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Research Article

Higher-Order Weakly Generalized Adjacent Epiderivatives and Applications to Duality of Set-Valued Optimization

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A new notion of higher-order weakly generalized adjacent epiderivative for a set-valued map is introduced. By virtue of the epiderivative and weak minimality, a higher-order Mond-Weir type dual problem and a higher-order Wolfe type dual problem are introduced for a constrained set-valued optimization problem, respectively. Then, corresponding weak duality, strong duality and converse duality theorems are established.

1. Introduction

In the last several decades, several notions of derivatives of set-valued maps have been proposed and used for the formulation of optimality conditions and duality in set-valued optimization problems. By using a contingent epiderivative of a set-valued map, Jahn and Rauh [1] obtained a unified necessary and sufficient optimality condition. Chen and Jahn [2] introduced a notion of a generalized contingent epiderivative of a set-valued map and obtained a unified necessary and sufficient conditions for a set-valued optimization problem. Lalitha and Arora [3] introduced a notion of a weak Clarke epiderivative and use it to establish optimality criteria for a constrained set-valued optimization problem. On the other hand, various kinds of differentiable type dual problems for set-valued optimization problems, such as Mond-Weir type and Wolfe type dual problems, have been investigated. By virtue of the tangent derivative of a set-valued map introduced in [4], Sach and Craven [5] discussed Wolfe type duality and Mond-Weir type duality problems for a set-valued optimization problem. By virtue of the codifferential of a set-valued map introduced in [6], Sach et al. [7] obtained Mond-Weir type and Wolfe type weak duality and strong duality theorems of set-valued optimization problems. As to other concepts of derivatives (epiderivatives) of set-valued maps and their applications, one can refer to [8–15]. Recently, Second-order derivatives have also been proposed, for example, see [16, 17] and so on.

Since higher-order tangent sets introduced in [4], in general, are not cones and convex sets, there are some difficulties in studying higher-order optimality conditions and duality for general set-valued optimization problems. Until now, there are only a few papers to deal with higher-order optimality conditions and duality of set-valued optimization problems by virtue of the higher-order derivatives or epiderivatives introduced by the higher-order tangent sets. Li et al. [18] studied some properties of higher-order tangent sets and higher-order derivatives introduced in [4], and then obtained higher-order necessary and sufficient optimality conditions for set-valued optimization problems under cone-concavity assumptions. By using these higher-order derivatives, they also discussed a higher-order Mond-Weir duality for a set-valued optimization problem in [19]. Li and Chen [20] introduced higher-order generalized contingent(adjacent) epiderivatives of set-valued maps, and obtained higher-order Fritz John type necessary and sufficient conditions for Henig efficient solutions to a constrained set-valued optimization problem.

Motivated by the work reported in [3, 5, 18–20], we introduce a notion of higher-order weakly generalized adjacent epiderivative for a set-valued map. Then, by virtue of the epiderivative, we discuss a higher-order Mond-Weir type duality problem and a higher-order Wolfe type duality problem to a constrained set-valued optimization problem, respectively.

The rest of the paper is organized as follows. In Section 2, we collect some of the concepts and some of their properties required for the paper. In Section 3, we introduce a generalized higher-order adjacent set of a set and a higher-order weakly generalized adjacent epiderivative of a set-valued map, and study some of their properties. In Sections 4 and 5, we introduce a higher-order Mond-Weir type dual problem and a higher-order Wolfe type dual problem to a constrained set-valued optimization problem and establish corresponding weak duality, strong duality and converse duality theorems, respectively.

2. Preliminaries and Notations

Throughout this paper, let X, Y, and Z be three real normed spaces, where the spaces Y and Z are partially ordered by nontrivial pointed closed convex cones $C \subset Y$ and $D \subset Z$ with $\mathrm{int}C \neq \emptyset$ and $\mathrm{int}D \neq \emptyset$, respectively. We assume that $0_X,\ 0_Y,\ 0_Z$ denote the origins of $X,\ Y,\ Z$, respectively, Y^* denotes the topological dual space of Y and C^* denotes the dual cone of C, defined by $C^* = \{\varphi \in Y^* \mid \varphi(y) \geq 0,\ \text{for all}\ y \in C\}$. Let M be a nonempty set in Y. The cone hull of M is defined by $\mathrm{cone}(M) = \{ty \mid t \geq 0,\ y \in M\}$. Let E be a nonempty subset of X, $F : E \to 2^Y$ and $G : E \to 2^Z$ be two given nonempty set-valued maps. The effective domain, the graph and the epigraph of F are defined respectively by $\mathrm{dom}(F) = \{x \in E \mid F(x) \neq \emptyset\}$, $\mathrm{graph}(F) = \{(x, y) \in X \times Y \mid x \in E,\ y \in F(x)\}$, and $\mathrm{epi}(F) = \{(x, y) \in X \times Y \mid x \in E,\ y \in F(x) + C\}$. The profile map $F_+ : E \to 2^Y$ is defined by $F_+(x) = F(x) + C$, for every $x \in \mathrm{dom}(F)$. Let $y_0 \in Y$, $F(E) = \bigcup_{x\in E} F(x)$ and $(F - y_0)(x) = F(x) - y_0 = \{y - y_0 \mid y \in F(x)\}$.

Definition 2.1. An element $y \in M$ is said to be a minimal point (resp., weakly minimal point) of M if $M \bigcap (y - C) = \{y\}$(resp., $M \bigcap (y - \mathrm{int}C) = \emptyset$). The set of all minimal point (resp., weakly minimal point) of M is denoted by $\mathrm{Min}_C M$ (resp., $W\mathrm{Min}_C M$).

Definition 2.2. Let $F : E \to 2^Y$ be a set-valued map.

