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Research Article

On The Hadamard's Inequality for Log-Convex Functions on the Coordinates

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Inequalities of the Hadamard and Jensen types for coordinated log-convex functions defined in a rectangle from the plane and other related results are given.

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1. Introduction

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds, this inequality is known as the Hermite-Hadamard inequality. For refinements, counterparts, generalizations and new Hadamard-type inequalities, see [1–8].

A positive function f is called log-convex on a real interval $I = [a, b]$, if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq f^{\lambda}(x) f^{1-\lambda}(y). \tag{1.2}$$

If f is a positive log-concave function, then the inequality is reversed. Equivalently, a function f is log-convex on I if f is positive and $\log f$ is convex on I. Also, if $f > 0$ and f'' exists on I, then f is log-convex if and only if $f \cdot f'' - (f')^2 \geq 0$.

The logarithmic mean of the positive real numbers a, b, $a \neq b$, is defined as

$$L(a,b) = \frac{b-a}{\log b - \log a}. \tag{1.3}$$

A version of Hadamard's inequality for log-convex (concave) functions was given in [9], as follows.

Theorem 1.1. *Suppose that f is a positive log-convex function on $[a,b]$, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq L(f(a), f(b)). \tag{1.4}$$

If f is a positive log-concave function, then the inequality is reversed.

For refinements, counterparts and generalizations of log-convexity see [9–13].

A convex function on the coordinates was introduced by Dragomir in [8]. A function $f : \Delta \to \mathbf{R}$ which is convex in Δ is called coordinated convex on Δ if the partial mapping $f_y : [a,b] \to \mathbf{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbf{R}$, $f_x(v) = f(x,v)$, are convex for all $y \in [c,d]$ and $x \in [a,b]$.

An inequality of Hadamard's type for coordinated convex mapping on a rectangle from the plane $\mathbf{R}^2$ established by Dragomir in [8], is as follows.

Theorem 1.2. *Suppose that $f : \Delta \to \mathbf{R}$ is coordinated convex on Δ, then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dy\,dx \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1.5}$$

The above inequalities are sharp.

The maximum modulus principle in complex analysis states that if f is a holomorphic function, then the modulus $|f|$ cannot exhibit a true local maximum that is properly within the domain of f. Characterizations of the maximum principle for sub(super)harmonic functions are considered in [14], as follows.

Theorem 1.3. *Let $G \subseteq \mathbb{R}^2$ be a region and let $f : G \to \mathbb{R}$ be a sub(super)harmonic function. If there is a point $\lambda \in G$ with $f(\lambda) \geq f(x)$, for all $x \in G$ then $f(x)$ is a constant function.*

Theorem 1.4. *Let $G \subseteq \mathbb{R}^2$ be a region and let f and g be bounded real-valued functions defined on G such that f is subharmonic and g is superharmonic. If for each point $a \in \partial_\infty G$*

$$\lim_{x \to a} \sup f(x) \leq \lim_{x \to a} \inf g(x), \tag{1.6}$$

then $f(x) < g(x)$ for all $x \in G$ or $f = g$ and f is harmonic.

In this paper, a new version of the maximum (minimum) principle in terms of convexity, and some inequalities of the Hadamard type are obtained.

2. On Coordinated Convexity and Sub(Super)Harmonic Functions

Consider the 2-dimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbf{R}^2$. A function $f : \Delta \to \mathbf{R}$ is called convex in Δ if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \tag{2.1}$$

holds for all $\mathbf{x}, \mathbf{y} \in \Delta$ and $\lambda \in [0,1]$.

As in [8], we define a log-convex function on the coordinates as follows: a function $f : \Delta \to \mathbf{R}_+$ will be called *coordinated log-convex* on Δ if the partial mappings $f_y : [a,b] \to \mathbf{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbf{R}$, $f_x(v) = f(x,v)$, are log-convex for all $y \in [c,d]$ and $x \in [a,b]$. A formal definition of a coordinated log-convex function may be stated as follows.

Definition 2.1. A function $f : \Delta \to \mathbf{R}_+$ will be called *coordinated log-convex* on Δ, for all $t, s \in [0,1]$ and $(x,y), (u,v) \in \Delta$, if the following inequality holds,

$$\begin{aligned} &f(tx + (1-t)y, su + (1-s)w) \\ &\quad \leq f^{ts}(x,u) f^{s(1-t)}(y,u) f^{t(1-s)}(x,w) f^{(1-t)(1-s)}(y,w). \end{aligned} \tag{2.2}$$

Equivalently, we can determine whether or not the function f is coordinated log-convex by using the following lemma.

