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Research Article

On Absolute Cesàro Summability

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Denote by $\mathcal{A}_k$ the sequence space defined by $\mathcal{A}_k = \{(s_n) : \sum_{n=1}^{\infty} n^{k-1}|a_n|^k < \infty, a_n = s_n - s_{n-1}\}$ for $k \geq 1$. In a recent paper by E. Savaş and H. Şevli (2007), they proved every Cesàro matrix of order α, for $\alpha > -1$, $(C, \alpha) \in B(\mathcal{A}_k)$ for $k \geq 1$. In this paper, we consider a further extension of absolute Cesàro summability.

1. Introduction

Let $\sum a_v$ denote a series with partial sums (s_n). For an infinite matrix T, t_n, the nth term of the T-transform of (s_n) is denoted by

$$t_n = \sum_{v=0}^{\infty} t_{nv} s_v. \tag{1.1}$$

A series $\sum a_v$ is said to be absolutely T-summable if $\sum_n |\Delta t_{n-1}| < \infty$, where Δ is the forward difference operator defined by $\Delta t_{n-1} = t_{n-1} - t_n$. Papers dealing with absolute summability date back at least as far as Fekete [1].

A sequence (s_n) is said to be of bounded variation (bv) if $\sum_n |\Delta s_n| < \infty$. Thus, to say that a series is absolutely summable by a matrix T is equivalent to saying that the T-transform the sequence is in bv. Necessary and sufficient conditions for a matrix $T : \mathrm{b}v \rightarrow \mathrm{b}v$ are known. (See, e.g., Stieglitz and Tietz [2]).

Let σ_n^{α} denote the nth terms of the transform of a Cesáro matrix (C, α) of a sequence (s_n). In 1957 Flett [3] made the following definition. A series $\sum a_n$, with partial sums (s_n), is said to be absolutely (C,α) summable of order $k \geq 1$, written $\sum a_n$ is summable $|C,\alpha|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| \sigma_{n-1}^{\alpha} - \sigma_n^{\alpha} \right|^k < \infty. \tag{1.2}$$

He then proved the following inclusion theorem.

Theorem 1.1 (see [3]). *If a series $\sum a_n$ is summable $|C,\alpha|_k$, then it is summable $|C,\beta|_r$ for each $r \geq k \geq 1$, $\alpha > -1$, $\beta > \alpha + 1/k - 1/r$.*

It then follows that if one chooses $r = k$, then a series $\sum a_n$, which is $|C,\alpha|_k$ summable, is also $|C,\beta|_k$ summable for $k \geq 1$, $\beta > \alpha > -1$.

Absolute Abel summability, written as $|A|$, was defined by Whittaker [4] as follows. A series $\sum a_n$ is said to be summable $|A|$ if the series $\sum a_n x^n$ is convergent for $0 \leq x < 1$ and its sum-function $\phi(x)$ satisfies the condition:

$$\int_0^1 \left| \phi'(x) \right| dx < \infty. \tag{1.3}$$

In the same paper, Flett extended this result to index k by replacing condition (1.3) by the condition:

$$\int_0^1 (1-x)^{k-1} \left| \phi'(x) \right|^k dx < \infty. \tag{1.4}$$

Thus the series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if the series $\sum a_n x^n$ is convergent for $0 \leq x < 1$ and its sum-function $\phi(x)$ satisfies condition (1.4). He then showed that summability $|A|_k$ is a weaker property than summability $|C,\alpha|_k$ for any $\alpha > -1$.

2. The Space $\mathcal{A}_k$

Let $\sum a_n$ be a series with partial sums (s_n). Denote by $\mathcal{A}_k$ the sequence space defined by

$$\mathcal{A}_k = \left\{ (s_n) : \sum_{n=1}^{\infty} n^{k-1} |a_n|^k < \infty,\ a_n = s_n - s_{n-1} \right\}. \tag{2.1}$$

If one sets $\alpha = 0$ in the inclusion statement involving (C,α) and (C,β), then one obtains the fact that $(C,\beta) \in B(\mathcal{A}_k)$ for each $\beta > 0$, where $B(\mathcal{A}_k)$ denotes the algebra of all matrices that map $\mathcal{A}_k$ to $\mathcal{A}_k$.

