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Research Article

On Uniqueness of Meromorphic Functions with Multiple Values in Some Angular Domains

Zu-Xing Xuan[1, 2]

[1] *Department of Mathematical Sciences, Tsinghua University, Haidian District, Beijing 100084, China*
[2] *Department of Basic Courses, Beijing Union University, No. 97 Bei Si Huan Dong Road, Chaoyang District, Beijing 100101, China*

Correspondence should be addressed to Zu-Xing Xuan, xuanzuxing@ss.buaa.edu.cn

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This article deals with problems of the uniqueness of transcendental meromorphic function with shared values in some angular domains dealing with the multiple values which improve a result of J. Zheng.

1. Introduction

A transcendental meromorphic function is meromorphic in the complex plane $\mathbb{C}$ and not rational. We assume that the readers are familiar with the Nevanlinna theory of meromorphic functions and the standard notations such as Nevanlinna deficiency $\delta(a, f)$ of $f(z)$ with respect to $a \in \widehat{\mathbb{C}}$ and Nevanlinna characteristic $T(r, f)$ of $f(z)$. And the lower order μ and the order λ are in turn defined as follows:

$$\mu = \mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r},$$
$$\lambda = \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}. \tag{1.1}$$

For the references, please see [1]. An $a \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called an IM (ignoring multiplicities) shared value in $X \subseteq \widehat{\mathbb{C}}$ of two meromorphic functions $f(z)$ and $g(z)$ if in X, $f(z) = a$ if and only if $g(z) = a$. It is Nevanlinna [2] who proved the first uniqueness theorem, called the Five Value Theorem, which says that two meromorphic functions $f(z)$ and $g(z)$ are identical if they have five distinct IM shared values in $X = \mathbb{C}$. After his very fundamental work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations (see [3]). Recently, Zheng in [4] suggested for the first time the investigation of uniqueness of a function meromorphic in a precise subset of $\widehat{\mathbb{C}}$, and this is an interesting topic.

Given m pair of real numbers $\{\alpha_j, \beta_j\}$ satisfying

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_m < \beta_m \leq \pi, \tag{1.2}$$

we define

$$\omega = \max\left\{\frac{\pi}{\beta_1 - \alpha_1}, \ldots, \frac{\pi}{\beta_m - \alpha_m}\right\}. \tag{1.3}$$

Zheng in [4] proved the following theorem.

Theorem A. *Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions, and let $f(z)$ be of finite order λ and such that for some $a \in \widehat{\mathbb{C}}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. For m pair of real numbers $\{\alpha_j, \beta_j\}$ satisfying* (1.2) *and*

$$\sum_{j=1}^{m} (\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta}{2}}, \tag{1.4}$$

where $\sigma = \max\{\omega, \mu\}$, assume that $f(z)$ and $g(z)$ have five distinct IM shared values in $X = \bigcup_{j=1}^{m}\{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f(z) \equiv g(z)$.

However, it was not discussed whether there are similar results dealing with multiple values in some angular domains. In this paper we investigate this problem.

We use $\overline{E}_{k)}(a, X, f)$ to denote the set of zeros of $f(z) - a$ in X, with multiplicities no greater than k, in which each zero counted only once.

Our main result is what follows.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions, and let $f(z)$ be of finite order λ and such that for some $a \in \widehat{\mathbb{C}}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. For m pair of real numbers $\{\alpha_j, \beta_j\}$ satisfying* (1.2) *and*

$$\sum_{j=1}^{m} (\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta}{2}}, \tag{1.5}$$

where $\sigma = \max\{\omega, \mu\}$, *assume that* a_j $(j = 1, 2, \ldots, q)$ *are* q *distinct complex numbers, and let* k_j $(j = 1, 2, \ldots, q)$ *be positive integers or* ∞ *satisfying*

$$k_1 \geq k_2 \geq \cdots \geq k_q, \tag{1.6}$$

$$\overline{E}_{k_j)}(a_j, X, f) = \overline{E}_{k_j)}(a_j, X, g), \tag{1.7}$$

$$\sum_{j=3}^{q} \frac{k_j}{k_j + 1} > 2, \tag{1.8}$$

where $X = \bigcup_{j=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\}$. *If* $\omega < \lambda(f)$, *then* $f(z) \equiv g(z)$.

2. Proof of Theorem 1.1

First we introduce several lemmas which are crucial in our proofs. The following result was proved in [5] (also see [6]).