Lemma 2.2. *Let $f : \Delta \to \mathbf{R}_+$. If f is twice differentiable then f is coordinated log-convex on Δ if and only if for the functions $f_y : [a,b] \to \mathbf{R}$, defined by $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbf{R}$, defined by $f_x(v) = f(x,v)$, we have*

$$f_x \cdot f_x'' - (f_x')^2 \geq 0, \qquad f_y \cdot f_y'' - \left(f_y'\right)^2 \geq 0. \tag{2.3}$$

Proof. The proof is straight forward using the elementary properties of log-convexity in one variable. □

Proposition 2.3. *Suppose that $g : [a,b] \to \mathbf{R}_+$ is twice differentiable on (a,b) and log-convex on $[a,b]$ and $h : [c,d] \to \mathbf{R}_+$ is twice differentiable on (c,d) and log-convex on $[c,d]$. Let $f : \Delta = [a,b] \times [c,d] \to \mathbf{R}_+$ be a twice differentiable function defined by $f(x,y) = g(x)h(y)$, then f is coordinated log-convex on Δ.*

Proof. This follows directly using Lemma 2.2. □

The following result holds.

Proposition 2.4. *Every log-convex function $f : \Delta = [a,b] \times [c,d] \to \mathbf{R}_+$ is log-convex on the coordinates, but the converse is not generally true.*

Proof. Suppose that $f : \Delta \to \mathbf{R}$ is convex in Δ. Consider the function $f_x : [c,d] \to \mathbf{R}_+$, $f_x(v) = f(x,v)$, then for $\lambda \in [0,1]$, and $v,w \in [c,d]$, we have

$$\begin{aligned} f_x(\lambda v + (1-\lambda)w) &= f(x, \lambda v + (1-\lambda)w) \\ &= f(\lambda x + (1-\lambda)x, \lambda v + (1-\lambda)w) \\ &\leq f^{\lambda}(x,v) f^{1-\lambda}(x,w) \\ &= f_x^{\lambda}(v) f_x^{1-\lambda}(w), \end{aligned} \tag{2.4}$$

which shows the log-convexity of f_x. The proof that $f_y : [a,b] \to \mathbf{R}_+$, $f_y(u) = f(u,y)$, is also log-convex on $[a,b]$ for all $y \in [c,d]$ follows likewise. Now, consider the mapping $f_0 : [0,1]^2 \to \mathbf{R}_+$ given by $f_0(x,y) = e^{xy}$. It is obvious that f_0 is log-convex on the coordinates but not log-convex on $[0,1]^2$. Indeed, if $(u,0), (0,w) \in [0,1]^2$ and $\lambda \in [0,1]$, we have:

$$\begin{gathered} \log f_0(\lambda(u,0) + (1-\lambda)(0,w)) = \log f_0(\lambda u, (1-\lambda)w) = \lambda(1-\lambda)uw, \\ \lambda \log f_0(u,0) + (1-\lambda) \log f_0(0,w) = 0. \end{gathered} \tag{2.5}$$

Thus, for all $\lambda \in (0,1)$ and $u,w \in (0,1)$, we have

$$\log f_0(\lambda(u,0) + (1-\lambda)(0,w)) > \lambda \log f_0(u,0) + (1-\lambda) \log f_0(0,w) \tag{2.6}$$

which shows that f_0 is not log-convex on $[0,1]^2$. □

In the following, a Jensen-type inequality for coordinated log-convex functions is considered.

Proposition 2.5. *Let f be a positive coordinated log-convex function on the open set $(a,b) \times (c,d)$ and let $x_i \in (a,b)$, $y_j \in (c,d)$. If $\alpha_i, \beta_j > 0$ and $\sum_{i=0}^{n} \alpha_i = 1$, $\sum_{j=0}^{m} \beta_j = 1$, then*

$$\log f\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{m} \beta_j y_j\right) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \log f(x_i, y_j). \tag{2.7}$$

