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Research Article

Perturbation Results on Semi-Fredholm Operators and Applications

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We give some results concerning stability in the Fredholm operators and Browder operators set, via the concept of measure of noncompactness. Moreover, we prove some localization results on the essential spectra of bounded operators on Banach space. As application, we describe the essential spectra of weighted shift operators. Finally, we describe the spectra of polynomially compact operators, and we use the obtained results to study the solvability for operator equations in Banach spaces.

1. Introduction

Throughout this paper, X denotes an infinite dimensional complex Banach space. We denote by $\mathcal{L}(X)$ the space of all bounded linear operators on X. The subspace of all compact operators of $\mathcal{L}(X)$ is denoted by $\mathcal{K}(X)$. We write $\mathcal{N}(T) \subseteq X$ for the null space and $\mathcal{R}(T) \subseteq X$ for the range of T. The nullity, $n(T)$, of T is defined as the dimension of $\mathcal{N}(T)$ and the deficiency, $d(T)$, of T is defined as the codimension of $\mathcal{R}(T)$ in X. The set of upper (lower) semi-Fredholm operators are defined, respectively by $\Phi_+(X) = \{T \in \mathcal{L}(X); n(T) < \infty$ and $\mathcal{R}(T)$ is closed in $X\}$and, respectively, $\Phi_-(X) = \{T \in \mathcal{L}(X);\ d(T) < \infty\}$. We use $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ for the set of Fredholm operators in $\mathcal{L}(X)$, and $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ for the set of semi-Fredholm operators in $\mathcal{L}(X)$. If $T \in \Phi_\pm(X)$, then $i(T) := n(T) - d(T)$ is called the index of T. It is well known that the index is a continuous function on the set of semi-Fredholm operators.

Various notions of essential spectrum appear in the applications of spectral theory (see, e.g., [1, 2]). We use $\sigma(T)$ for the spectrum of $T \in \mathcal{L}(X), \sigma_e(T) = \{\lambda \in \mathbb{C};\ \lambda - T \notin \Phi(X)\}$ for Wolf essential spectrum, $\sigma_{\mathrm{ess}}(T) = \mathbb{C} \setminus \{\lambda \in \mathbb{C};\ \lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0\}$ for Schechter essential spectrum, and $\sigma_a(T) = \{\lambda \in \mathbb{C}; \inf_{\|x\|=1} \|(\lambda - T)(x)\| = 0\}$ for approximate point spectrum.

Recall that $a(T)$ (resp., $\delta(T)$), the ascent (resp., the descent) of $T \in \mathcal{L}(X)$, is the smallest nonnegative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ (resp. $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$). If no such n exists, then $a(T) = +\infty$ (resp. $\delta(T) = +\infty$). The sets of upper and lower semi-Browder operators are defined, respectively by $\mathcal{B}_+(X) = \{T \in \mathcal{L}(X);\ \ T \in \Phi_+(X)$ and $a(T) < \infty\}, \mathcal{B}_-(X) = \{T \in \mathcal{L}(X);\ \ T \in \Phi_-(X)$ and $\delta(T) < \infty\}$. The set of Browder operators on X is $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. The corresponding spectrum is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C};\ \ \lambda - T \notin \mathcal{B}(X)\}$.

We are interested in this paper (Section 2) to the study of the stability problem in Fredholm operators set and semi-Fredholm operators set. In the past few years, a lot of work has been done along these lines, [3–5] and others. A well-known fact is that $\Phi_+(X)$ is an open set. An important question is to characterize, for a given $S \in \Phi_+(X)$, the class of bounded operators T on X, such that $S + T$ still belongs to $\Phi_+(X)$. Recall that if $T \in \mathcal{K}(X)$, then $S + T \in \Phi_+(X)$ (see [2, Theorem 16.9]). More generally, this fact holds true also for T a strictly singular operator (see [6, Proposition 2.c.10]). Noncompactness measures provide advanced techniques to obtain current precise results along this line; see for example [7, 8]. By means of the Kuratowski measure, for a given $S \in \Phi_+(X)$, we describe in Theorem 2.2 a class of bounded operators T on X, for which $S+T \in \Phi_+(X)$. We should notice that, in general, the size of the perturbation depends upon S. This key-result permits to prove in Corollary 2.3 some localization results about the essential spectra σ_e and σ_{ess} of bounded operators on X. Next, we investigate the stability in the semi-Browder operators set. In [9], Grabiner proves that $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ are closed under commuting perturbation. In [4], Rakočević extends this result to the perturbation classes associated with the sets of semi-Fredholm operators. In Theorem 2.4, by means of the Kuratowski measure, we characteriz for a given $S \in \mathcal{B}_+(X)$, a class of bounded operators T on X, that commute with S, such that $S + T \in \mathcal{B}_+(X)$. As the corollary of this theorem we obtain the main result of Grabiner. As the application of the obtained results, we describe the essential spectra of weighted shift operators.

In Section 3, we are interested in the study of polynomially compact operators. Consider $\mathcal{P}(\mathcal{K}(X)) := \{T \in \mathcal{L}(X)$ such that $P(T) \in \mathcal{K}(X)$ for some nonzero complex polynomial $P\}$. For $T \in \mathcal{P}(\mathcal{K}(X))$ there exists a unique unitary polynomial $m_T(z)$ of least degree such that $m_T(T)$ is compact. This polynomial will be called the minimal polynomial of T. In this section, we describe $\sigma_e(S - T)$, for $T, S \in \mathcal{L}(X)$ with compact commutator such that $T \in \mathcal{P}(\mathcal{K}(X))$. Next, we show that if there exists an analytic function f in a neighborhood of $\sigma(T)$ such that $f(T)$ is compact, then $T \in \mathcal{P}(\mathcal{K}(X))$. As application, we use the obtained results to investigate the solvability for operator equations in Banach spaces, $S\varphi - T\varphi = \psi$. For $T \in \mathcal{P}(\mathcal{K}(X))$, we give affirmative answer under several sufficient conditions on S. This result extends the analysis started in [10, 11] and generalizes the result obtained, in case $S = \lambda I$, in [12, Theorem 2.2].

