \in \underline{u} \wedge K$.

Definition 5.2. A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (5.1) if the following holds:

(1) $F(\overline{u}) \in L^q(\Omega)$;

(2) $\langle A\overline{u} + F(\overline{u}), w - \overline{u}\rangle + \int_\Omega j_1^0(\cdot, \overline{u}; w - \overline{u})dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma\overline{u}; \gamma w - \gamma\overline{u})d\sigma \ge 0$, $\forall w \in \overline{u} \vee K$.

The conditions for Clarke's generalized gradient $s \mapsto \partial j_k(x,s)$ and the functions j_k, $k = 1,2$, are the same as in (j1)–(j3). We only change the property (F1) to the following.

(F2) (i) $x \mapsto f(x,r,u(x),\xi)$ is measurable for all $r \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, and for all measurable functions $u : \Omega \to \mathbb{R}$.

(ii) $(r,\xi) \mapsto f(x,r,s,\xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for all $s \in \mathbb{R}$ and for a.a. $x \in \Omega$.

(iii) $s \mapsto f(x,r,s,\xi)$ is decreasing for all $(r,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and for a.a. $x \in \Omega$.

(iv) There exist a constant $c_2 > 0$ and a function $k_2 \in L_+^q(\Omega)$ such that

$$\left|f(x,r,s,\xi)\right| \le k_2(x) + c_0|\xi|^{p-1} \tag{5.3}$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, and for all $r,s \in [\underline{u}(x), \overline{u}(x)]$.

By [31] the mapping $x \mapsto f(x,u(x),u(x),\nabla u(x))$ is measurable for $u \in W^{1,p}(\Omega)$; however, the associated Nemytskij operator $F : W^{1,p}(\Omega) \subset L^p(\Omega) \to L^q(\Omega)$ is not necessarily continuous. An important tool in extending the previous result to discontinuous Nemytskij operators is the next fixed point result. The proof of this lemma can be found in [30, Theorem 1.1.1].

Lemma 5.3. *Let P be a subset of an ordered normed space, $G : P \to P$ an increasing mapping, and $G[P] = \{Gx \mid x \in P\}$.*

(1) *If $G[P]$ has a lower bound in P and the increasing sequences of $G[P]$ converge weakly in P, then G has the least fixed point x_*, and $x_* = \min\{x \mid Gx \le x\}$.*

(2) *If $G[P]$ has an upper bound in P and the decreasing sequences of $G[P]$ converge weakly in P, then G has the greatest fixed point x^*, and $x^* = \max\{x \mid x \le Gx\}$.*

Our main result of this section is the following theorem.

Theorem 5.4. *Assume that hypotheses (A1′)–(A3′), (j1)–(j3), (F2), and* (4.24) *are valid, and let $\underline{u}$ and $\overline{u}$ be sub- and supersolutions of* (5.1) *satisfying $\underline{u} \le \overline{u}$ and* (2.1)*. Then there exist extremal solutions u^* and u_* of* (5.1) *with $\underline{u} \le u_* \le u^* \le \overline{u}$.*

Proof. We consider the following auxiliary problem:

$$u \in K : \langle Au + F_z(u), v - u\rangle + \int_\Omega j_1^0(\cdot, u; v - u)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K, \tag{5.4}$$

where $F_z(u)(x) = f(x, u(x), z(x), \nabla u(x))$, and we define the set $H := \{z \in W^{1,p}(\Omega) : z \in [\underline{u}, \overline{u}]$, and z is a supersolution of (5.1) satisfying $z \wedge K \subset K\}$. On H we introduce the fixed point operator $L : H \to K$ by $z \mapsto u^* =: Lz$, that is, for a given supersolution $z \in H$, the element Lz is the greatest solution of (5.4) in $[\underline{u}, z]$, and thus, it holds $\underline{u} \le Lz \le z$ for all $z \in H$. This implies $L : H \to [\underline{u}, \overline{u}] \cap K$. Because of (4.24), Lz is also a supersolution of (5.4) satisfying