Proof. Let $x_i \in (a,b)$, $\alpha_i > 0$ be such that $\sum_{j=0}^{m} \alpha_i = 1$, and let $y_i \in (c,d)$, $\beta_j > 0$ be such that $\sum_{j=0}^{m} \beta_j = 1$, then we have,

$$f\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j\right) \leq \prod_{i=1}^{n} f^{\alpha_i}\left(x_i, \sum_{j=1}^{m} \beta_j y_j\right) \leq \prod_{i=1}^{n} \prod_{j=1}^{m} f^{\alpha_i \beta_j}(x_i, y_j), \tag{2.8}$$

and, since f is positive,

$$\log f\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j\right) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \log f(x_i, y_j), \tag{2.9}$$

which is as required. □

Remark 2.6. Let $f(x,y) = xy$, then the following inequality holds:

$$\log\left[\left(\sum_{i=1}^{n}\alpha_i x_i\right)\left(\sum_{j=1}^{m}\beta_j y_j\right)\right] \leq \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j \log x_i y_j. \tag{2.10}$$

The above result may be generalized to the integral form as follows.

Proposition 2.7. *Let f be a positive coordinated log-convex function on the $\overset{\circ}{\Delta} := (a,b) \times (c,d)$, and let $x(t) : [r_1, r_2] \to \mathbf{R}$ be integrable with $a < x(t) < b$, and let $y(t) : [s_1, s_2] \to \mathbf{R}$ be integrable with $c < y(t) < d$. If $\alpha : [r_1, r_2] \to \mathbf{R}$ is positive, $\int_{r_1}^{r_2}\alpha(t)dt = 1$, and $(\alpha x)(t)$ is integrable on $[r_1, r_2]$ and $\beta : [s_1, s_2] \to \mathbf{R}$ is positive, $\int_{s_1}^{s_2}\beta(t)dt = 1$, and $(\beta y)(t)$ is integrable on $[s_1, s_2]$, then*

$$\begin{aligned} &\log f\left(\int_{r_1}^{r_2}\alpha(t)x(t)dt, \int_{s_1}^{s_2}\beta(u)y(u)du\right) \\ &\quad\leq \int_{r_1}^{r_2}\int_{s_1}^{s_2}\alpha(t)\beta(u)\log f(x(t), y(u))du\,dt. \end{aligned} \tag{2.11}$$

Proof. Applying Jensen's integral inequality in one variable on the x-coordinate and on the y-coordinate we get the required result. The details are omitted. □

Theorem 2.8. *Let $f : \Delta \to \mathbf{R}_+$ be a positive coordinated log-convex function in Δ, then for all distinct $x_1, x_2, x_3 \in [a,b]$, such that $x_1 < x_2 < x_3$ and distinct $y_1, y_2, y_3 \in [c,d]$ such that $y_1 < y_2 < y_3$, the following inequality holds:*

$$\begin{aligned} &f^{x_2y_2+y_3x_3}(x_1,y_1)\cdot f^{y_1x_2+y_2x_3}(x_1,y_3)\cdot f^{x_1y_2+x_2y_3}(x_3,y_1) \\ &\quad\cdot f^{x_1y_1+y_2x_2}(x_3,y_3)\cdot f^{x_1y_3+x_3y_1}(x_2,y_2) \\ &\geq f^{x_2y_3+y_2x_3}(x_1,y_1)\cdot f^{y_1x_3+x_2y_2}(x_1,y_3)\cdot f^{x_1y_3+x_2y_2}(x_3,y_1) \\ &\quad\cdot f^{x_1y_2+y_1x_2}(x_3,y_3)\cdot f^{x_1y_1+x_3y_3}(x_2,y_2). \end{aligned} \tag{2.12}$$

Proof. Let x_1, x_2, x_3 be distinct points in $[a,b]$ and let y_1, y_2, y_3 be distinct points in $[c,d]$. Setting $\alpha = (x_3 - x_2)/(x_3 - x_1)$, $x_2 = \alpha x_1 + (1-\alpha)x_3$ and let $\beta = (y_3 - y_2)/(y_3 - y_1)$, $y_2 = \beta y_1 + (1-\beta)y_3$, we have

$$\begin{aligned} \log f(x_2,y_2) &= \log f(\alpha x_1 + (1-\alpha)x_3, \beta y_1 + (1-\beta)y_3) \\ &\leq \alpha\beta\log f(x_1,y_1) + \alpha(1-\beta)\log f(x_1,y_3) \\ &\quad + \beta(1-\alpha)\log f(x_3,y_1) + (1-\alpha)(1-\beta)\log f(x_3,y_3) \\ &= \frac{x_3-x_2}{x_3-x_1}\frac{y_3-y_2}{y_3-y_1}\log f(x_1,y_1) + \frac{x_3-x_2}{x_3-x_1}\frac{y_2-y_1}{y_3-y_1}\log f(x_1,y_3) \\ &\quad + \frac{x_2-x_1}{x_3-x_1}\frac{y_3-y_2}{y_3-y_1}\log f(x_3,y_1) + \frac{x_2-x_1}{x_3-x_1}\frac{y_2-y_1}{y_3-y_1}\log f(x_3,y_3), \end{aligned} \tag{2.13}$$