(i) F is said to be C-convex on a convex set E, if for any $x_1, x_2 \in E$ and $\lambda \in (0,1)$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + C. \tag{2.1}$$

(ii) F is said to be C-convex like on a nonempty subset E, if for any $x_1, x_2 \in E$ and $\lambda \in (0,1)$, there exists $x_3 \in E$ such that $\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(x_3) + C$.

Remark 2.3. (i) If F is C-convex on a convex set E, then F is C-convex like on E. But the converse does not hold.

(ii) If F is C-convex like on a nonempty subset E, then $F(E) + C$ is convex.

Suppose that m is a positive integer, X is a normed space supplied with a distance d and K is a subset of X. We denote by $d(x, K) = \inf_{y\in K} d(x, y)$ the distance from x to K, where we set $d(x, \emptyset) = +\infty$.

Definition 2.4 (see [4]). Let x belong to a subset K of a normed space X and let $u_1, \dots, u_{m-1}$ be elements of X. We say that the subset

$$\begin{aligned} T_K^{\flat(m)}(x, u_1, \dots, u_{m-1}) &= \liminf_{h\to 0^+} \frac{K - x - hu_1 - \cdots - h^{m-1}u_{m-1}}{h^m} \\ &= \left\{ y \in X \mid \lim_{h\to 0^+} d\left(y, \frac{K - x - hu_1 - \cdots - h^{m-1}u_{m-1}}{h^m} \right) = 0 \right\} \end{aligned} \tag{2.2}$$

is the mth-order adjacent set of K at $(x, u_1, \dots, u_{m-1})$.

From [18, Propositions 3.2], we have the following result.

Proposition 2.5. *If K is convex, $x \in K$, and $u_i \in X$, $i = 1, \dots, m-1$, then $T_K^{\flat(m)}(x, u_1, \dots, u_{m-1})$ is convex.*

3. Higher-Order Weakly Generalized Adjacent Epiderivatives

Definition 3.1. Let x belong to a subset K of X and let $u_1, \dots, u_{m-1}$ be elements of X. The subset

$$\begin{aligned} G - T_K^{\flat(m)}(x, u_1, \dots, u_{m-1}) &= \liminf_{h\to 0^+} \frac{\operatorname{cone}(K - x) - hu_1 - \cdots - h^{m-1}u_{m-1}}{h^m} \\ &= \left\{ y \in X \mid \lim_{h\to 0^+} d\left(y, \frac{\operatorname{cone}(K - x) - hu_1 - \cdots - h^{m-1}u_{m-1}}{h^m} \right) = 0 \right\} \end{aligned} \tag{3.1}$$

is said to be the mth-order generalized adjacent set of K at $(x, u_1, \dots, u_{m-1})$.

Definition 3.2. The mth-order weakly generalized adjacent epiderivative $d_w^{\flat(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in \operatorname{graph}(F)$ with respect to (with respect to) vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} &d_w^{\flat(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\ &\quad = W\mathrm{Min}_C\Big\{ y \in Y : (x, y) \in G - T_{\mathrm{epi}(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \Big\}. \end{aligned} \tag{3.2}$$

Definition 3.3 (see [3, 21]). The weak domination property (resp., domination property) is said to hold for a subset H of Y if $H \subset W\mathrm{Min}_C H + \mathrm{int}C \cup \{0_Y\}$(resp., $H \subset \mathrm{Min}_C H + C$).

To compare our derivative with well-known derivatives, we recall some notions.

Definition 3.4 (see [4]). The mth-order adjacent derivative $D^{\flat(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in \operatorname{graph}(F)$ with respect to vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} &\operatorname{graph}\Big(D^{\flat(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\Big) \\ &\quad = T_{\operatorname{graph}(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}). \end{aligned} \tag{3.3}$$

Definition 3.5 (see [19]). The C-directed mth-order adjacent derivative $D_C^{\flat(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in \operatorname{graph}(F)$ with respect to vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the mth-order adjacent derivative of set-valued mapping F_+ at (x_0, y_0) with respect to $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$.

Definition 3.6 (See [20]). The mth-order generalized adjacent epiderivative $D_g^{\flat(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in \operatorname{graph}(F)$ with respect to vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} &D_g^{\flat(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\ &\quad = \mathrm{Min}_C\Big\{ y \in Y : (x, y) \in T_{\mathrm{epi}(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \Big\}, \\ &\quad\ x \in \operatorname{dom}\Big[D^{\flat(m)}F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\Big]. \end{aligned} \tag{3.4}$$

Using properties of higher-order adjacent sets [4], we have the following result.

Proposition 3.7. *Let $(x_0, y_0) \in graph(F)$. If $d_w^{\flat(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) \neq \emptyset$ and the set $\{y \in Y \mid (x - x_0, y) \in G\text{-}T_{epi(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\}$ fulfills the weak domination property for all $x \in E$, then for any $x \in E$,*

(i)

$$D^{\flat(m)}F(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})(x-x_0) \subseteq d_w^{\flat(m)}F(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})(x-x_0)+C, \tag{3.5}$$

(ii)

$$D_C^{\flat(m)}F(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})(x-x_0) \subseteq d_w^{\flat(m)}F(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})(x-x_0)+C, \tag{3.6}$$

(iii)

$$D_g^{\flat(m)}F(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})(x-x_0) \subseteq d_w^{\flat(m)}F(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})(x-x_0)+C. \tag{3.7}$$

Remark 3.8. The reverse inclusions in Proposition 3.7 may not hold. The following examples explain the case, where we only take $m=2$.