$$\langle ALz + F_z(Lz), w - Lz\rangle + \int_\Omega j_1^0(\cdot, Lz; w - Lz)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma Lz; \gamma w - \gamma Lz)d\sigma \ge 0 \tag{5.5}$$

for all $w \in Lz \vee K$. By the monotonicity of f with respect to its third argument, $Lz \le z$, and using the representation $w = Lz + (v - Lz)^+$ for any $v \in K$ we obtain

$$\begin{aligned} 0 &\le \langle ALz + F_z(Lz), (v - Lz)^+\rangle + \int_\Omega j_1^0(\cdot, Lz; (v - Lz)^+)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma Lz; \gamma(v - Lz)^+)d\sigma \\ &\le \langle ALz + F_{Lz}(Lz), (v - Lz)^+\rangle + \int_\Omega j_1^0(\cdot, Lz; (v - Lz)^+)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma Lz; \gamma(v - Lz)^+)d\sigma \end{aligned} \tag{5.6}$$

for all $v \in K$. Consequently, Lz is a supersolution of (5.1). This shows $L : H \to H$.

Let $v_1, v_2 \in H$, and assume that $v_1 \le v_2$. Then we have the following.

$$\begin{aligned} &Lv_1 \in [\underline{u}, v_1] \text{ is the greatest solution of} \\ &\quad \langle Au + F_{v_1}(u), v - u\rangle + \int_\Omega j_1^0(\cdot, u; v - u)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K. \end{aligned} \tag{5.7}$$

$$\begin{aligned} &Lv_2 \in [\underline{u}, v_2] \text{ is the greatest solution of} \\ &\quad \langle Au + F_{v_2}(u), v - u\rangle + \int_\Omega j_1^0(\cdot, u; v - u)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K. \end{aligned} \tag{5.8}$$

Since $v_1 \le v_2$, it follows that $Lv_1 \le v_2$, and due to (4.24), Lv_1 is also a subsolution of (5.7), that is, (5.7) holds, in particular, for $v \in Lv_1 \wedge K$, that is,

$$\begin{aligned} 0 \ge \langle ALv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+\rangle &- \int_\Omega j_1^0(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\ - \int_{\partial\Omega} j_2^0(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma \end{aligned} \tag{5.9}$$

for all $v \in K$. Using the monotonicity of f with respect to its third argument s yields

$$\begin{aligned}
0 &\geq \langle ALv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_\Omega j_1^0(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\
&\quad - \int_{\partial\Omega} j_2^0(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma \\
&\geq \langle ALv_1 + F_{v_2}(Lv_1), (Lv_1 - v)^+ \rangle - \int_\Omega j_1^0(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\
&\quad - \int_{\partial\Omega} j_2^0(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma
\end{aligned} \tag{5.10}$$

for all $v \in K$. Hence, Lv_1 is a subsolution of (5.8). By Theorem 4.5, we know that there exists the greatest solution of (5.8) in $[Lv_1, v_2]$. But Lv_2 is the greatest solution of (5.8) in $[\underline{u}, v_2] \supseteq [Lv_1, v_2]$ and therefore, $Lv_1 \leq Lv_2$. This shows that L is increasing.

In the last step we have to prove that any decreasing sequence of $L(H)$ converges weakly in H. Let $(u_n) = (Lz_n) \subset L(H) \subset H$ be a decreasing sequence. Then $u_n(x) \searrow u(x)$ a.e. $x \in \Omega$ for some $u \in [\underline{u}, \overline{u}]$. The boundedness of u_n in $W^{1,p}(\Omega)$ can be shown similarly as in Section 4. Thus the compact imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ along with the monotony of u_n as well as the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ implies

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega), \\
u_n \longrightarrow u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega, \\
\gamma u_n \longrightarrow \gamma u \quad \text{in } L^p(\partial\Omega) \text{ and a.e. pointwise in } \partial\Omega.
\end{gathered} \tag{5.11}$$