and we can write

$$\log \frac{f^{x_2y_2+y_3x_3}(x_1,y_1)f^{y_1x_2+y_2x_3}(x_1,y_3)f^{x_1y_2+x_2y_3}(x_3,y_1)f^{x_1y_1+y_2x_2}(x_3,y_3)f^{x_1y_3+x_3y_1}(x_2,y_2)}{f^{x_2y_3+y_2x_3}(x_1,y_1)f^{y_1x_3+x_2y_2}(x_1,y_3)f^{x_1y_3+x_2y_2}(x_3,y_1)f^{x_1y_2+y_1x_2}(x_3,y_3)f^{x_1y_1+x_3y_3}(x_2,y_2)} \geq 0. \tag{2.14}$$

From this inequality it is easy to deduce the required result (2.12). □

The subharmonic functions exhibit many properties of convex functions. Next, we give some results for the coordinated convexity and sub(super)harmonic functions.

Proposition 2.9. *Let $f : \Delta \subseteq \mathbf{R}^2 \to \mathbf{R}$ be coordinated convex (concave) on Δ. If f is a twice differentiable on Δ°, then f is sub(super)harmonic on Δ°.*

Proof. Since f is coordinated convex on Δ then the partial mappings $f_y : [a,b] \to \mathbf{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbf{R}$, $f_x(v) = f(x,v)$, are convex for all $y \in [c,d]$ and $x \in [a,b]$. Equivalently, since f is differentiable we can write

$$0 \leq f_x'' = \frac{\partial^2 f}{\partial^2 y} \tag{2.15}$$

for all $y \in (c,d)$, and

$$0 \leq f_y'' = \frac{\partial^2 f}{\partial^2 x} \tag{2.16}$$

for all $x \in (a,b)$, which imply that

$$f_x'' + f_y'' = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} \geq 0 \tag{2.17}$$

which shows that f is subharmonic. If f is coordinated concave on Δ, replace "$\leq$" by "$\geq$" above, we get that f is superharmonic on Δ°. □

We now give two version(s) of the Maximum (Minimum) Principle theorem using convexity on the coordinates.

Theorem 2.10. *Let $f : \Delta \subseteq \mathbf{R}^2 \to \mathbf{R}$ be a coordinated convex (concave) function on Δ. If f is twice differentiable in Δ° and there is a point $a = (a_1, a_2) \in \Delta^\circ$ with $f(a_1,a_2) \geq (\leq) f(x,y)$, for all $(x,y) \in \Delta$ then f is a constant function.*

Proof. By Proposition 2.9, we get that f is sub(super)harmonic. Therefore, by Theorem 1.3 and the maximum principal the required result holds (see [14, page 264]). □

Theorem 2.11. *Let f and g be two twice differentiable functions in Δ°. Assume that f and g are bounded real-valued functions defined on Δ such that f is coordinated convex and g is coordinated concave. If for each point $a = (a_1, a_2) \in \partial_{\infty}\Delta$*

$$\lim_{(x,y)\to(a_1,a_2)} \sup f(x,y) \leq \lim_{(x,y)\to(a_1,a_2)} \inf g(x,y), \tag{2.18}$$

then $f(x,y) < g(x,y)$ for all $(x,y) \in \Delta$ or $f = g$ and f is harmonic.

Proof. By Proposition 2.9, we get that f is subharmonic and g is superharmonic. Therefore, by Theorem 1.4 and using the maximum principal the required result holds, (see [14, page 264]). □

Remark 2.12. The above two results hold for log-convex functions on the coordinates, simply, replacing f by $\log f$, to obtain the results.

3. Some Inequalities and Applications

In the following we develop a Hadamard-type inequality for coordinated log-convex functions.

Corollary 3.1. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbf{R}_+$ is log-convex on the coordinates of Δ, then*

$$\begin{aligned} \log f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \log f(x,y)\,dy\,dx \\ &\leq \log \sqrt[4]{f(a,c) f(a,d) f(b,c) f(b,d)}. \end{aligned} \tag{3.1}$$

For a positive coordinated log-concave function f, the inequalities are reversed.