Example 3.9. Let $X=Y=R$, $E=X$, $C=R_+$, $F(x)=\{y\in R : y\geq x^{4/3}\}$, for all $x\in E$, $(x_0,y_0)=(0,0)$ and $(u,v)=(1,0)$. Then for any $x\in E$, $T^{\flat(2)}_{\mathrm{graph}(F)}(x_0,y_0,u,v)(x-x_0)=T^{\flat(2)}_{\mathrm{epi}(F)}(x_0,y_0,u,v)(x-x_0)=\emptyset$ and $G\text{-}T^{\flat(2)}_{\mathrm{epi}(F)}(x_0,y_0,u,v)(x-x_0)=\{y\mid y\geq 0\}$. Therefore, for any $x\in E$, $D^{\flat(2)}F(x_0,y_0,u,v)(x-x_0)$, $D_C^{\flat(2)}F(x_0,y_0,u,v)(x-x_0)$ and $D_g^{\flat(2)}F(x_0,y_0,u,v)(x-x_0)$ do not exist, but

$$d_w^{\flat(2)}F(x_0,y_0,u,v)(x-x_0)=\{0\}. \tag{3.8}$$

Example 3.10. Let $X=R$, $Y=R^2$, $E=X$, $C=R^2_+$, $F(x)=\{(y_1,y_2)\in R^2\mid y_1\geq x^{4/3},\ y_2\in R\}$, for all $x\in E$, $(x_0,y_0)=(0,(0,0))\in\mathrm{graph}(F)$ and $(u,v)=(1,(0,0))$. Then, $T^{\flat(2)}_{\mathrm{graph}(F)}(x_0,y_0,u,v)=T^{\flat(2)}_{\mathrm{epi}(F)}(x_0,y_0,u,v)=\emptyset$, $G\text{-}T^{\flat(2)}_{\mathrm{epi}(F)}(x_0,y_0,u,v)=R\times(R_+\times R)$. Hence, for any $x\in E$, $D^{\flat(2)}F(x_0,y_0,u,v)(x-x_0)$, $D_C^{\flat(2)}F(x_0,y_0,u,v)(x-x_0)$ and $D_g^{\flat(2)}F(x_0,y_0,u,v)(x-x_0)$ do not exist. But

$$d_w^{\flat(2)}F(x_0,y_0,u,v)(x-x_0)=\left\{(y_1,y_2)\in R^2\mid y_1=0,\ y_2\in R\right\}. \tag{3.9}$$

Example 3.11. Suppose that $X=R$, $Y=R^2$, $E=X$, $C=R^2_+$. Let $F:E\to 2^{R^2}$ be a set-valued map with $F(x)=\{(y_1,y_2)\in R^2\mid y_1\geq x^6,\ y_2\geq x^2\}$, $(x_0,y_0)=(0,(0,0))\in\mathrm{graph}(F)$

and $(u,v) = (1,(0,0))$. Then $T^{b(2)}_{\mathrm{graph}(F)}(x_0,y_0,u,v) = T^{b(2)}_{\mathrm{epi}(F)}(x_0,y_0,u,v) = R \times (R_+ \times [1,+\infty))$, $G\text{-}T^{b(2)}_{\mathrm{epi}(F)}(x_0,y_0,u,v) = R \times (R_+ \times R_+)$. Therefore for any $x \in E$,

$$\begin{aligned} D^{b(2)}F(x_0,y_0,u,v)(x-x_0) = D^{b(2)}_C F(x_0,y_0,u,v)(x-x_0) = R_+ \times [1,+\infty), \\ D^{b(2)}_g F(x_0,y_0,u,v)(x-x_0) = \{(0,1)\}, \\ d^{b(2)}_w F(x_0,y_0,u,v)(x-x_0) = \{(y_1,0) \mid y_1 \geq 0\} \bigcup \{(0,y_2) \mid y_2 \geq 0\}. \end{aligned} \tag{3.10}$$

Now we discuss some crucial propositions of the mth-order weakly generalized adjacent epiderivative.

Proposition 3.12. *Let $x, x_0 \in E$, $y_0 \in F(x_0)$, $(u_i,v_i) \in \{0_X\} \times C$. If the set $P(x-x_0) := \{y \in Y \mid (x-x_0,y) \in G\text{-}T^{b(m)}_{epi(F)}(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})\}$ fulfills the weak domination property for all $x \in E$, then for all $x \in E$,*

$$F(x) - y_0 \subset d^{b(m)}_w F(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})(x-x_0) + C. \tag{3.11}$$

Proof. Take any $x \in E$, $y \in F(x)$ and an arbitrary sequence $\{h_n\}$ with $h_n \to 0^+$. Since $y_0 \in F(x_0)$,

$$h^m_n(x-x_0,\ y-y_0) \in \mathrm{cone}(\mathrm{epi}(F) - (x_0,y_0)). \tag{3.12}$$

It follows from $(u_i,v_i) \in \{0_X\} \times C$, $i = 1,2,\ldots,m-1$, and C is a convex cone that

$$\begin{aligned} &h_n(u_1,v_1) + \cdots + h^{m-1}_n(u_{m-1},v_{m-1}) \in \{0_X\} \times C, \\ &(x_n,y_n) := h_n(u_1,v_1) + \cdots + h^{m-1}_n(u_{m-1},v_{m-1}) \\ &\quad + h^m_n(x-x_0,y-y_0) \in \mathrm{cone}(\mathrm{epi}(F) - (x_0,y_0)). \end{aligned} \tag{3.13}$$

We get

$$(x-x_0,y-y_0) = \frac{(x_n,y_n) - h_n(u_1,v_1) - \cdots - h^{m-1}_n(u_{m-1},v_{m-1})}{h^m_n}, \tag{3.14}$$

which implies that

$$(x-x_0,y-y_0) \in G\text{-}T^{b(m)}_{\mathrm{epi}(F)}(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1}), \tag{3.15}$$

that is, $y - y_0 \in P(x - x_0)$. By the definition of mth-order weakly generalized adjacent epiderivative and the weak domination property, we have

$$P(x - x_0) \subset d_w^{b(m)}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x - x_0) + C. \tag{3.16}$$

Thus $F(x) - y_0 \subset d_w^{b(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x - x_0) + C$. □

Remark 3.13. Since the cone-convexity and cone-concavity assumptions are omitted, Proposition 3.12 improves [18, Theorem 4.1] and [20, Proposition 3.1].