Let A be a sequence to sequence transformation mapping, the sequence (s_n) into (t_n). If whenever (s_n) converges absolutely, (t_n) converges absolutely, A is called absolutely conservative. If the absolute convergence of (s_n) implies the absolute convergence of (t_n) to the same limit, A is called absolutely regular.

In 1970, using the same definition as Flett, Das [5] defined such a matrix to be absolutely kth power conservative for $k \geq 1$, if $T \in B(\mathcal{A}_k)$; that is, if (s_n) is a sequence satisfying

$$\sum_{n=1}^{\infty} n^{k-1} |s_n - s_{n-1}|^k < \infty, \tag{2.2}$$

then

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{2.3}$$

For $k = 1$, condition (2.2) guarantees the convergence of (s_n). Note that when $k > 1$, (2.2) does not necessarily imply the convergence of (s_n). For example, take

$$s_n = \sum_{v=1}^{n} \frac{1}{v \log(v+1)}. \tag{2.4}$$

Then (2.2) holds but (s_n) does not converge. Thus, since the limit of (s_n) needs not to exist, we cannot introduce the concept of absolute kth power regularity when $k > 1$.

In that same paper, Das proved that every conservative Hausdorff matrix $H \in B(\mathcal{A}_k)$, which contains as a special case the fact that $(C, \beta) \in B(\mathcal{A}_k)$ for $\beta > 0$. We know that if $\beta \geq 0$, then (C, β) is regular, and if $\beta < 0$, then (C, β) is neither conservative nor regular. In [6], the result of Flett and Das was extended by the following theorem.

Theorem 2.1 (see [6]). *It holds that $(C, \alpha) \in B(\mathcal{A}_k)$ for each $\alpha > -1$.*

Remark 2.2. In [6], when $-1 < \alpha < 0$ it should be added the condition

$$\sum_{n=1}^{\infty} n^{k-\alpha-1} |a_n|^k = O(1). \tag{2.5}$$

in the statement of Theorem 2.1. Also, it should be added the absolute values of the binomial coefficients in the proof of Theorem 2.1 for the case $-1 < \alpha < 0$.

Since summability $|A|_k$ is a weaker property than summability $|C, \alpha|_k$ for any $\alpha > -1$, from Theorem 2.1, we obtain the following theorem.

Theorem 2.3. *If $(s_n) \in \mathcal{A}_k$ then $\sum a_n$ is summable $|A|_k$, $k \geq 1$.*

3. The Main Results

In this paper we consider a further extension of absolute Cesàro summability. If one sets $\alpha = 0$ in Theorem 1.1, then one obtains the fact that $(C, \beta) \in (\mathcal{A}_k, \mathcal{A}_r)$ for each $r \geq k \geq 1$, $\beta > 1/k - 1/r$. It is the purpose of this work to extend this result to the case $\beta > -k/r$.

We will use the following Lemma.

Lemma 3.1 (see [7]). *If $\theta > -1$ and $\theta - \varphi > 0$, then*

$$\sum_{n=v}^{\infty} \frac{E_{n-v}^{\varphi}}{nE_n^{\theta}} = \frac{1}{vE_v^{\theta-\varphi-1}}, \quad E_n^{\theta} = \frac{\Gamma(\theta+n+1)}{\Gamma(n+1)\Gamma(\theta+1)} \approx \frac{n^{\theta}}{\Gamma(\theta+1)}. \tag{3.1}$$

We now prove the following theorem.