2. Some New Properties in Fredholm Theory by Means of the Kuratowski Measure of Noncompactness

In this section, we give some results concerning the classes of Fredholm operators and Browder operators via the concept of measures of noncompactness. General definition can be found in [13]. We write M_X for the family of all nonempty and bounded subset of X. We deal with a specific measure: the Kuratowski measure of noncompactness defined on M_X as follows (see [14]):

$$\gamma(A) = \inf\{\varepsilon > 0 :\ A \ \text{ may be covered by finitely many set of diameter} \le \varepsilon\}. \tag{2.1}$$

For $T \in \mathcal{L}(X)$, we define the two nonnegative quantities (see [15]) associated with T by

$$\alpha(T) = \sup\left\{ \frac{\gamma(T(A))}{\gamma(A)}; A \in M_X,\ \gamma(A) > 0 \right\}, \qquad \beta(T) = \inf\left\{ \frac{\gamma(T(A))}{\gamma(A)}; A \in M_X,\ \gamma(A) > 0 \right\}. \tag{2.2}$$

Let X_1 be an infinite dimensional subspace of X and let J_{X_1} be the natural embedding of X_1 into X. The disc (resp., circle) with center 0 and radius r is denoted by $\mathbb{D}(0,r)$ (resp., $\mathcal{C}(0,r)$). We write $\overline{\mathbb{D}}(0,r)$ for the closure of $\mathbb{D}(0,r)$ and we use $\mathcal{C}[r_1,r_2] := \overline{\mathbb{D}}(0,r_2) \setminus \mathbb{D}(0,r_1)$, for $r_1 \leq r_2$.

We start this section by some fundamental properties satisfied by α and β which will be useful in the remainder of the text. For more detail, we refer to [15].

Proposition 2.1. *Let T, S be in $\mathcal{L}(X)$. Then one has the following.*

(i) $\alpha(tT) = |t|\alpha(T)$ *and* $\beta(tT) = |t|\beta(T)$, *for all* $t \in \mathbb{R}$.

(ii) $|\alpha(T) - \alpha(S)| \leq \alpha(T+S) \leq \alpha(T) + \alpha(S)$.

(iii) $\alpha(T \circ S) \leq \alpha(T)\alpha(S)$ *and* $\beta(T \circ S) \geq \beta(T)\beta(S)$.

(iv) $\beta(T) - \alpha(S) \leq \beta(T+S) \leq \beta(T) + \alpha(S)$.

(v) *If T is an isomorphism, then* $\alpha(T^{-1})\beta(T) = 1$.

(vi) $\beta(T) > 0$ *if and only if* $T \in \Phi_+(X)$.

(vii) $\alpha(T) \leq \|T\|$ *and* $\beta(T) \geq \liminf_{\|x\| \to +\infty} \|T(x)\| / \|x\|$.

In the following theorem we establish a stability property in the upper semi-Fredholm operators set. This result provides, in particular, an extension of Theorem 6.1 in [8].

Theorem 2.2. *Let T, S be two bounded operators on X and let f be an analytic function in a neighborhood Ω of $\sigma(S) \cup \sigma(T)$ not vanishing on a connected component of $\sigma(S) \cup \sigma(T)$.*

(i) *If $\alpha(T) < \beta(S)$, then $T + S \in \Phi_+(X)$ and $i(T+S) = i(S)$.*

Suppose moreover that the commutator $[T,S] \in \mathcal{K}(X)$ and $\alpha(f(T)) < \beta(f(S))$, then one has the following.

(ii) $T - S \in \Phi_+(X)$.

(iii) $f(S) \in \Phi(X)$ *implies that* $T - S \in \Phi(X)$.

(iv) $f(z) = z^n$, *for some* $n \in \mathbb{N}^*$, *implies that* $i(T-S) = i(S)$.

Proof. By Proposition 2.1, we have for all $t \in [0,1]$, $\beta(tT+S) \geq \beta(S) - t\alpha(T) > 0$, then $tT + S \in \Phi_+(X)$, for all $t \in [0,1]$, in particular, $T + S \in \Phi_+(X)$. By the continuity of the index on $\Phi_+(X)$, we get $i(T+S) = i(S)$, and this proves (i).

Now, assume that $\alpha(f(T)) < \beta(f(S))$, applying (i), we get $f(T) - f(S) \in \Phi_+(X)$ and $i(f(T) - f(S)) = i(f(S))$. Let ω be an open set with closure $\overline{\omega} \subset \Omega$ and whose boundary $\partial\omega$ consists of finite number of simple closed curves that do not intersect, and such that $\sigma(S) \cup \sigma(T) \subset \omega$. Then we have

$$f(T) - f(S) = \frac{1}{2i\pi} \int_{\partial\omega} \left((z-T)^{-1} - (z-S)^{-1} \right) f(z)\, dz. \tag{2.3}$$

Since $[S,T] \in \mathcal{K}(X)$, then there exist compact operators K_1 and K_2 such that

$$(z-T)^{-1}-(z-S)^{-1}=(T-S)(z-T)^{-1}(z-S)^{-1}+K_1=(z-T)^{-1}(z-S)^{-1}(T-S)+K_2. \tag{2.4}$$

Integrating along $\partial\omega$, we get

$$f(T)-f(S)=(T-S)L+K_3=L(T-S)+K_4, \tag{2.5}$$

where $L=(1/(2i\pi))\int_{\partial\omega}(z-T)^{-1}(z-S)^{-1}f(z)dz$. It is easily checked that $L \in \mathcal{L}(X)$ and $K_3, K_4 \in \mathcal{K}(X)$. This leads to $T-S \in \Phi_+(X)$. If $f(S) \in \Phi(X)$, then $f(T)-f(S) \in \Phi(X)$. By (2.5), we conclude that $T-S \in \Phi(X)$.