Lemma 2.1 (see [5]). *Let* $f(z)$ *be transcendental and meromorphic in* $\mathbb{C}$ *with the lower order* $0 \leq \mu < \infty$ *and the order* $0 < \lambda \leq \infty$. *Then for arbitrary positive number* σ *satisfying* $\mu \leq \sigma \leq \lambda$ *and a set* E *with finite linear measure, there exists a sequence of positive numbers* $\{r_n\}$ *such that*

(1) $r_n \overline{\in} E$, $\lim_{n\to\infty}(r_n/n) = \infty$,

(2) $\liminf_{n\to\infty}(\log T(r_n, f)/\log r_n) \geq \sigma$,

(3) $T(t, f) < (1 + o(1))(t/r_n)^{\sigma} T(r_n, f)$, $t \in [r_n/n, nr_n]$.

A sequence r_n satisfying (1), (2), and (3) in Lemma 2.1 is called Polya peak of order σ outside E in this article. For $r > 0$ and $a \in \mathbb{C}$ define

$$D(r, a) := \left\{\theta \in [-\pi, \pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \frac{1}{\log r} T(r, f)\right\}, \tag{2.1}$$

$$D(r, \infty) := \left\{\theta \in [-\pi, \pi) : \log^+ \left|f\left(re^{i\theta}\right)\right| > \frac{1}{\log r} T(r, f)\right\}. \tag{2.2}$$

The following result is a special version of the main result of Baernstein [7].

Lemma 2.2. *Let* $f(z)$ *be transcendental and meromorphic in* $\mathbb{C}$ *with the finite lower order* μ *and the order* $0 < \lambda \leq \infty$ *and for some* $a \in \widehat{\mathbb{C}}$, $\delta = \delta(a, f) > 0$. *Then for arbitrary Polya peak* r_n *of order* $\sigma > 0$, $\mu \leq \sigma \leq \lambda$, *we have*

$$\liminf_{n\to\infty} mes\, D(r_n, a) \geq \min\left\{2\pi, \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta}{2}}\right\}. \tag{2.3}$$

Although Lemma 2.2 was proved in [7] for the Polya peak of order μ, the same argument of Baernstein [7] can derive Lemma 2.2 for the Polya peak of order σ, $\mu \leq \sigma \leq \lambda$.

Nevanlinna theory on angular domain will play a key role in the proof of theorems. Let $f(z)$ be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\le \arg z\le\beta\}$, where $0<\beta-\alpha\le 2\pi$. Nevanlinna defined the following notations (see [8]):

$$\begin{aligned}
A_{\alpha,\beta}(r,f) &= \frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^{\omega}}-\frac{t^{\omega}}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right|+\log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t},\\
B_{\alpha,\beta}(r,f) &= \frac{2\omega}{\pi r^{\omega}}\int_{\alpha}^{\beta}\log^+\left|f\left(re^{i\theta}\right)\right|\sin\omega(\theta-\alpha)d\theta,\\
C_{\alpha,\beta} &= 2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^{\omega}}-\frac{|b_n|^{\omega}}{r^{2\omega}}\right)\sin\omega(\theta_n-\alpha),
\end{aligned}\tag{2.4}$$

where $\omega=\pi/(\beta-\alpha)$ and $b_n=|b_n|e^{i\theta_n}$ are the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ appearing according to their multiplicities. $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\overline{\Omega}(\alpha,\beta)$ and Nevanlinna's angular characteristic is defined as follows:

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).\tag{2.5}$$

Throughout, we denote by $R_{\alpha,\beta}(r,*)$ a quantity satisfying

$$R_{\alpha,\beta}(r,*)=O\{\log(rS_{\alpha,\beta}(r,*))\},\quad r\overline{\in}E,\tag{2.6}$$

where E denotes a set of positive real numbers with finite linear measure. It is not necessarily the same for every occurrence in the context [9].

Lemma 2.3. *Let $f(z)$ be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(\frac{1}{f-a}\right)=S_{\alpha,\beta}(r,f)+O(1),\tag{2.7}$$

and for an integer $p\ge 0$,

$$\begin{aligned}
S_{\alpha,\beta}\left(r,f^{(p)}\right) &\le 2^pS_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f),\\
A_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right) &= R_{\alpha,\beta}(r,f),
\end{aligned}\tag{2.8}$$

and $R_{\alpha,\beta}(r,f^{(p)})=R_{\alpha,\beta}(r,f)$.

Lemma 2.4. *Let $f(z)$ be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then for arbitrary q distinct $a_j \in \widehat{\mathbb{C}}$ $(1 \le j \le q)$, we have*

$$(q-2)S_{\alpha,\beta}(r,f) \le \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + R(r,f), \tag{2.9}$$

where the term $\overline{C}_{\alpha,\beta}(r,1/(f-a_j))$ will be replaced by $\overline{C}_{\alpha,\beta}(r,f)$ when some $a_j=\infty$.