Theorem 3.2. *Let $r \geq k \geq 1$.*

(i) *It holds that $(C,\alpha) \in (\mathcal{A}_k, \mathcal{A}_r)$ for each $\alpha > 1 - k/r$.*

(ii) *If $\alpha = 1 - k/r$ and the condition $\sum_{n=1}^{\infty} n^{k-1} \log n |a_n|^k = O(1)$ is satisfied then $(C,\alpha) \in (\mathcal{A}_k, \mathcal{A}_r)$.*

(iii) *If the condition $\sum_{n=1}^{\infty} n^{k+(r/k)(1-\alpha)-2} |a_n|^k = O(1)$ is satisfied then $(C,\alpha) \in (\mathcal{A}_k, \mathcal{A}_r)$ for each $-k/r < \alpha < 1 - k/r$.*

Proof. Let σ_n^{α} denote the nth term of the Cesáro mean of order α of a sequence (s_n); that is,

$$\sigma_n^{\alpha} = \frac{1}{E_n^{\alpha}} \sum_{v=0}^{n} E_{n-v}^{\alpha-1} s_v. \tag{3.2}$$

We will show that $(\sigma_n^{\alpha}) \in \mathcal{A}_r$; that is,

$$\sum_{n=1}^{\infty} n^{r-1} \left|\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}\right|^r < \infty. \tag{3.3}$$

Let τ_n^{α} denote the nth term of the Cesáro mean of order α $(\alpha > -1)$ of the sequence (na_n); that is,

$$\tau_n^{\alpha} = \frac{1}{E_n^{\alpha}} \sum_{v=1}^{n} E_{n-v}^{\alpha-1} v a_v. \tag{3.4}$$

Since $\tau_n^{\alpha} = n(\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha})$ (see [8]), condition (3.3) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |\tau_n^{\alpha}|^r < \infty. \tag{3.5}$$

It follows from Hölder's inequality that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} |\tau_n^{\alpha}|^r &= \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{1}{E_n^{\alpha}} \sum_{v=1}^{n} E_{n-v}^{\alpha-1} v a_v \right|^r \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n(E_n^{\alpha})^r} \left\{ \sum_{v=1}^{n} \left|E_{n-v}^{\alpha-1}\right| v^k |a_v|^k \right\}^{r/k} \times \left\{ \sum_{v=1}^{n} \left|E_{n-v}^{\alpha-1}\right| \right\}^{(k-1)r/k}. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned}\sum_{v=1}^{n}\left|E_{n-v}^{\alpha-1}\right| &= \left|E_{0}^{\alpha-1}\right| + \sum_{v=1}^{n-1}\left|E_{n-v}^{\alpha-1}\right| = \left|E_{0}^{\alpha-1}\right| + \left|\sum_{v=1}^{n-1}E_{n-v}^{\alpha-1}\right| \\ &= \left|E_{0}^{\alpha-1}\right| + \left|\sum_{v=0}^{n}E_{n-v}^{\alpha-1} - E_{n}^{\alpha-1} - E_{0}^{\alpha-1}\right| = \left|E_{0}^{\alpha-1}\right| + \left|E_{n-1}^{\alpha} - E_{0}^{\alpha-1}\right|,\end{aligned} \tag{3.7}$$

and using the fact that

$$\left|\frac{E_{n-1}^{\alpha}}{E_{n}^{\alpha}}\right| = O(1), \tag{3.8}$$

we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left|\tau_{n}^{\alpha}\right|^{r} &\leq \sum_{n=1}^{\infty}\frac{\left(E_{n}^{\alpha}\right)^{(k-1)r/k}}{n\left(E_{n}^{\alpha}\right)^{r}}\left\{\sum_{v=1}^{n}\left|E_{n-v}^{\alpha-1}\right|v^{k}\left|a_{v}\right|^{k}\right\}^{r/k} \\ &\leq \sum_{n=1}^{\infty}\frac{\left(E_{n}^{\alpha}\right)^{-r/k}}{n}\left\{\sum_{v=1}^{n}\left|E_{n-v}^{\alpha-1}\right|v^{1-k/r+k^{2}/r}\left|a_{v}\right|^{k^{2}/r}v^{-(r-k)+k(r-k)/r}\left|a_{v}\right|^{k(r-k)/r}\right\}^{r/k}.\end{aligned} \tag{3.9}$$