Proof. In Theorem 1.2, replace f by $\log f$ and we get the required result. □

Lemma 3.2. *For $A, B, C \in \mathbb{R}^+$ with $A, B, C > 1$, the function*

$$\psi(\beta) = C^{\beta} \frac{A^{\beta} B - 1}{\ln(A^{\beta} B)}, \quad 0 \leq \beta \leq 1 \tag{3.2}$$

is convex for all $\beta \in [0,1]$. Moreover,

$$\int_0^1 \psi(\beta)\,d\beta \leq \frac{\psi(0) + \psi(1)}{2}, \tag{3.3}$$

for all $A, B, C > 1$.

Proof. Since ψ is twice differentiable for all $\beta \in (0,1)$ with $A, B, C > 1$, we note that for all $0 < \beta_1 \leq \beta_2 < 1$, $\psi(\beta_1) \leq \psi(\beta_2)$, which shows that ψ is increasing and thus ψ' is nonnegative which is equivalent to saying that ψ' is increasing and hence ψ is convex. Now, using inequality (1.1), we get

$$\int_0^1 C^\beta \frac{A^\beta B-1}{\ln(A^\beta B)}d\beta = \int_0^1 \psi(\beta)d\beta \le \frac{\psi(0)+\psi(1)}{2} = \frac{1}{2}\left[\frac{B-1}{\ln(B)} + C\cdot\frac{AB-1}{\ln(AB)}\right], \tag{3.4}$$

which completes the proof. □

Theorem 3.3. *Suppose that $f : \Delta = [a,b]\times[c,d] \to \mathbf{R}_+$ is log-convex on the coordinates of Δ. Let*

$$A = \frac{f(a,c)}{f(b,c)}\frac{f(b,d)}{f(a,d)}, \qquad B = \frac{f(a,d)}{f(b,d)}, \qquad C = \frac{f(b,c)}{f(b,d)}, \tag{3.5}$$

then the inequalities

$$\begin{aligned} I &= \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dx\,dy \\ &\le f(b,d)\times\begin{cases} 1, & A=B=C=1,\\ \left(\frac{B-1}{\ln B}\right)\left(\frac{C-1}{\ln C}\right), & A=1,\\ H(C), & B=1,\\ H(B), & C=1,\\ \frac{C-1}{\ln C}, & A=B=1,\\ \frac{B-1}{\ln B}, & A=C=1,\\ -\frac{\gamma+\ln(-\ln A)+Ei(1,-\ln A)}{\ln A}, & B=C=1,\\ \frac{1}{2}\left[\frac{B-1}{\ln(B)}+C\cdot\frac{AB-1}{\ln(AB)}\right], & A,B,C>1,\\ \int_0^1 C^\beta\frac{A^\beta B-1}{\ln(A^\beta B)}d\beta, & \textit{otherwise} \end{cases} \end{aligned} \tag{3.6}$$

hold, where γ is the Euler constant,

$$\begin{aligned} H(x) &= \frac{Ei(1,-\ln x)+\ln(\ln x)-Ei(1,-\ln(Ax))-\ln(\ln(Ax))}{\ln A} \\ &\quad + \begin{cases} \frac{2\ln(\ln A)-\ln(-\ln A)}{\ln A}, & \frac{\ln x}{\ln A}<0,\ -\frac{\ln x}{\ln A}<1,\\ 0, & \textit{otherwise}, \end{cases} \\ Ei(x) &= V.P.\int_{-x}^{\infty}\frac{e^{-t}}{t}dt \end{aligned} \tag{3.7}$$

is the exponential integral function. For a coordinated log-concave function f, the inequalities are reversed.