Proposition 3.14. *Let E be a nonempty convex subset of X, $x, x_0 \in E$, $y_0 \in F(x_0)$. Let $F - y_0$ be C-convex like on E, $u_i \in E$, $v_i \in F(u_i) + C$, $i = 1, 2, \ldots, m-1$. If the set $q(x - x_0) := \{y \in Y \mid (x - x_0, y) \in G\text{-}T_{epi(F)}^{b(m)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ fulfills the weak domination property for all $x \in E$, then*

$$F(x) - y_0 \subset d_w^{b(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C. \tag{3.17}$$

Proof. Take any $x \in E$, $y \in F(x)$ and an arbitrary sequence $\{h_n\}$ with $h_n \to 0^+$. Since E is convex and $F - y_0$ be C-convex like on E, we get that $\text{epi}(F) - (x_0, y_0)$ is a convex subset and $\text{cone}(\text{epi}(F) - (x_0, y_0))$ is a convex cone. Therefore

$$\begin{aligned} &h_n(u_1 - x_0, v_1 - y_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0) \\ &\quad = \left(h_n + \cdots + h_n^{m-1}\right)\left(\frac{h_n u_1 + \cdots + h_n^{m-1} u_{m-1}}{h_n + \cdots + h_n^{m-1}} - x_0, \frac{h_n v_1 + \cdots + h_n^{m-1} v_{m-1}}{h_n + \cdots + h_n^{m-1}} - y_0\right) \\ &\quad \in \text{cone}\left(\text{epi}F - (x_0, y_0)\right). \end{aligned} \tag{3.18}$$

It follows from $h_n > 0$, E is convex and $\text{cone}(\text{epi}F - (x_0, y_0))$ is a convex cone that

$$\begin{aligned} (x_n, y_n) := h_n(u_1 - x_0, v_1 - y_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0) \\ + h_n^m(x - x_0, y - y_0) \in \text{cone}\left(\text{epi}F - (x_0, y_0)\right). \end{aligned} \tag{3.19}$$

We obtain that

$$\begin{aligned} &(x - x_0, y - y_0) \\ &\quad = \frac{(x_n, y_n) - h_n(u_1 - x_0, v_1 - y_0) - \cdots - h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0)}{h_n^m}, \end{aligned} \tag{3.20}$$

which implies that

$$(x - x_0, y - y_0) \in G\text{-}T_{\text{epi}(F)}^{b(m)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0), \tag{3.21}$$

that is, $y - y_0 \in q(x - x_0)$. By the definition of mth-order weakly generalized adjacent epiderivative and the weak domination property, we have

$$q(x - x_0) \subset d_w^{b(m)}\big(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0\big)(x - x_0) + C. \tag{3.22}$$

Thus $F(x) - y_0 \subset d_w^{b(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C$, and the proof is complete. □

Remark 3.15. Since the cone-convexity assumptions are replaced by cone-convex likeness assumptions, Proposition 3.14 improves [20, Proposition 3.1].

4. Higher-Order Mond-Weir Type Duality

In this section, we introduce a higher-order Mond-Weir type dual problem for a constrained set-valued optimization problem by virtue of the higher-order weakly generalized adjacent epiderivative and discuss its weak duality, strong duality and converse duality properties. The notation $(F, G)(x)$ is used to denote $F(x) \times G(x)$. Firstly, we recall the definition of interior tangent cone of a set and state a result regarding it from [16].

The interior tangent cone of K at x_0 is defined as

$$IT_K(x_0) = \big\{u \in X \mid \exists \lambda > 0,\ \forall t \in (0, \lambda),\ \forall u' \in B_X(u, \lambda),\ x_0 + tu' \in K\big\}, \tag{4.1}$$

where $B_X(u, \lambda)$ stands for the closed ball centered at $u \in X$ and of radius λ.

Lemma 4.1 (see [16]). *If $K \subset X$ is convex, $x_0 \in K$ and* $\operatorname{int} K \neq \emptyset$, *then*

$$IT_{\operatorname{int} K}(x_0) = \mathit{intcone}(K - x_0). \tag{4.2}$$

Consider the following set-valued optimization problem:

$$(SP)\begin{cases} \min & F(x), \\ \text{s.t.} & G(x) \bigcap (-D) \neq \emptyset, \quad x \in E. \end{cases} \tag{4.3}$$

Set $K := \{x \in E \mid G(x) \bigcap (-D) \neq \emptyset\}$. A point $(x_0, y_0) \in X \times Y$ is said to be a feasible solution of (SP) if $x_0 \in K$ and $y_0 \in F(x_0)$.

Definition 4.2. A point (x_0, y_0) is said to be a weakly minimal solution of (SP) if $(x_0, y_0) \in K \times F(K)$ satisfying $y_0 \in F(x_0)$ and $(F(K) - y_0) \bigcap (-\operatorname{int} C) = \emptyset$.