Now, if $f(z)=z^n$, then $\alpha(T^n)<\beta(S^n)$ yields $\alpha((tT)^n)<\beta(S^n)$, $\forall t \in [0,1]$. Therefore, by (ii), $tT-S \in \Phi_+(X)$, for all $t \in [0,1]$. By the continuity of the index function on $\Phi_+(X)$, we get $i(T-S)=i(S)$. □

For $T \in \mathcal{L}(X)$, define $\beta_0(T)$ (resp., $\alpha_0(T)$) to be the limit of the sequence $(\beta(T^n))^{1/n}$ (resp., $(\alpha(T^n))^{1/n}$). For the existence of these limits see [2, Lemma 1.21].

Corollary 2.3. *Let T be a bounded operator on X, the one has the following.*

(i) $\sigma_{\text{ess}}(T) \subset \overline{\mathbb{D}}(0,\alpha_0(T))$.

(ii) *If $T \notin \Phi_-(X)$, then $\overline{\mathbb{D}}(0,\beta_0(T)) \subset \sigma_e(T)$.*

(iii) *If $T \in \Phi_-(X)$, then $\sigma_e(T) \subset C[\beta_0(T),\alpha_0(T)]$.*

(iv) *If $0 \notin \sigma_{\text{ess}}(T)$, then $\sigma_{\text{ess}}(T) \subset C[\beta_0(T),\alpha_0(T)]$.*

(v) *If $0 \in \sigma_{\text{ess}}(T)$, then $\overline{\mathbb{D}}(0,\beta_0(T)) \subset \sigma_{\text{ess}}(T)$.*

Proof. Let $n \in \mathbb{N}^*$ and suppose that $|\lambda|^n > \alpha(T^n)$, then, by Theorem 2.2(iv), we have $\lambda - T \in \Phi(X)$ and $i(\lambda-T)=0$. Hence, if $|\lambda|>\alpha_0(T)$, then $\lambda \notin \sigma_{\text{ess}}(T)$, and this proves (i).

Notice that if $\beta(T)=0$, then $\beta_0(T)=0$ and the results are all trivial. Suppose that $\beta(T)>0$. For $|\lambda|<\beta_0(T)$, there exists $n \in \mathbb{N}^*$ such that $|\lambda|^n<\beta(T^n)$. Then, by Theorem 2.2(iv), we have $\lambda-T \in \Phi_+(X)$ and $i(\lambda-T)=i(T)$. Hence, we get easily (ii)–(v). □

2.1. Stability in the Browder and the Semi-Browder Operators

The following theorem uses the measure of noncompactness to establish stability in the semi-Browder operators set. More precisely, we have the following.

Theorem 2.4. *Suppose that S and T are commuting bounded linear operators on the Banach space X. Assume that $\alpha(T)<\beta(S)$, then*

$$a(S)<\infty \text{ implies that } a(T+S)<\infty. \tag{2.6}$$

Proof. For $t \in [0,1]$, we have $\alpha(tT) < \beta(S)$, and then, by Theorem 2.2(i), $tT + S \in \Phi_+(X)$. Set $\mathcal{N}^\infty(T) = \bigcup_n \mathcal{N}(T^n)$ and $\mathcal{R}^\infty(T) = \bigcap_n \mathcal{R}(T^n)$. Since S and T are commuting, then according to [16, Theorem 3], for all $t \in [0,1]$, there exists $\varepsilon(t) > 0$ such that, for all s in the disk $\mathbb{D}(t, \varepsilon(t))$,

$$\overline{\mathcal{N}^\infty(tT+S)} \cap \mathcal{R}^\infty(tT+S) = \overline{\mathcal{N}^\infty(sT+S)} \cap \mathcal{R}^\infty(sT+S). \tag{2.7}$$

Hence, $\overline{\mathcal{N}^\infty(tT+S)} \cap \mathcal{R}^\infty(tT+S)$ is a locally constant function of t on the interval $[0,1]$. Since every locally constant function on a connected set is constant, then

$$\forall t \in [0,1], \quad \overline{\mathcal{N}^\infty(tT+S)} \cap \mathcal{R}^\infty(tT+S) = \overline{\mathcal{N}^\infty(S)} \cap \mathcal{R}^\infty(S). \tag{2.8}$$

Now, since $a(S) < \infty$, then from [5, Proposition 1.6(i)]

$$\mathcal{N}^\infty(S) \cap \mathcal{R}^\infty(S) = \overline{\mathcal{N}^\infty(S)} \cap \mathcal{R}^\infty(S) = \{0\}. \tag{2.9}$$

Thus, $\mathcal{N}^\infty(T+S) \cap \mathcal{R}^\infty(T+S) = \{0\}$, and again by [5, Proposition 1.6(i)], it follows that $a(T+S) < \infty$. □

Remark 2.5. Theorem 2.4 extends the results of Grabiner [9, Theorem 2]. Indeed, if T is compact, we obtain $0 = \alpha(T) < \beta(S) = \beta(T+S)$. Hence, Theorem 2.4 yields $a(S) < \infty$ if and only if $a(T+S) < \infty$. This proves that $\mathcal{B}_+(X)$ is closed under commuting compact perturbation. By duality argument, we prove the closeness of $\mathcal{B}_-(X)$.

Corollary 2.6. *Let S, T be commuting bounded operators on X. Suppose that there exists $n \in \mathbb{N}^*$ such that $\alpha(T^n) < \beta(S^n)$.*

(i) *If $S \in \mathcal{B}_+(X)$, then $T + S \in \mathcal{B}_+(X)$.*

(ii) *If $S \in \mathcal{B}(X)$, then $T + S \in \mathcal{B}(X)$.*

Proof. (i) Let $t \in [0,1]$. Since $\alpha((tT)^n) < \beta(S^n)$, then from Theorem 2.2, $tT + S \in \Phi(X)$. Arguing as in the proof of Theorem 2.4, we get the result.

(ii) Since $S \in \mathcal{B}(X)$, then $i(S) = 0$. By Theorem 2.2, $i(T+S) = 0$. On the other hand, (i) yields $a(T+S) < \infty$. According to [17, Theorem 4.5(d)], we get $\delta(T+S) < \infty$. □

Corollary 2.7. *Let T be a bounded operator on X, thene one has the following.*

(i) $\sigma_b(T) \subset \mathbb{D}(0, \alpha_0(T))$.