We use $\overline{C}^{k)}_{\alpha,\beta}(r,1/(f-a))$ to denote the zeros of $f(z)-a$ in $\overline{\Omega}(\alpha,\beta)$ whose multiplicities are no greater than k and are counted only once. Likewise, we use $\overline{C}^{(k+1}_{\alpha,\beta}(r,1/(f-a))$ to denote the zeros of $f(z)-a$ in $\overline{\Omega}(\alpha,\beta)$ whose multiplicities are greater than k and are counted only once.

Lemma 2.5. *Let $f(z)$ be meromorphic on $\overline{\Omega}(\alpha,\beta)$, and let k_j $(j=1,2,\ldots,q)$ be q positive integers. Then for arbitrary q distinct $a_j \in \widehat{\mathbb{C}}$ $(1 \le j \le q)$, we have*

$$\begin{aligned} &(i) \quad (q-2)S_{\alpha,\beta}(r,f) < \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{C}^{k_j)}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + \sum_{j=1}^{q}\frac{1}{k_j+1}C_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + R(r,f), \\ &(ii) \quad \left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)S_{\alpha,\beta}(r,f) < \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{C}^{k_j)}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + R(r,f), \end{aligned} \tag{2.10}$$

where the term $\overline{C}_{\alpha,\beta}(r,1/(f-a_j))$ will be replaced by $\overline{C}_{\alpha,\beta}(r,f)$ when some $a_j=\infty$.

Proof. According to our notations, we have

$$\begin{aligned} \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) &= \overline{C}^{k)}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) + \overline{C}^{(k+1}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) \\ &= \frac{k}{k+1}\overline{C}^{k)}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) + \frac{1}{k+1}\overline{C}^{k)}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) + \overline{C}^{(k+1}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) \\ &\le \frac{k}{k+1}\overline{C}^{k)}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) + \frac{1}{k+1}C^{k)}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) + \frac{1}{k+1}C^{(k+1}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) \\ &= \frac{k}{k+1}\overline{C}^{k)}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) + \frac{1}{k+1}C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right). \end{aligned} \tag{2.11}$$

By Lemma 2.4,

$$\begin{aligned}(q-2)S_{\alpha,\beta}(r,f) &\le \sum_{j=1}^{q}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+R(r,f)\\ &\le \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{C}^{k_j)}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{q}\frac{1}{k_j+1}C_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+R(r,f),\end{aligned} \tag{2.12}$$

and (i) follows.

Furthermore, $C_{\alpha,\beta}(r,1/(f-a_j)) < S_{\alpha,\beta}(r,f)$, and on combining this with (i), we get (ii). □

Proof of Theorem 1.1. Suppose $f(z)\not\equiv g(z)$. For convenience, below we omit the subscript of all the notations, such as $S(r,*)$ and $C(r,*)$. By applying Lemma 2.5 to g and (1.6), we have

$$\begin{aligned}\left(\sum_{j=3}^{q}\frac{k_j}{k_j+1}+\frac{2k_2}{k_2+1}-2\right)S(r,g) &\le \frac{k_2}{k_2+1}\sum_{j=1}^{q}\overline{C}^{k_j)}\left(r,\frac{1}{g-a_j}\right)+R(r,g)\\ &\le \frac{k_2}{k_2+1}C\left(r,\frac{1}{f-g}\right)+R(r,g)\\ &\le \frac{k_2}{k_2+1}S(r,f-g)+R(r,g)\\ &\le \frac{k_2}{k_2+1}S(r,f)+\frac{k_2}{k_2+1}S(r,g)+R(r,g),\end{aligned} \tag{2.13}$$

so that

$$\left(\sum_{j=3}^{q}\frac{k_j}{k_j+1}+\frac{k_2}{k_2+1}-2\right)S(r,g)-R(r,g) < \frac{k_2}{k_2+1}S(r,f). \tag{2.14}$$

This implies that $R(r,g)=R(r,f)$. We have also (2.14) for alternation of f and g, then

$$\left(\sum_{j=3}^{q}\frac{k_j}{k_j+1}+\frac{k_2}{k_2+1}-2\right)S(r,f)-R(r,f) < \frac{k_2}{k_2+1}S(r,g) \le S(r,f)+R(r,f). \tag{2.15}$$

By (1.8), we have

$$S(r,f)=O(\log r), \quad r\notin E. \tag{2.16}$$

We assume that $a \in \mathbb{C}$. By the same argument we can show Theorem 1.1 for the case when $a = \infty$. By applying Lemma 2.3 and (2.16), we estimate

$$\begin{aligned}
B\left(r, \frac{1}{f^{(p)} - a}\right) &\leq S\left(r, f^{(p)}\right) + O(1) \\
&= (A+B)\left(r, \frac{f^{(p)}}{f}\right) + (A+B)(r,f) + p\overline{C}(r,f) + C(r,f) + O(1) \\
&\leq (p+1)S(r,f) + R(r,f) = O(\log r), \quad r \notin E.
\end{aligned} \tag{2.17}$$

The following method comes from [10]. But we quote it in detail here because of its independent significance. Note that $\lambda(f) > \omega$. We need to treat two cases.