Proof. Since $f : \Delta = [a,b] \times [c,d] \to \mathbf{R}_+$ is log-convex on the coordinates of Δ, then

$$\begin{aligned} &f(\alpha a + (1-\alpha)b, \beta c + (1-\beta)d) \\ &\quad\leq f^{\alpha\beta}(a,c) f^{\beta(1-\alpha)}(b,c) f^{\alpha(1-\beta)}(a,d) f^{(1-\alpha)(1-\beta)}(b,d) \\ &\quad= f^{\alpha\beta}(a,c) f^{\beta}(b,c) f^{-\alpha\beta}(b,c) f^{\alpha}(a,d) f^{-\alpha\beta}(a,d) \\ &\qquad\times f(b,d) f^{-\beta}(b,d) f^{-\alpha}(b,d) f^{\alpha\beta}(b,d) \\ &\quad= \left[\frac{f(a,c)}{f(b,c)} \frac{f(b,d)}{f(a,d)}\right]^{\alpha\beta} \left[\frac{f(a,d)}{f(b,d)}\right]^{\alpha} \left[\frac{f(b,c)}{f(b,d)}\right]^{\beta} f(b,d). \end{aligned} \tag{3.8}$$

Integrating the previous inequality with respect to α and β on $[0,1]^2$, we have,

$$\begin{aligned} &\int_0^1\int_0^1 f(\alpha a + (1-\alpha)b, \beta c + (1-\beta)d)\, d\alpha\, d\beta \\ &\quad\leq f(b,d) \int_0^1\int_0^1 \left[\frac{f(a,c)}{f(b,c)} \frac{f(b,d)}{f(a,d)}\right]^{\alpha\beta} \left[\frac{f(a,d)}{f(b,d)}\right]^{\alpha} \left[\frac{f(b,c)}{f(b,d)}\right]^{\beta} d\alpha\, d\beta. \end{aligned} \tag{3.9}$$

Therefore, by (3.9) and for nonzero, positive A, B, C, we have the following cases.

(1) If $A = B = C = 1$, the result is trivial.

(2) If $A = 1$, then

$$\begin{aligned} &\int_0^1\int_0^1 f(\alpha a + (1-\alpha)b, \beta c + (1-\beta)d)\, d\alpha\, d\beta \\ &\quad\leq f(b,d) \int_0^1\int_0^1 \left[\frac{f(a,d)}{f(b,d)}\right]^{\alpha} \left[\frac{f(b,c)}{f(b,d)}\right]^{\beta} d\alpha\, d\beta \\ &\quad= f(b,d) \left(\int_0^1 B^{\alpha} d\alpha\right)\left(\int_0^1 C^{\beta} d\beta\right) \\ &\quad= f(b,d) \left(\frac{B-1}{\ln B}\right)\left(\frac{C-1}{\ln C}\right). \end{aligned} \tag{3.10}$$

(3) If $B = 1$, then

$$\begin{aligned}
&\int_0^1\int_0^1 f(\alpha a + (1-\alpha)b, \beta c + (1-\beta)d)\,d\alpha\,d\beta \\
&\quad \le f(b,d)\int_0^1\int_0^1 A^{\alpha\beta}C^{\beta}\,d\alpha\,d\beta \\
&\quad = f(b,d)\int_0^1 \frac{A^{\alpha}C-1}{\ln(A^{\alpha}C)}\,d\alpha \\
&\quad = f(b,d)\left[\left(\begin{cases} \dfrac{2\ln(\ln A)-\ln(-\ln A)}{\ln A}, & \dfrac{\ln C}{\ln A}<0,\ -\dfrac{\ln C}{\ln A}<1 \\ 0, & \text{otherwise} \end{cases}\right)\right. \\
&\qquad\qquad \left. + \frac{Ei(1,-\ln C)+\ln(\ln C)-Ei(1,-\ln(AC))-\ln(\ln(AC))}{\ln A}\right].
\end{aligned} \tag{3.11}$$

(4) If $C = 1$, then

$$\begin{aligned}
&\int_0^1\int_0^1 f(\alpha a + (1-\alpha)b, \beta c + (1-\beta)d)\,d\alpha\,d\beta \\
&\quad \le f(b,d)\int_0^1\int_0^1 A^{\alpha\beta}B^{\alpha}\,d\alpha\,d\beta \\
&\quad = f(b,d)\int_0^1 \frac{A^{\beta}B-1}{\ln(A^{\beta}B)}\,d\beta \\
&\quad = f(b,d)\left[\left(\begin{cases} \dfrac{2\ln(\ln A)-\ln(-\ln A)}{\ln A}, & \dfrac{\ln B}{\ln A}<0,\ -\dfrac{\ln B}{\ln A}<1 \\ 0, & \text{otherwise} \end{cases}\right)\right. \\
&\qquad\qquad \left. + \frac{Ei(1,-\ln B)+\ln(\ln C)-Ei(1,-\ln(AB))-\ln(\ln(AB))}{\ln A}\right].
\end{aligned} \tag{3.12}$$