(ii) *If $0 \notin \sigma_b(T)$, then $\sigma_b(T) \subset C[\beta_0(T), \alpha_0(T)]$.*

Proof. (i) For $|\lambda| > \alpha_0(T)$, there exists $n \in \mathbb{N}^*$ such that $|\lambda|^n > \alpha(T^n)$. By Corollary 2.6, we have $\lambda - T \in \mathcal{B}(X)$. The result follows since we can choose n arbitrary large.

(ii) Since $0 \notin \sigma_b(T)$, then $T \in \Phi(X)$ and hence $\beta(T) > 0$. For $|\lambda| < \beta_0(T)$, there exists $n \in \mathbb{N}^*$ such that $|\lambda|^n < \beta(T^n)$. Corollary 2.6 implies that $\lambda - T \in \mathcal{B}(X)$ since $T \in \mathcal{B}(X)$. □

2.2. Application: Weighted Shift Operators

Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be a bounded complex sequence. Consider the unilateral backward weighted shift operator $W(\omega, p)$ defined on $X = l^r(\mathbb{N}, \mathbb{C}), r \geq 1$, by $W(\omega, p)(x_0, x_1, \ldots) =$

$(\omega_p x_p, \omega_{p+1} x_{p+1}, \ldots)$. In [18, Proposition 1.6.15], the authors give a localization results for the spectrum and the approximate point spectrum of unilateral backward weighted shift operator. In this section, we investigate the Wolf essential spectrum of $W(\omega, p)$.

Proposition 2.8. *The following statements hold true.*

(i) $\alpha(W(\omega,p)) \le \omega_+ := \limsup_{n\to+\infty}|\omega_n|$ *and* $\beta(W(\omega,p)) \ge \omega_- := \liminf_{n\to+\infty}|\omega_n|$.

(ii) $\sigma_e(W(\omega,p)) \subset C[\omega_-, \omega_+]$.

Proof. For $\varepsilon > 0$, the set $E = \{n \in \mathbb{N};\ \ |\omega_n| > \varepsilon + \omega_+ \text{ or } |\omega_n| < -\varepsilon + \omega_-\}$ is finite. Consider

$$X_1 = \{(x_n)_n \in X; x_n = 0,\ \forall n \in E\}, \qquad X_2 = \{(x_n)_n \in X;\ \ x_n = 0,\ \forall n \notin E\}. \tag{2.10}$$

We have $X = X_1 \oplus X_2$. Since X_2 is finite dimensional subspace, then

$$\alpha(W(\omega,p)) = \alpha(W(\omega,p)J_{X_1}) \le \left\|W(\omega,p)J_{X_1}\right\| \le \varepsilon + \omega_+. \tag{2.11}$$

Otherwise, by Proposition 2.1,

$$\beta(W(\omega,p)) = \beta(W(\omega,p)J_{X_1}) \ge \liminf_{\|x\|\to+\infty} \frac{\left\|W(\omega,p)J_{X_1}(x)\right\|}{\|x\|} \ge -\varepsilon + \omega_-. \tag{2.12}$$

Since we can choose ε arbitrary small, then we get (i).

We should notice that if 0 is a cluster point for the sequence $(|\omega_n|)_n$, then $\omega_- = 0$ and (ii) follows from Corollary 2.3(i). If not, then $F_0 = \{n \ge p \text{ such that } \omega_n = 0\}$ is a finite set and $W(\omega,p)$ is a Fredholm operator with index p. More precisely, $n(W(\omega,p)) = p + \operatorname{card}(F_0)$ and $d(W(\omega,p)) = \operatorname{card}(F_0)$, here $\operatorname{card}(F_0)$ denotes the cardinal of F_0. Now, by Corollary 2.3(iii), we get $\sigma_e(W(\omega,p)) \subset C[\omega_-, \omega_+]$, which proves the proposition. □

Remark 2.9. Notice that if $(|\omega_n|)_n$ converges to l, then according to Proposition 2.8, we get $\alpha(W(\omega,p)) = \beta(W(\omega,p)) = l$ and $\sigma_e(W(\omega,p)) \subset \mathcal{C}(0,l)$. Since $i(W(\omega,p)) \neq 0$, then by the continuity of the index function on $\Phi(X)$, we obtain $\sigma_e(W(\omega,p)) = \mathcal{C}(0,l)$. This is a well-known fact (see, e.g., [19, Proposition 27.7, page 139]).

In what follows, we investigate more precisely the essential spectrum of $W(\omega,p)$. For this end define $\mathcal{A}^{(0)}(|\omega|)$ to be the limit set of $(|\omega|)_n$, that is, the set of all cluster points of the sequence $(|\omega_n|)_n$, and $\mathcal{A}^{(k+1)}(|\omega|)$ to be the limit set of $\mathcal{A}^{(k)}(|\omega|)$ for $k \ge 0$.

Proposition 2.10. *Suppose that* $\mathcal{A}^{(0)}(|\omega|) = \{0 \le l_1 < \cdots < l_N\}$ *is finite, then*

$$\sigma_e(W(\omega,p)) \subset \bigcup_{1\le i\le N} \mathcal{C}(0,l_i). \tag{2.13}$$

Proof. For $0 < \varepsilon < (1/2)\inf_{i\neq j}|l_i - l_j|$, consider $A_i = \{n \in \mathbb{N}; ||\omega_n| - l_i| > \varepsilon\}, i = 1, \ldots, N,\ A_0 = \bigcap_{i=1}^{N} A_i$ and $X_i = \{(x_n)_n \in X;\ \ x_n = 0,\ \forall n \in A_i\},\ i = 0, \ldots, N$. We can write $X = \oplus_{i=0}^{N} X_i$. For $i = 0, \ldots, N$, define the operator S_i by

$$S_i = W(\omega, p) \quad \text{on } X_i,$$
$$S_i = 0 \quad \text{on } \oplus_{j\neq i} X_j. \tag{2.14}$$