Suppose that $(u_i, v_i, w_i) \in X \times Y \times Z$, $i = 1, 2, \ldots, m-1$, $(x_0, y_0) \in \mathrm{graph}(F)$, $z_0 \in G(x_0) \bigcap (-D)$, and $\Omega = \mathrm{dom}[d_w^{b(m)}(F,G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)]$. We introduce a higher-order Mond-Weir type dual problem(DSP) of (SP) as follows:

$$\begin{aligned} \max \quad & y_0 \\ \text{s.t.} \quad & \phi(y) + \psi(z) \geq 0, \\ & (y,z) \in d_w^{b(m)}(F,G)\big(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0\big)(x), \quad x \in \Omega, \end{aligned} \tag{4.4}$$

$$\psi(z_0) \geq 0, \tag{4.5}$$

$$\phi \in C^* \setminus \{0_{Y^*}\}, \tag{4.6}$$

$$\psi \in D^*. \tag{4.7}$$

Let $H = \{y_0 \in F(x_0) \mid (x_0, y_0, z_0, \phi, \psi)$ satisfy conditions (4.4)–(4.7)$\}$. A point $(x_0, y_0, z_0, \phi, \psi)$ satisfying (4.4)–(4.7) is called a feasible solution of (DSP). A feasible solution $(x_0, y_0, z_0, \phi, \psi)$ is called a weakly maximal solution of (DSP) if $(H - y_0) \bigcap \mathrm{int}C = \emptyset$.

Theorem 4.3 (weak duality). *Let $(x_0, y_0) \in graph(F), z_0 \in G(x_0) \bigcap (-D)$ and $(u_i, v_i, w_i + z_0) \in \{0_X\} \times C \times D$, $i = 1, 2, \ldots, m-1$. Let the set $\{(y,z) \in Y \times Z \mid (x,y,z) \in G\text{-}T_{epi(F,G)}^{b(m)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)$ fulfill the weak domination property for all $x \in \Omega$. If $(\overline{x}, \overline{y})$ is a feasible solution of (SP) and $(x_0, y_0, z_0, \phi, \psi)$ is a feasible solution of (DSP), then*

$$\phi(\overline{y}) \geq \phi(y_0). \tag{4.8}$$

Proof. It follows from Proposition 3.12 that

$$\begin{aligned} &(F,G)(\overline{x}) - (y_0, z_0) \\ &\quad \subset d_w^{b(m)}(F,G)\big(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0\big)(\overline{x} - x_0) + C \times D. \end{aligned} \tag{4.9}$$

Since $(\overline{x}, \overline{y})$ is a feasible solution of (SP), $G(\overline{x}) \bigcap (-D) \neq \emptyset$. Take $\overline{z} \in G(\overline{x}) \bigcap (-D)$. Then, it follows from (4.5) and (4.7) that

$$\psi(\overline{z} - z_0) \leq 0. \tag{4.10}$$

By (4.4), (4.6), (4.7), (4.9) and (4.10), we get

$$\phi(\overline{y}) \geq \phi(y_0). \tag{4.11}$$

Thus, the proof is complete. □

Remark 4.4. In Theorem 4.3, cone-convexity assumptions of [19, Theorem 4.1] are omitted.

By the similiar proof method of Theorem 4.3, it follows from Proposition 3.14 that the following theorem holds.

Theorem 4.5 (weak duality). *Let* $(x_0, y_0) \in graph(F)$, $z_0 \in G(x_0) \bigcap (-D)$ *and* $(u_i, v_i, w_i + z_0) \in epi(F, G) - (x_0, y_0, z_0)$, $i = 1, 2, \ldots, m-1$. *Suppose that* (F, G) *is* $C \times D$*-convex like on a nonempty convext subset E. Let the set* $\{(y, z) \in Y \times Z \mid (x, y, z) \in G\text{-}T^{b(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ *fulfill the weak domination property for all* $x \in \Omega$. *If* $(\overline{x}, \overline{y})$ *is a feasible solution of* (SP) *and* $(x_0, y_0, z_0, \phi, \psi)$ *is a feasible solution of* (DSP), *then*

$$\phi(\overline{y}) \geq \phi(y_0). \tag{4.12}$$

Lemma 4.6. *Let* $(x_0, y_0) \in graph(F)$, $z_0 \in G(x_0) \bigcap (-D)$, $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, 2, \ldots, m-1$. *Let the set* $P(x) := \{(y, z) \in Y \times Z \mid (x, y, z) \in G\text{-}T^{b(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ *fulfill the weak domination property for all* $x \in \Omega$. *If* (x_0, y_0) *is a weakly minimal solution of* (SP), *then*

$$\begin{aligned} &\Big[d^{b(m)}_{w}(F, G)\big(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0\big)(x) \\ &\quad + C \times D + (0_Y, z_0)\Big] \bigcap (-\text{int}(C \times D)) = \emptyset, \end{aligned} \tag{4.13}$$

for all $x \in \Omega$.

Proof. Since (x_0, y_0) is a weakly minimal solution of (SP), $(F(K) - y_0) \bigcap -\text{int}C = \emptyset$. Then,

$$\text{cone}\big(F(K) + C - y_0\big) \bigcap -\text{int}C = \emptyset. \tag{4.14}$$

Assume that the result (4.13) does not hold. Then there exist $\overline{c} \in C$, $\overline{d} \in D$ and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Z$ with $\overline{x} \in \Omega$ such that

$$(\overline{y}, \overline{z}) \in d^{b(m)}_{w}(F, G)\big(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0\big)(\overline{x}), \tag{4.15}$$

$$(\overline{y}, \overline{z}) + \left(\overline{c}, \overline{d}\right) + (0_Y, z_0) \in -\text{int}(C \times D). \tag{4.16}$$

It follows from (4.15) and the definition of mth-order weakly generalized adjacent epiderivative that

$$(\overline{x}, \overline{y}, \overline{z}) \in G\text{-}T^{b(m)}_{\text{epi}(F,G)}(F, G)\big(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0\big). \tag{4.17}$$