(I) $\lambda(f) > \mu$. Then $\lambda(f^{(p)}) = \lambda(f) > \sigma \geq \mu = \mu(f^{(p)})$. And by the inequality (1.5), we can take a real number $\epsilon > 0$ such that

$$\sum_{j=1}^{m} (\alpha_{j+1} - \beta_j + 2\epsilon) + 2\epsilon < \frac{4}{\sigma + 2\epsilon} \arcsin\sqrt{\frac{\delta}{2}}, \tag{2.18}$$

where $\alpha_{m+1} = 2\pi + \alpha_1$, and

$$\lambda\left(f^{(p)}\right) > \sigma + 2\epsilon > \mu. \tag{2.19}$$

Applying Lemma 2.1 to $f^{(p)}(z)$ gives the existence of the Polya peak r_n of order $\sigma + 2\epsilon$ of $f^{(p)}$ such that $r_n \notin E$, and then from Lemma 2.2 for sufficiently large n we have

$$mesD(r_n, a) > \frac{4}{\sigma + 2\epsilon} \arcsin\sqrt{\frac{\delta}{2}} - \epsilon, \tag{2.20}$$

since $\sigma + 2\epsilon > 1/2$. We can assume for all the n, (13) holds. Set

$$K := mes\left(D(r_n, a) \bigcap \bigcup_{j=1}^{m} (\alpha_j + \epsilon, \beta_j - \epsilon) \right). \tag{2.21}$$

Then from (2.18) and (2.20) it follows that

$$\begin{aligned}
K &\geq mes(D(r_n, a)) - mes\left([0, 2\pi) \setminus \bigcup_{j=1}^{m} (\alpha_j + \epsilon, \beta_j - \epsilon) \right) \\
&= mes(D(r_n, a)) - mes\left(\bigcup_{j=1}^{m} (\beta_j - \epsilon, \alpha_{j+1} + \epsilon) \right) \\
&= mes(D(r_n, a)) - \sum_{j=1}^{m} (\alpha_{j+1} - \beta_j + 2\epsilon) > \epsilon > 0.
\end{aligned} \tag{2.22}$$

It is easy to see that there exists a j_0 such that for infinitely many n, we have

$$mes\Big(D(r_n,a)\bigcap(\alpha_{j_0}+\epsilon,\beta_{j_0}-\epsilon)\Big)>\frac{K}{q}. \tag{2.23}$$

We can assume for all the n, (2.23) holds. Set $E_n = D(r_n,a)\bigcap(\alpha - j_0+\epsilon,\beta_{j_0}-\epsilon)$. Thus from the definition (2.1) of $D(r,a)$ it follows that

$$\begin{aligned}\int_{\alpha_{j_0}+\epsilon}^{\beta_{j_0}-\epsilon}\log^+\frac{1}{|f^{(p)}(r_ne^{i\theta})-a|}d\theta &\geq \int_{E_n}\log^+\frac{1}{|f^{(p)}(r_ne^{i\theta})-a|}d\theta\\ &\geq mes(E_n)\frac{T(r_n,f^{(p)})}{\log r_n}\\ &> \frac{K}{m}\frac{T(r_n,f^{(p)})}{\log r_n}.\end{aligned} \tag{2.24}$$

On the other hand, by the definition (2.4) of $B_{\alpha,\beta}(r,*)$ and (2.14), we have

$$\begin{aligned}\int_{\alpha_{j_0}+\epsilon}^{\beta_{j_0}-\epsilon}\log^+\frac{1}{|f^{(p)}(r_ne^{i\theta})-a|}d\theta &\leq \frac{\pi}{2\omega_{j_0}\sin(\epsilon\omega_{j_0})}r^{\omega_{j_0}}B_{\alpha_{j_0},\beta_{j_0}}\left(r,\frac{1}{f^{(p)}-a}\right)\\ &< \widetilde{K}_{j_0}r^{\omega_{j_0}}\log r, \quad r\notin E.\end{aligned} \tag{2.25}$$

Combining (2.24) with (2.25) gives

$$T\Big(r_n,f^{(p)}\Big)\leq\frac{m\widetilde{K}_{j_0}}{K}r_n^{\omega_{j_0}}\log^2 r_n. \tag{2.26}$$

Thus from (1.5) in Lemma 2.1 for $\sigma+2\epsilon$, we have

$$\sigma+\epsilon\leq\limsup_{n\to\infty}\frac{\log T(r_n,f^{(p)})}{\log r_n}\leq\omega_{j_0}\leq\sigma+\epsilon. \tag{2.27}$$

This is impossible.