Since, For all $i \neq j,\ S_i \circ S_j = 0$ and $W(\omega, p) = \sum_{i=0}^{n} S_i$, then

$$\prod_{i\in\{0,\ldots,N\}} (\lambda - S_i) = \lambda^{N-1}(\lambda - W(\omega, p)). \tag{2.15}$$

This yields

$$\sigma_e(W(\omega, p)) \setminus \{0\} = \left[\bigcup_{0\leq i\leq N} \sigma_e(S_i)\right] \setminus \{0\}. \tag{2.16}$$

Observe that X_0 is finite dimensional and then S_0 is finite rank. Hence, $\sigma_e(S_0) = \{0\}$. It remains to prove that, For all $i = 1, \ldots, N,\ \sigma_e(S_i) \subset \mathcal{C}(0, l_i)$.Consider the operator $S'_i : X \to X$ defined by

$$S'_i = 0 \quad \text{on } X_i$$
$$S'_i = l_i S \quad \text{on } \oplus_{j\neq i} X_j, \tag{2.17}$$

where $S = W(1, p)$ is the corresponding un-weighted shift operator. We have $S_i + S'_i = W(v, p)$, with $v = (v_n)_n$ being the sequence defined by

$$v_n = \omega_n \quad \text{for} \quad n \notin A_i,$$
$$v_n = l_i \quad \text{for} \quad n \in A_i. \tag{2.18}$$

Observe that $(|v_n|)_n$ converges and $\lim_{n\to+\infty}|v_n| = l_i$, then $\sigma_e(W(v, p)) = \mathcal{C}(0, l_i)$. Since $S_i \circ S'_i = S'_i \circ S_i = 0$, then, as above, $\sigma_e(W(v, p)) \setminus \{0\} = (\sigma_e(S_i) \cup \sigma_e(S'_i)) \setminus \{0\}$. Hence, $\sigma_e(S_i) \subset \mathcal{C}(0, l_i)$, and this completes the proof. □

Now, we prove the following result.

Theorem 2.11. *Suppose that there exists $k \geq 0$ such that $\mathcal{A}^{(k)}(|\omega|)$ is a finite set, then*

$$\sigma_e(W(\omega, p)) \subset \bigcup_{l\in\mathcal{A}^{(0)}(|\omega|)} \mathcal{C}(0, l). \tag{2.19}$$

Proof (by induction). For $k = 0$, the result follows by Proposition 2.10. Let $k \geq 0$ be an integer and suppose that if $\mathcal{A}^{(k)}(|\omega|)$ is a finite set, then (2.19) holds true. Suppose now that

$\mathcal{A}^{(k+1)}(|\omega|) = \{l_1, \ldots, l_N\}$ is a finite set. For $0 < \varepsilon < (1/2)\inf_{i \neq j}|l_i - l_j|$ and $i = 1, \ldots, N$, we consider $B_i = \{n \in \mathbb{N}; ||\omega_n| - l_i| > \varepsilon\}$ and $B_0 = \bigcap_{i=1}^N B_i$. Define the sequence $u^i = (u_n^i)_n$, $0 \leq i \leq N$ by

$$\begin{aligned} u_n^i &= \omega_n, \quad \forall n \notin B_i, \\ u_n^i &= 0, \quad \forall n \in B_i. \end{aligned} \tag{2.20}$$

Since $W(\omega, p) = \sum_{i=0}^N W(u^i, p)$ and $W(u^i, p) \circ W(u^j, p) = 0$, for all $i \neq j$, then

$$\sigma_e(W(\omega, p)) \setminus \{0\} \subset \left[\bigcup_{i=0}^N \sigma_e\left(W\left(u^i, p\right)\right)\right] \setminus \{0\}. \tag{2.21}$$

Observe that $\mathcal{A}^{(k)}(|u^0|)$ is a finite set and $\mathcal{A}^{(0)}(|u^0|) \subset \mathcal{A}^{(0)}(|\omega|) \cup \{0\}$. Hence

$$\sigma_e\left(W\left(u^0, p\right)\right) \subset \bigcup_{l \in \mathcal{A}^{(0)}(|\omega|)} \mathcal{C}(0, l) \cup \{0\}. \tag{2.22}$$

Now, consider the sequence $v^i = (v_n^i)_n$, $i = 1, \ldots, N$ defined by

$$\begin{aligned} v_n^i &= l_i, \quad \forall n \in B_i, \\ v_n^i &= 0, \quad \forall n \notin B_i. \end{aligned} \tag{2.23}$$

Clearly, $\liminf_{n \to +\infty} |u_n^i + v_n^i| \geq l_i - \varepsilon$ and $\limsup_{n \to +\infty} |u_n^i + v_n^i| \leq l_i + \varepsilon$. Hence, by Proposition 2.8, $\sigma_e(W(u_n^i + v_n^i, p)) \subset \mathcal{C}[l_i - \varepsilon, l_i + \varepsilon]$. Since $W(u^i, p) \circ W(v^i, p) = W(v^i, p) \circ W(u^i, p) = 0$ and $W(u^i, p) + W(v^i, p) = W(u^i + v^i, p)$, then

$$\sigma_e\left(W\left(u^i + v^i, p\right)\right) \setminus \{0\} = \left[\sigma_e\left(W\left(u^i, p\right)\right) \cup \sigma_e\left(W\left(v^i, p\right)\right)\right] \setminus \{0\}. \tag{2.24}$$

Hence, we get, for $i = 1, \ldots, N$,

$$\sigma_e\left(W\left(u^i, p\right)\right) \subset \mathcal{C}[l_i - \varepsilon, l_i + \varepsilon]. \tag{2.25}$$

Since we can choose $\varepsilon > 0$ arbitrary small, then by (2.21), (2.22), and (2.25), we get (2.19). $\square$

Finally, consider the superposition of two weighted shift operators $W(\omega, p) + W(u, k)$. Suppose that $\limsup_{n \to +\infty} |u_n| < \liminf_{n \to +\infty} |\omega_n|$, then, by Proposition 2.8, $\alpha(W(u, k)) < \beta(W(\omega, p))$. By Theorem 2.2, $W(\omega, p) + W(u, k) \in \Phi(X)$ and $i(W(\omega, p) + W(u, k)) = i(W(\omega, p))$.