Thus, for an arbitrary sequence $\{h_n\}$ with $h_n \to 0^+$, there exists a sequence $\{(x_n, y_n, z_n)\} \subseteq \text{cone}(\text{epi}(F, G) - (x_0, y_0, z_0))$ such that

$$\frac{(x_n, y_n, z_n) - h_n(u_1, v_1, w_1 + z_0) - \cdots - h_n^{m-1}(u_{m-1}, v_{m-1}, w_{m-1} + z_0)}{h_n^m} \longrightarrow (\overline{x}, \overline{y}, \overline{z}). \tag{4.18}$$

From (4.16) and (4.18), there exists a sufficiently large N_1 such that

$$y_n - h_n v_1 - \cdots - h_n^{m-1} v_{m-1} + h_n^m \bar{c} \in -\mathrm{int}C, \quad \text{for } n > N_1, \tag{4.19}$$

$$\begin{aligned}\tilde{z}_n &:= \frac{z_n - h_n(w_1 + z_0) - \cdots - h_n^{m-1}(w_{m-1} + z_0)}{h_n^m} \\ &= \frac{h_n + \cdots + h_n^{m-1}}{h_n^m}\left(\frac{z_n - h_n w_1 - \cdots - h_n^{m-1} w_{m-1}}{h_n + \cdots + h_n^{m-1}} - z_0\right) \longrightarrow \bar{z} \\ &\in -\left(\mathrm{int}D + z_0 + \bar{d}\right) \subset -\mathrm{intcone}(D + z_0).\end{aligned} \tag{4.20}$$

Since $v_1, \ldots, v_{m-1}$, $-\bar{c} \in -C$, $h_n > 0$ and C is a convex cone,

$$h_n v_1 + \cdots + h_n^{m-1} v_{m-1} - h_n^m \bar{c} \in -C. \tag{4.21}$$

It follows from (4.19) and (4.21) that

$$y_n \in -\mathrm{int}C, \quad \text{for } n > N_1. \tag{4.22}$$

By (4.20) and Lemma 4.1, we have $-\bar{z} \in IT_{\mathrm{int}D}(-z_0)$. Then, it follows from the definition of $IT_{\mathrm{int}D}(-z_0)$ that $\exists \lambda > 0$, for all $t \in (0, \lambda)$, for all $u' \in B_X(-\bar{z}, \lambda)$, $-z_0 + tu' \in \mathrm{int}D$. Since $h_n \to 0^+$, there exists a sufficiently large N_2 such that

$$\frac{h_n^m}{h_n + \cdots + h_n^{m-1}} \in (0, \lambda), \quad \text{for } n > N_2. \tag{4.23}$$

Then, from (4.20), we have

$$\frac{z_n - h_n w_1 - \cdots - h_n^{m-1} w_{m-1}}{h_n + \cdots + h_n^{m-1}} \in -\mathrm{int}D, \quad \text{for } n > N_2. \tag{4.24}$$

It follows from $h_n > 0, w_1, \ldots, w_{m-1}, \in -D$, and D is a convex cone that

$$z_n \in -\mathrm{int}D, \quad \text{for } n > N_2. \tag{4.25}$$

Since $z_n \in \mathrm{cone}(G(x_n) + D - z_0)$, there exist $\lambda_n \geq 0, \bar{z}_n \in G(x_n), d_n \in D$ such that $z_n = \lambda_n(\bar{z}_n + d_n - z_0)$. It follows from (4.25) that $\bar{z}_n \in G(x_n) \bigcap (-D)$, for $n > N_2$, and then

$$x_n \in K, \quad \text{for any } n > N_2. \tag{4.26}$$

It follows from (4.22) that

$$y_n \in \operatorname{cone}\left(F(K)+C-y_0\right) \bigcap -\operatorname{int}C, \quad \text{for } n > \max\{N_1, N_2\}, \tag{4.27}$$

which contradicts (4.14). Thus (4.13) holds and the proof is complete. □

Theorem 4.7 (strong duality). *Suppose that* $(x_0, y_0) \in graph(F)$, $z_0 \in G(x_0) \bigcap (-D)$ *and the following conditions are satisfied:*

(i) $(u_i, v_i, w_i + z_0) \in epi(F,G) - (x_0, y_0, z_0)$, $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, 2, \ldots, m-1$;

(ii) (F,G) *is* (C,D)*-convex like on a nonempty convex subset* E;

(iii) (x_0, y_0) *is a weakly minimal solution of* (SP);

(iv) $P(x) := \{(y,z) \in Y \times Z \mid (x,y,z) \in G\text{-}T^{\flat(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ *fulfills the weak domination property for all* $x \in E$ *and* $(0_Y, 0_Z) \in P(0_X)$;

(v) *There exists an* $x' \in E$ *such that* $G(x') \bigcap (-\operatorname{int}D) \neq \emptyset$.

Then there exist $\phi \in (C^* \setminus \{0_{Y^*}\})$ *and* $\psi \in D^*$ *such that* $(x_0, y_0, z_0, \phi, \psi)$ *is a weakly maximal solution of* (DSP).

Proof. Define

$$M = \bigcup_{x \in \Omega} d_w^{\flat(m)}(F,G)\left(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1}\right)(x) + C \times D + (0_Y, z_0). \tag{4.28}$$

By the similar proof method for the convexity of M in [20, Theorem 5.1], just replacing mth-order generalized adjacent epiderivative by mth-order weakly generalized adjacent epiderivative, we have that M is a convex set. It follows from Lemma 4.6 that

$$M \bigcap (-\operatorname{int}(C \times D)) = \emptyset. \tag{4.29}$$

By the separation theorem of convex sets, there exist $\phi \in Y^*$ and $\psi \in Z^*$, not both zero functionals, such that

$$\phi(\overline{y}) + \psi(\overline{z}) \geq \phi(y) + \psi(z), \quad \forall (\overline{y}, \overline{z}) \in M, \ (y,z) \in -\operatorname{int}(C \times D). \tag{4.30}$$

It follows from (4.30) that

$$\phi(y) \leq \psi(z), \quad \forall (y,z) \in (-\operatorname{int}C) \times \operatorname{int}D, \tag{4.31}$$

$$\phi(\overline{y}) + \psi(\overline{z}) \geq 0, \quad \forall (\overline{y}, \overline{z}) \in M. \tag{4.32}$$

From (4.31), we obtain that ψ is bounded below on the $\operatorname{int}D$. Then, $\psi(z) \geq 0$, for all $z \in \operatorname{int}D$. Naturally, $\psi \in D^*$. By the similar proof method for $\psi \in D^*$, we get $\phi \in C^*$.