(II) $\lambda(f)=\mu$. Then $\sigma=\mu=\lambda(f)=\lambda(f^{(p)})=\mu(f^{(p)})$. By the same argument as in (I) with all the $\sigma+2\epsilon$ replaced by $\sigma=\mu$, we can derive

$$\max\{\omega,\mu\}=\sigma\leq\omega<\lambda(f). \tag{2.28}$$

This is impossible. Theorem 1.1 follows. □

Remark 2.6. In Theorem A, $q=5$, $k_1=k_2=k_3=k_4=k_5=\infty$, then

$$\frac{k_3}{k_3+1}+\frac{k_4}{k_4+1}+\frac{k_5}{k_5+1}=3>2, \tag{2.29}$$

so Theorem A is a special case of Theorem 1.1. Meanwhile, Zheng in [4, pages 153–154] gave some examples to indicate that the conditions are necessary. So the conditions in theorem are also necessary.

Corollary 2.7. *In Theorem 1.1,*

(i) *if* $q = 7$, *then* $f(z) \equiv g(z)$,

(ii) *if* $q = 6$, $k_3 \geq 2$, *then* $f(z) \equiv g(z)$,

(iii) *if* $q = 5$, $k_3 \geq 3$, $k_5 \geq 2$, *then* $f(z) \equiv g(z)$,

(iv) *if* $q = 5$, $k_4 \geq 4$, *then* $f(z) \equiv g(z)$,

(v) *if* $q = 5$, $k_3 \geq 5$, *then* $f(z) \equiv g(z)$,

(vi) *if* $q = 5$, $k_3 \geq 6$, $k_4 \geq 2$, *then* $f(z) \equiv g(z)$,

Corollary 2.8. *Let* $f(z)$ *and* $g(z)$ *be both transcendental meromorphic functions and let* $f(z)$ *be of finite lower order* μ *and such that for some* $a \in \widehat{\mathbb{C}}$ *and an integer* $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. *For* m *pair of real numbers* $\{\alpha_j, \beta_j\}$ *satisfying* (1.2) *and*

$$\sum_{j=1}^{m} (\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}}, \tag{2.30}$$

where $\sigma = \max\{\omega, \mu\}$, *assume that* a_j $(j = 1, 2, \ldots, q)$ *are* $q(= 5 + [2/k])$ *distinct complex numbers satisfying that* $\overline{E}_{k)}(a_j, X, f) = \overline{E}_{k)}(a_j, X, g)$ $(j = 1, 2, \ldots, q)$, *where* k *is an integer or* ∞. *If* $\omega < \lambda(f)$, *then* $f(z) \equiv g(z)$.

Corollary 2.9. *Let* $f(z)$ *and* $g(z)$ *be both transcendental meromorphic functions and let* $f(z)$ *be of finite lower order* μ *and such that for some* $a \in \widehat{\mathbb{C}}$ *and an integer* $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. *For* m *pair of real numbers* $\{\alpha_j, \beta_j\}$ *satisfying* (1.2) *and*

$$\sum_{j=1}^{m} (\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}}, \tag{2.31}$$

where $\sigma = \max\{\omega, \mu\}$, *assume that* a_j $(j = 1, 2, \ldots, q)$ *are* $q = 5$ *distinct complex numbers satisfying that* $\overline{E}_{3)}(a_j, X, f) = \overline{E}_{3)}(a_j, X, g)$ $(j = 1, 2, 3)$, $\overline{E}_{2)}(a_j, X, f) = \overline{E}_{2)}(a_j, X, g)$ $(j = 4, 5)$, *then* $f(z) \equiv g(z)$.

Question 1. For two meromorphic functions defined in $\mathbb{C}$, there are many uniqueness theorems when they share small functions ($a(z)$ is called a small function of $f(z)$ if $T(r, a(z)) = o(T(r, f))(r \to \infty)$) (see [3]). So we ask an interesting question: are there similar results when they share small functions in some precise domain $X \subseteq \mathbb{C}$?

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