To close this section, we define a special class of bounded operators on a Banach space X, that presents some interesting properties. Set

$$\mathcal{L}_0(X) := \left\{T \in \mathcal{L}(X);\ \alpha(T) = \beta(T)\right\}. \tag{2.26}$$

First, we observe that $\mathcal{K}(X) \subset \mathcal{L}_0(X)$, and $\lambda I \in \mathcal{L}_0(X)$ for all $\lambda \in \mathbb{C}$. Also, we notice that if $\omega = (\omega_n)_n$ is a complex sequence that converges, then the weighted shift operator $W(\omega, p)$ is a nontrivial element of $\mathcal{L}_0(l^r(\mathbb{N}, \mathbb{C}))$. Now, we prove the following result.

Proposition 2.12.

(i) *For all $T_1, T_2 \in \mathcal{L}_0(X)$ one has $T_1T_2 \in \mathcal{L}_0(X)$, and $\alpha(T_1T_2) = \alpha(T_1)\alpha(T_2)$.*

(ii) *If $T \in \mathcal{L}_0(X)$ is invertible, then $T^{-1} \in \mathcal{L}_0(X)$.*

Proof. We observe, by Proposition 2.1, that

$$\alpha(T_1)\alpha(T_2) = \beta(T_1)\beta(T_2) \le \beta(T_1T_2) \le \alpha(T_1T_2) \le \alpha(T_1)\alpha(T_2). \tag{2.27}$$

This proves the statement (i). Again, by Proposition 2.1, for T invertible, we have $\alpha(T^{-1}) = 1/\beta(T) = 1/\alpha(T) = \beta(T^{-1})$. This proves (ii). □

As an immediate result we get, for all T being in $\mathcal{L}_0(X), \alpha_0(T) = \alpha(T) = \beta_0(T) = \beta(T)$. In the following proposition we describe the essential spectra for a given $T \in \mathcal{L}_0(X)$.

Proposition 2.13. *Let T be in $\mathcal{L}_0(X)$ and suppose that $0 \in \sigma_{\text{ess}}(T)$, then one has the following.*

(i) $\sigma_b(T) = \sigma_{\text{ess}}(T) = \overline{\mathbb{D}}(0, \alpha(T))$.

(ii) *If $T \notin \Phi_-(X)$, then $\sigma_e(T) = \overline{\mathbb{D}}(0, \alpha(T))$.*

(iii) *If $T \in \Phi_-(X)$, then $\sigma_e(T) = \mathcal{C}(0, \alpha(T))$.*

Proof. According to Corollary 2.3, we have $\overline{\mathbb{D}}(0, \alpha(T)) \subset \sigma_{\text{ess}}(T)$. By Corollary 2.7, we get $\sigma_b(T) \subset \mathbb{D}(0, \alpha(T))$. Since $\sigma_{\text{ess}}(T) \subset \sigma_b(T)$, then we get (i). The assertion (ii) follows from Corollary 2.3(i)–(ii). For (iii), on one hand, by Corollary 2.3(iii), we have $\sigma_e(T) \subset \mathcal{C}(0, \alpha(T))$, on the other hand, the boundary $\partial\sigma_{\text{ess}}(T) \subset \sigma_e(T)$. □

Notice that if $\omega = (\omega_n)_n$ is a complex sequence that converges to l, then by Proposition 2.13 (i),

$$\sigma_b(W(\omega, p)) = \sigma_{\text{ess}}(W(\omega, p)) = \overline{\mathbb{D}}(0, l). \tag{2.28}$$

3. Fredholm Theory for Polynomially Compact Operators

In this section, we present a spectral analysis for polynomially compact operators. We begin by proving an important result about perturbation by polynomially compact operators in the general context of normed spaces. First, we make the following definition.

Definition 3.1. Let Y be a normed space, let $T \in \mathcal{P}(\mathcal{K}(Y)), m_T$ be the minimal polynomial of T, and let $S \in \mathcal{L}(Y)$. We say that T and S communicate if There exists a continuous map $\varphi : [0,1] \to \mathbb{C};\ \ \varphi(0) = 0$ and $\varphi(1) = 1$, such that, for all λ zero of $m_T,\ \varphi(t)\lambda \in \rho_e(S),$ for all $t \in [0,1]$.

Theorem 3.2. *Let T, S be two bounded operators on a normed space Y with compact commutator. Suppose that $T \in \mathcal{P}(\mathcal{K}(Y))$ and $m_T(\lambda) \neq 0$, for all $\lambda \in \sigma_e(S)$. Then $T - S \in \Phi(Y)$.*

If moreover, T and S communicate, then $i(T - S) = i(S)$.

Proof. Since $m_T(\lambda) \neq 0$ for all $\lambda \in \sigma_e(S)$, then we can write $m_T(S) = \prod_{i=1}^{N}(S - \lambda_i)$, with $\lambda_i \notin \sigma_e(S)$. This yields $m_T(S) \in \Phi(Y)$. On the other hand $m_T(T)$ is compact, then $m_T(S) - m_T(T) \in \Phi(Y)$. Writing $m_T(S) - m_T(T) = (S - T)L + K_1 = L(S - T) + K_2$, with $L \in \mathcal{L}(Y)$ and $K_1, K_2 \in \mathcal{K}(Y)$, we conclude that $S - T \in \Phi(Y)$.