(5) If $A = B = 1$, then

$$\begin{aligned}
&\int_0^1\int_0^1 f(\alpha a + (1-\alpha)b, \beta c + (1-\beta)d)\,d\alpha\,d\beta \\
&\quad \le f(b,d)\int_0^1\int_0^1 A^{\alpha\beta}B^{\alpha}C^{\beta}\,d\alpha\,d\beta = f(b,d)\int_0^1 C^{\beta}\,d\beta = f(b,d)\frac{C-1}{\ln C}.
\end{aligned} \tag{3.13}$$

(6) If $A = C = 1$, then

$$\begin{aligned}
&\int_0^1\int_0^1 f(\alpha a + (1-\alpha)b, \beta c + (1-\beta)d)\,d\alpha\,d\beta \\
&\quad \le f(b,d)\int_0^1\int_0^1 A^{\alpha\beta}B^{\alpha}C^{\beta}\,d\alpha\,d\beta = f(b,d)\int_0^1 B^{\alpha}\,d\alpha = f(b,d)\frac{B-1}{\ln B}.
\end{aligned} \tag{3.14}$$

(7) If $B = C = 1$, then

$$\begin{aligned}
&\int_0^1\int_0^1 f(\alpha a + (1-\alpha)b, \beta c + (1-\beta)d)\,d\alpha\,d\beta \\
&\quad \le f(b,d)\int_0^1\int_0^1 A^{\alpha\beta}B^{\alpha}C^{\beta}\,d\alpha\,d\beta \\
&\quad = f(b,d)\int_0^1\int_0^1 \left(A^{\beta}\right)^{\alpha}\,d\alpha\,d\beta \\
&\quad = f(b,d)\int_0^1 \frac{A^{\alpha}-1}{\ln A^{\alpha}}\,d\alpha \\
&\quad = -f(b,d)\frac{\gamma + \ln(-\ln A) + Ei(1, -\ln A)}{\ln A}.
\end{aligned} \tag{3.15}$$

(8) If $A, B, C > 1$, then

$$f(b,d)\int_0^1\int_0^1 A^{\alpha\beta}B^{\alpha}C^{\beta}\,d\alpha\,d\beta = f(b,d)\int_0^1 C^{\beta}\left[\frac{A^{\beta}B-1}{\ln\left(A^{\beta}B\right)}\right]d\beta. \tag{3.16}$$

Therefore, by Lemma 3.2, we deduce that

$$f(b,d)\int_0^1\int_0^1 A^{\alpha\beta}B^{\alpha}C^{\beta}\,d\alpha\,d\beta \le \frac{f(b,d)}{2}\left[\frac{B-1}{\ln(B)} + C\cdot\frac{AB-1}{\ln(AB)}\right]. \tag{3.17}$$

(9) If $A, B, C \neq 1$, we have

$$f(b,d)\int_0^1\int_0^1 A^{\alpha\beta}B^{\alpha}C^{\beta}\,d\alpha\,d\beta = f(b,d)\int_0^1 C^{\beta}\left[\frac{A^{\beta}B-1}{\ln\left(A^{\beta}B\right)}\right]d\beta, \tag{3.18}$$

which is difficult to evaluate because it depends on the values of A, B, and C. □

Remark 3.4. The integrals in (3), (4), and (7) in the proof of Theorem 2.11 are evaluated using Maple Software.

Corollary 3.5. *In Theorem 3.3, if*

(1) $f(x,y)=f(x)$, *then*

$$\frac{1}{b-a}\int_a^b f(x)dx \le L\big(f(a),f(b)\big), \tag{3.19}$$

and for instance, if $f_1(x)=e^{x^p}$, $p\ge 1$ *we deduce*

$$\frac{1}{b-a}\int_a^b e^{x^p}dx \le L\Big(e^{a^p},e^{b^p}\Big). \tag{3.20}$$

(2) $f(x,y)=f_1(x)f_2(y)$, *then*

$$I \le L\big(f_1(a),f_1(b)\big)L\big(f_2(c),f_2(d)\big), \tag{3.21}$$

and for instance, if $f_1(x,y)=e^{x^p+y^q}$, $p,q\ge 1$, *we deduce*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d e^{x^p+y^q}dx\,dy \le L\Big(e^{a^p},e^{b^p}\Big)L\Big(e^{c^p},e^{d^p}\Big). \tag{3.22}$$

Proof. Follows directly by applying inequality (1.4). □

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