Now we show that $\phi \neq 0_{Y^*}$. Suppose that $\phi = 0_{Y^*}$. Then $\psi \in D^* \setminus \{0_{Z^*}\}$. By Proposition 3.14 and condition (v), there exists a point $(y', z') \in (F, G)(x')$ such that $z' \in -\mathrm{int}D$ and

$$(y', z') - (y_0, z_0) \in d_w^{b(m)}(F, G)\left(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0\right)\left(x' - x_0\right) + C \times D. \tag{4.33}$$

Thus it follows from (4.32) that $\psi(z') \geq 0$. Since $z' \in -\mathrm{int}D$ and $\psi \in D^* \setminus \{0_{Z^*}\}$, we have $\psi(z') < 0$, which leads to a contradiction. So $\phi \neq 0_{Y^*}$.

From (4.32) and assumption (iv), we have $\psi(z_0) \geq 0$. Since $z_0 \in -D$ and $\psi \in D^*$, $\psi(z_0) \leq 0$. Therefore

$$\psi(z_0) = 0. \tag{4.34}$$

It follows from (4.32), (4.34), $\phi \in C^* \setminus \{0_{Y^*}\}$ and $\psi \in D^*$ that $\phi(y) + \psi(z) \geq 0$, for all $(y, z) \in d_w^{b(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x)$. So $(x_0, y_0, z_0, \phi, \psi)$ is a feasible solution of (DSP).

Finally, we prove that $(x_0, y_0, z_0, \phi, \psi)$ is a weakly maximal solution of (DSP).

Suppose that $(x_0, y_0, z_0, \phi, \psi)$ is not a weakly maximal solution of (DSP). Then there exists a feasible solution $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\phi}, \tilde{\psi})$ of (DSP) such that

$$\tilde{y} > y_0. \tag{4.35}$$

According to $\phi \in C^* \setminus \{0_{Y^*}\}$, we get

$$\phi(\tilde{y}) > \phi(y_0). \tag{4.36}$$

Since (x_0, y_0) is a weakly minimal solution of (SP), it follows from Theorem 4.5 that

$$\phi(\tilde{y}) \leq \phi(y_0), \tag{4.37}$$

which contradicts (4.36). Thus the conclusion holds and the proof is complete. □

Now we give an example to illustrate the Strong Duality. we only take $m = 2$.

Example 4.8. Let $X = Y = Z = R$, $E = [-1, 1] \subset X$, $C = D = R_+$. Let $F : E \to 2^Y$ be a set-valued map with

$$F(x) = \begin{cases} \{y \in R \mid y \geq x^{4/3},\ x \in [-1, 1)\}, \\ \left\{y \in R \mid y \geq \dfrac{1}{2},\ x = 1\right\}, \end{cases} \tag{4.38}$$

and $G : E \to Z$ be a set-valued map with

$$G(x) = \begin{cases} \left\{ z \in R \mid z \geq x^{6/5} - \dfrac{1}{4},\ x \in [-1,1) \right\}, \\ \left\{ z \in R \mid z \geq \dfrac{1}{2},\ x = 1 \right\}. \end{cases} \tag{4.39}$$

Naturally, (F,G) is a $R_+ \times R_+$-convex like map on the convex set E.

Let $(x_0,y_0) = (0,0) \in \operatorname{graph}(F)$. Then (x_0,y_0) is a weakly minimal solution of (SP). Take $z_0 = -1/12 \in G(x_0) \bigcap(-D)$, $(u_1,v_1,w_1) = (0,0,-1/12) \in X \times (-C) \times (-D)$. Then $(u_1,v_1,w_1+z_0) \in \operatorname{epi}(F,G) - (x_0,y_0,z_0)$, $d_w^{\flat(2)}(F,G)(x_0,y_0,z_0,u_1,v_1,w_1+z_0)(x) = \{(y,z) \in R^2 : y = 0,\ z \in R\}$, for $x \in X$. The dual problem (DSP) becomes

$$\begin{aligned} \max \quad & y_0 \\ \text{s.t.} \quad & \phi(y) + \varphi(z) \geq 0,\ (y,z) \in d_w^{\flat(2)}(F,G)\big(x_0,y_0,z_0,u_1,v_1,w_1+z_0\big)(x), \quad x \in X, \\ & \varphi(z_0) \geq 0, \\ & \phi \in C^* \setminus \{0_{Y^*}\}, \\ & \varphi \in D^*. \end{aligned} \tag{4.40}$$

Therefore the conditions of Theorem 4.7 are satisfied. Simultaneous, take $\phi = 1/2 \in C^*$ and $\varphi = 0$. Obviously, $(x_0,y_0,z_0,\phi,\varphi)$ is a feasible solution of (DSP). It follows from Theorem 4.5 that $(x_0,y_0,z_0,\phi,\varphi)$ is a weakly maximal solution of (DSP).

Since neither of F and G is R_+-convex map on the E, the assumptions of [19, Theorem 4.3] are not satisfied. Therefore, [19, Theorem 4.3] is unusable here.