Since $u_n \in K$, it follows $u \in K$. From (5.4) with u replaced by u_n and v by u, and using the fact that $(s, r) \mapsto j_k^0(x, s; r)$, $k = 1, 2$, is upper semicontinuous, we obtain by applying Fatou's Lemma

$$\begin{aligned}
\limsup_{n\to\infty} \langle Au_n, u_n - u \rangle &\leq \limsup_{n\to\infty} \langle F_{z_n}(u_n), u - u_n \rangle + \limsup_{n\to\infty} \int_\Omega j_1^0(\cdot, u_n; u - u_n)dx \\
&\quad + \limsup_{n\to\infty} \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n)d\sigma \\
&\leq \underbrace{\limsup_{n\to\infty} \langle F_{z_n}(u_n), u - u_n \rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j_1^0(\cdot, u_n; u - u_n)}_{\leq j_1^0(\cdot, u; 0) = 0} dx \\
&\quad + \int_{\partial\Omega} \underbrace{\limsup_{n\to\infty} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n)}_{\leq j_2^0(\cdot, \gamma u; \gamma 0) = 0} d\sigma \\
&\leq 0.
\end{aligned} \tag{5.12}$$

The S_+-property of A provides the strong convergence of (u_n) in $W^{1,p}(\Omega)$. As $Lz_n = u_n$ is also a supersolution of (5.4) Definition 5.2 yields

$$\left\langle Au_n + F_{z_n}(u_n), (v-u_n)^+\right\rangle + \int_\Omega j_1^0(\cdot, u_n; (v-u_n)^+)\,dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma(v-u_n)^+)\,d\sigma \ge 0 \tag{5.13}$$

for all $v \in K$. Due to $z_n \ge u_n \ge u$ and the monotonicity of f we get

$$\begin{aligned} 0 &\le \langle Au_n + F_{z_n}(u_n), (v-u_n)^+\rangle + \int_\Omega j_1^0(\cdot, u_n; (v-u_n)^+)\,dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma(v-u_n)^+)\,d\sigma \\ &\le \langle Au_n + F_u(u_n), (v-u_n)^+\rangle + \int_\Omega j_1^0(\cdot, u_n; (v-u_n)^+)\,dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma(v-u_n)^+)\,d\sigma \end{aligned} \tag{5.14}$$

for all $v \in K$, and since the mapping $u \mapsto u^+ = \max(u, 0)$ is continuous from $W^{1,p}(\Omega)$ to itself (cf. [29]), we can pass to the upper limit on the right-hand side for $n \to \infty$. This yields

$$\langle Au + F_u(u), (v-u)^+\rangle + \int_\Omega j_1^0(\cdot, u; (v-u)^+)\,dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma(v-u)^+)\,dx \ge 0, \quad \forall v \in K, \tag{5.15}$$

which shows that u is a supersolution of (5.1), that is, $u \in H$. As $\overline{u}$ is an upper bound of $L(H)$, we can apply Lemma 5.3, which yields the existence of the greatest fixed point u^* of L in H. This implies that u^* must be the the greatest solution of (5.1) in $[\underline{u}, \overline{u}]$. By analogous reasoning, one shows the existence of the smallest solution u_* of (5.1). This completes the proof of the theorem. □

Remark 5.5. Sub- and supersolutions of problem (5.1) have been constructed in [32] under the conditions (A1′)–(A3′), (j1)–(j2) and (F2)(i)–(F2)(iii), where the gradient dependence of f has been dropped, meaning that $f(x, r, s) := f(x, r, s, \xi)$. Further, it is assumed that $A = -\Delta_p$ which is the negative p-Laplacian defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{where } \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right). \tag{5.16}$$

The coefficients a_i, $i = 1, \dots, N$ are given by

$$a_i(x, s, \xi) = |\xi|^{p-2}\xi_i. \tag{5.17}$$

Thus, hypothesis (A1′) is satisfied with $k_0 = 0$ and $c_0 = 1$. Hypothesis (A2′) is a consequence of the inequalities from the vector-valued function $\xi \mapsto |\xi|^{p-2}\xi$ (see [7, page 37]), and (A3′) is satisfied with $c_1 = 1$ and $k_1 = 0$. The construction is done by using solutions of simple auxiliary elliptic boundary value problems and the eigenfunction of the p-Laplacian which belongs to its first eigenvalue.

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