Now, consider $Q_t(z) = \prod_{i=1}^{N}(z - \lambda_i\varphi(t))$, then $Q_t(\varphi(t)T) = (\varphi(t))^N m_T(T)$. Thus, $Q_t(\varphi(t)T)$ is compact and, for all $\lambda \in \sigma_e(S), Q_t(\lambda) \neq 0$. This yields

$$\varphi(t)T - S \in \Phi(Y), \quad \forall t \in [0,1]. \tag{3.1}$$

By the continuity of the index function on $\Phi(Y)$, we get $i(\varphi(t)T - S)$ constant for all $t \in [0,1]$. In particular, $i(T - S) = i(S)$. □

Remark 3.3. Theorem 3.2 is an improvement of [12, Theorem 2.1]. Indeed, if $\sigma_e(S)$ is a discrete set of $\mathbb{C}$, then T and S communicate. In the particular case where $S = \lambda I$, we have $\sigma_e(S) = \{\lambda\}$. Therefore, $(\lambda I - T)$ is a Fredholm operator of index zero.

We notice that if, for some $p \in \mathbb{N}^*$, $m_T(z) = z^p$, then, for all $S \in \Phi(Y)$, S and T communicate. Hence, we obtain the following.

Corollary 3.4. *Let T, S be two bounded operators on a normed space Y with compact commutator. Suppose that $T^p \in \mathcal{K}(Y)$, for some $p \in \mathbb{N}^*$. If $S \in \Phi(Y)$, then $T - S \in \Phi(Y)$ and $i(T - S) = i(S)$.*

Corollary 3.5. *Let T, S be two commuting bounded operators on the Banach space X. Suppose that $T \in \mathcal{P}(\mathcal{K}(X))$, $S \in \mathcal{B}(X)$, and assume that T and S communicate, then $T + S \in \mathcal{B}(X)$.*

Proof. As in the proof of Theorem 3.2, (3.1) we obtain $S - \varphi(t)T \in \Phi(X)$. Arguing as in the proof of Theorem 2.4, we get $a(S - T) < \infty$. Now, by Theorem 3.2, we have $i(T - S) = i(S) = 0$. Therefore, according to [17, Theorem 4.5(d)] we get $\delta(S - T) < \infty$. □

The following proposition is a well-know result, see [12, 20]. Here, we present a simple proof for this fact.

Proposition 3.6. *Let $T \in \mathcal{P}(\mathcal{K}(X))$ and let m_T be the minimal polynomial of T. Then*

$$\sigma_e(T) = \sigma_b(T) = \{\lambda \in \mathbb{C} \text{ such that } m_T(\lambda) = 0\}. \tag{3.2}$$

Proof. Since $m_T(T)$ is compact, then $\sigma_b(m_T(T)) = \{0\}$. By [3, Theorem 1], $\sigma_b(\mathrm{m}_T(T)) = m_T(\sigma_b(T))$. Hence, $\sigma_e(T) \subset \sigma_b(T) \subset \{\lambda \in \mathbb{C};\ \ m_T(\lambda) = 0\}$. Let $\lambda \in \mathbb{C}$ be such that $m_T(\lambda) = 0$, we can write $m_T(T) = (T - \lambda)Q(T) = Q(T)(T - \lambda)$. Since $m_T(T)$ is compact and, by the minimality of m_T, $Q(T)$ is not compact, then $(T - \lambda) \notin \Phi(X)$. Hence, $\{\lambda \in \mathbb{C}; m_T(\lambda) = 0\} \subset \sigma_e(T)$. □

Proposition 3.7. *Let T, S be two bounded operators on X with compact commutator.*

(i) *If $T \in \mathcal{P}(\mathcal{K}(X))$, then $\sigma_e(S - T) \subset \sigma_e(S) - \sigma_e(T)$.*

(ii) *If there exists $p \in \mathbb{N}^*$ such that $T^p \in \mathcal{K}(X)$, then $\sigma_e(S - T) = \sigma_e(S) - \sigma_e(T)$.*

Proof. (i) If $\lambda \in \sigma_e(S - T)$, then $S - T - \lambda \notin \Phi(X)$. On the other hand $T + \lambda I \in \mathcal{P}(\mathcal{K}(X))$, and $[T + \lambda I, S] = [T, S]$ is compact. According to Theorem 3.2, there exists $\lambda_S \in \sigma_e(S)$ such that

$m_{T+\lambda}(\lambda_S) = 0$, where $m_{T+\lambda}(z) = m_T(z-\lambda)$ is the minimal polynomial of $T+\lambda$. Hence $\lambda = \lambda_S - \lambda_T$, where $m_T(\lambda_T) = 0$. Finally, the result follows from Proposition 3.6

(ii) By (i), $\sigma_e(S-T) \subset \sigma_e(S) - \sigma_e(T)$. Since $T^p \in \mathcal{K}(X)$, then $\sigma_e(T) = \{0\}$, and we obtain $\sigma_e(S-T) \subset \sigma_e(S) = \sigma_e(S-T-(-T)) \subset \sigma_e(S-T)$. □

Notice that in general, the converse inclusion in (i) does not hold.

Example 3.8. Consider the unweighted shift operator $S = W(1,p)$. According to Remark 2.9, we have $\sigma_e(S) = \mathcal{C}(0,1)$, the unit circle. Let $\lambda = (\lambda_n)_n$ be a bounded complex sequence and let $K_\lambda : l^r(\mathbb{N},\mathbb{C}) \to l^r(\mathbb{N},\mathbb{C})$ be defined by $K_\lambda((x_n)_n) = (\lambda_n x_n)_n$. Suppose that $\lambda_{i+p} = \lambda_i$, for all $i \geq 0$, then $K_\lambda S = SK_\lambda$. Consider $P(z) = \prod_{i=0}^{p-1}(z-\lambda_i)$, then $P(K_\lambda) = 0 \in \mathcal{K}(X)$. Suppose that $|\lambda_i| \neq 1$, for all $i \in \{0,\ldots,p-1\}$, then, applying Theorem 3.2, we get that $S_\lambda = S - K_\lambda$ is a Fredholm operator. By Proposition 3.7, we get $\sigma_e(S_\lambda) \subset \bigcup_{i=0}^{p-1} \mathcal{C}(-\lambda_i, 1)$.