Applying Hölder's inequality with indices r/k, $r/(r-k)$, we deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\tau_{n}^{\alpha}\right|^{r} \leq \sum_{n=1}^{\infty}\frac{\left(E_{n}^{\alpha}\right)^{-r/k}}{n}\sum_{v=1}^{n}\left|E_{n-v}^{\alpha-1}\right|^{r/k}v^{k-1+r/k}\left|a_{v}\right|^{k}\left\{\sum_{v=1}^{n}v^{k-1}\left|a_{v}\right|^{k}\right\}^{(r-k)/k}. \tag{3.10}$$

Since $(s_n) \in \mathcal{A}_k$, we have

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\tau_{n}^{\alpha}\right|^{r} = O(1)\sum_{v=1}^{\infty}v^{k-1}\left|a_{v}\right|^{k}v^{r/k}\sum_{n=v}^{\infty}\frac{\left|E_{n-v}^{\alpha-1}\right|^{r/k}}{n\left(E_{n}^{\alpha}\right)^{r/k}}. \tag{3.11}$$

From Lemma 3.1, if $\alpha > 1 - k/r$, then

$$\sum_{n=v}^{\infty}\frac{\left(E_{n-v}^{\alpha-1}\right)^{r/k}}{n\left(E_{n}^{\alpha}\right)^{r/k}} = O\left(v^{-r/k}\right), \tag{3.12}$$

therefore

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\tau_{n}^{\alpha}\right|^{r} = O(1)\sum_{v=1}^{\infty}v^{k-1}\left|a_{v}\right|^{k} = O(1). \tag{3.13}$$

If $\alpha = 1 - k/r$, then (See Lemma 5 of [[9]]).

$$\sum_{n=v}^{\infty} \frac{\left|E_{n-v}^{\alpha-1}\right|^{r/k}}{n\left(E_n^{\alpha}\right)^{r/k}} = O\left(v^{-r/k} \log v\right), \tag{3.14}$$

and then

$$\sum_{n=1}^{\infty} \frac{1}{n} |\tau_n^{\alpha}|^r = O(1) \sum_{v=1}^{\infty} v^{k-1} \log v |a_v|^k = O(1). \tag{3.15}$$

If $-k/r < \alpha < 1 - k/r$, then (See Lemma 5 of [[9]])

$$\sum_{n=v}^{\infty} \frac{\left|E_{n-v}^{\alpha-1}\right|^{r/k}}{n\left(E_n^{\alpha}\right)^{r/k}} = O\left(v^{-\alpha(r/k)-1}\right), \tag{3.16}$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{n} |\tau_n^{\alpha}|^r = O(1) \sum_{v=1}^{\infty} v^{k+(r/k)(1-\alpha)-2} |a_v|^k = O(1). \tag{3.17}$$

□

Theorem 3.2 includes Theorem 2.1 with the special case $r = k$.

Theorem 3.3. *If $(s_n) \in \mathcal{A}_k$, then $\sum a_n$ is summable $|A|_r, r \geq k \geq 1$.*

Proof. Using the fact that the summability $|A|_k$ is a weaker property than summability $|C,\alpha|_k$ for any $\alpha > -1$, then the proof follows from Theorem 3.2. □

Now we give some negative results.

Corollary 3.4. *Let $k < r$. Then $(s_n) \in \mathcal{A}_r$ does not imply that the series $\sum a_n$ is summable $|A|_k$.*

Proof. Let p be any number such that $k < p < r$ and let $a_n = 1/n(\log n)^{1/p}$. Then, we have $(s_n) \in \mathcal{A}_r$. As in the proof of Flett, since $\int_0^1 (1-x)^{k-1} |\phi'(x)|^k dx$ is divergent, $\sum a_n$ is not summable $|A|_k$. □

Corollary 3.5. *Let $k < r$. Then $(C,\alpha) \notin (\mathcal{A}_r, \mathcal{A}_k)$ for any $\alpha > -1$.*

Proof. The proof follows Theorem 3.3 and Corollary 3.4. □

Corollary 3.6. *Let $k < r$. Then $(C,\alpha) \notin (\mathcal{A}_k, \mathcal{A}_r)$ for any $-1 < \alpha < -k/r$.*

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