The index of S_λ depends on the position of λ_i with respect to $\mathcal{C}(0,1)$. If $|\lambda_i| < 1$, for all $i \in \{0,\ldots,p-1\}$, then K_λ and S communicate and by Theorem 3.2, $i(S_\lambda) = i(S) = p$. If we suppose that $\lambda_i = \lambda$, for all $i \in \{0,\ldots,p-1\}$ with $|\lambda| > 1$, then $K_\lambda = \lambda I$, and S_λ is invertible. In this case $i(S_\lambda) = 0 \neq i(S)$. Observe that in this case, K_λ and S do not communicate.

Theorem 3.9. *Let T be a bounded operator on X. Suppose that there exists an analytic function f in a neighborhood of $\sigma(T)$ which does not vanish on a connected component of $\sigma(T)$ such that $f(T) \in \mathcal{K}(X)$, then $T \in \mathcal{P}(\mathcal{K}(X))$.*

Proof. From [3, Theorem 1] , we have $\sigma_b(f(T)) = f(\sigma_b(T))$. Since $f(T) \in \mathcal{K}(X)$, then $\sigma_b(f(T)) = \{0\}$. Hence, $\sigma_b(T) \subset \sigma(T) \cap \{\lambda \in \mathbb{C}; f(\lambda) = 0\}$ and therefore, $\sigma_b(T)$ is a finite set $\{\lambda_1,\ldots,\lambda_n\}$. Write $f(z) = P(z)g(z)$, where $P(z) := \prod_{i=1}^{n}(z-\lambda_i)^{\alpha_i}$ and g is an analytic function with $g(\lambda_i) \neq 0$, for all $i \in \{1,\ldots,n\}$. Since g does not vanish on $\sigma_b(T)$, then $0 \notin \sigma_b(g(T))$. Thus, $g(T) \in \Phi(X)$ and $P(T) \in \mathcal{K}(X)$. □

3.1. Application: Solvability of Operator Equations

In the following theorem, we treat the question of the solvability of operator equations. We will prove, under several sufficient conditions, that if the homogeneous equation $S\varphi - T\varphi = 0$ only has the trivial solution $\varphi = 0$, then for all $\psi \in X$ the nonhomogeneous equation $S\varphi - T\varphi = \psi$ has a unique solution $\varphi \in X$, and this solution depends continuously on ψ.

Theorem 3.10. *Let Y be a normed space and let T, S be two communicating commuting bounded operators on Y. Suppose that $T \in \mathcal{P}(\mathcal{K}(Y))$ and let m_T be the minimal polynomial of T. Assume that $0 \notin \sigma_{\mathrm{ess}}(S) \cup \sigma_a(m_T(S))$.*

If $F := S - T$ is injective, then the inverse operator $F^{-1} : Y \to Y$ exists and is bounded.

Proof. F is injective, then $\mathcal{N}(F) = \{0\}$, thus $n(F) = 0$. Applying Theorem 3.2, we get $i(F) = i(S) = 0$. It follows that $d(T) = 0$ and therefore, the operator F is surjective. Hence, the inverse operator $F^{-1} = (S-T)^{-1} : Y \to Y$ exists. Since Y is not necessary a Banach space, we have to prove that F^{-1} is bounded. Suppose that it is not so, then there exists $(f_n)_n \subset X$ with $\|f_n\| = 1$ and the sequence $\varphi_n = F^{-1}f_n$ satisfies: $\|\varphi_n\| \to +\infty$ as $n \to +\infty$. Set $g_n := f_n/(\|\varphi_n\|)$ and $\psi_n := \varphi_n/\|\varphi_n\|$, $n \in \mathbb{N}$. Then $g_n \to 0$ as $n \to 0$, and $\|\psi_n\| = 1$. Since $F\psi_n = g_n$ and $FS = SF$, then there exists $L \in \mathcal{L}(Y)$ such that

$$m_T(T)\psi_n = m_T(S)\psi_n + L(g_n). \tag{3.3}$$

Since $m_T(T)$ is compact, we can choose a subsequence $(\psi_{n(k)})_k$ such that $(m_T(T)\psi_{n(k)}) \to \psi$ as $k \to +\infty$. Using (3.3), we observe that $m_T(S)\psi_{n(k)} \to \psi$ as $k \to +\infty$. On the one hand, $Fm_T(S)\psi_{n(k)} = m_T(S)F\psi_{n(k)} = m_T(S)(g_n) \to 0$ as $k \to +\infty$. On the other hand, $Fm_T(S)\psi_{n(k)} \to F(\psi)$. Hence, $F(\psi) = 0$ which implies that $\psi = 0$. This is in contradiction with $0 \notin \sigma_a(m_T(S))$. □

Theorem 3.11. *Let $T \in \mathcal{P}(\mathcal{K}(X))$ and $S \in \mathcal{B}(X)$ be communicating, commuting operators on the Banach space X. Suppose that $0 \notin \sigma_a(m_T(S))$, and set $F = S - T$. Then the projection $P : X \to \mathcal{N}(F^{a(F)})$ defined by the decomposition $X = \mathcal{N}(F^{a(F)}) \bigoplus \mathcal{R}(F^{a(F)})$ is compact, and the operator $F - P$ is bijective.*

Proof. First we notice that by Corollary 3.5, $F \in \mathcal{B}(X)$, then $F \in \Phi(X)$ and $a(F) < \infty$. Thus, $F^{a(F)} \in \Phi(X)$, which implies that $\mathcal{N}(F^{a(F)})$ is finite dimensional. Hence, the projection P is continuous and compact. Now, we claim that $F - P$ is bijective. Let $\varphi \in \mathcal{N}(F - P)$. Since $P\varphi \in \mathcal{N}(F^{a(F)})$, then $F^{a(F)+1}(\varphi) = 0$, which implies that $F^{a(F)}(\varphi) = 0$. Thus $P(\varphi) = \varphi$. Since $F(\varphi) = P(\varphi)$, then $F(\varphi) = \varphi$. We get by iteration $F^{a(F)}(\varphi) = \varphi = 0$. On the other hand, from Theorem 3.10 applied to the operator $T + P$, we conclude that $F - P$ is surjective. □

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