Theorem 4.9 (converse duality). *Suppose that $(x_0,y_0) \in graph(F)$, $z_0 \in G(x_0) \bigcap(-D)$, and the following conditions are satisfied:*

(i) $(u_i,v_i,w_i+z_0) \in \{0_X\} \times C \times D,\ i = 1,2,\ldots,m-1$;

(ii) *the set $\{(y,z) \in Y \times Z \mid (x,y,z) \in G\text{-}T^{\flat(m)}_{epi(F,G)}(x_0,y_0,z_0,u_1,v_1,w_1+z_0,\ldots,u_{m-1},v_{m-1},w_{m-1}+z_0)\}$ fulfills the weak domination property for all $x \in \Omega$;*

(iii) *there exist $\phi \in (C^* \setminus \{0_{Y^*}\})$ and $\varphi \in D^*$ such that $(x_0,y_0,z_0,\phi,\varphi)$ is a weakly maximal solution of (DSP).*

Then (x_0,y_0) is a weakly minimal solution of (SP).

Proof. It follows from Proposition 3.12 that

$$\begin{aligned} & (y-y_0, z-z_0) \\ & \quad \in d_w^{\flat(m)}(F,G)\big(x_0,y_0,z_0,u_1,v_1,w_1+z_0,\ldots,u_{m-1},v_{m-1},w_{m-1}+z_0\big)(x-x_0) + C \times D, \end{aligned} \tag{4.41}$$

for all $x \in K,\ y \in F(x),\ z \in G(x)$. Then,

$$\phi(y - y_0) + \psi(z - z_0) \geq 0, \quad \forall x \in K,\ y \in F(x),\ z \in G(x). \tag{4.42}$$

It follows from $x \in K$ that there exists $\overline{z} \in G(x)$ such that $\overline{z} \in -D$. So $\psi(\overline{z}) \leq 0$. Then, from (4.5) and (4.42), we get

$$\phi(y) \geq \phi(y_0), \quad \forall x \in K,\ y \in F(x). \tag{4.43}$$

We now show that (x_0, y_0) is a weakly minimal solution of (SP). Assume that (x_0, y_0) is not a weakly minimal solution of (SP). Then there exists $y_1 \in F(K)$ such that $y_1 - y_0 \in -\mathrm{int}C$. It follows from $\phi \in C^* \setminus \{0_{Y^*}\}$ that $\phi(y_1) < \phi(y_0)$, which contradicts (4.43). Thus (x_0, y_0) is a weakly minimal solution of (SP) and the proof is complete. □

Theorem 4.10 (converse duality). *Suppose that $(x_0, y_0) \in graph(F)$, $z_0 \in G(x_0) \bigcap (-D)$, and the following conditions are satisfied:*

(i) $(u_i, v_i, w_i + z_0) \in epi(F, G) - (x_0, y_0, z_0),\ i = 1, 2, \ldots, m-1$;

(ii) *the set* $\{(y, z) \in Y \times Z \mid (x, y, z) \in G\text{-}T^{b(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ *fulfills the weak domination property for all* $x \in \Omega$;

(iii) *there exist* $\phi \in (C^* \setminus \{0_{Y^*}\})$ *and* $\psi \in D^*$ *such that* $(x_0, y_0, z_0, \phi, \psi)$ *is a weakly maximal solution of* (DSP).

Then (x_0, y_0) is a weakly minimal solution of (SP).

Proof. By the similar proof method for Theorem 4.9, it follows from Proposition 3.14 that the conclusion holds. □

5. Higher-Order Wolfe Type Duality

In this section, we introduce a kind of higher-order Wolf type dual problem for a constrained set-valued optimization problem by virtue of the higher-order weakly generalized adjacent epiderivative and discuss its weak duality, strong duality and converse duality properties.

Suppose that $(u_i, v_i, w_i) \in X \times Y \times Z$, $i = 1, 2, \ldots, m-1$, $(x_0, y_0) \in \mathrm{graph}(F)$, $z_0 \in G(x_0) \bigcap (-D)$, and $\Omega = \mathrm{dom}[d^{b(m)}_w(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)]$. We introduce a higher-order Wolfe type dual problem$(WDSP)$ of (SP) as follows:

$$\max \quad \Phi(x_0, y_0, z_0, \phi, \psi) = \phi(y_0) + \psi(z_0)$$

$$\text{s.t.} \quad \phi(y) + \psi(z) \geq 0, (y, z) \in d^{b(m)}_w(F, G) \tag{5.1}$$

$$\times (x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x), \quad x \in \Omega,$$

$$\phi \in C^* \setminus \{0_{Y^*}\}, \tag{5.2}$$

$$\psi \in D^*. \tag{5.3}$$

A point $(x_0, y_0, z_0, \phi, \psi)$ satisfying (5.1)–(5.3) is called a feasible solution of $(WDSP)$. A feasible solution $(x_0, y_0, z_0, \phi_0, \psi_0)$ is called an optimal solution of $(WDSP)$ if, for any feasible solution (x, y, z, ϕ, ψ), $\Phi(x_0, y_0, z_0, \phi_0, \psi_0) \geq \Phi(x, y, z, \phi, \psi)$.

Theorem 5.1 (weak duality). *Let $(x_0, y_0) \in graph(F)$, $z_0 \in G(x_0) \bigcap (-D)$, $(u_i, v_i, w_i) \in \{0_X\} \times C \times D$, $i = 1, 2, \ldots, m-1$. Let the set $\{(y, z) \in Y \times Z \mid (x, y, z) \in G\text{-}T^{b(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)$ fulfill the weak domination property for all $x \in \Omega$. If $(\overline{x}, \overline{y})$ is a feasible solution of (SP) and $(x_0, y_0, z_0, \phi, \psi)$ is a feasible solution of $(WDSP)$, then*

$$\phi(\overline{y}) \geq \phi(y_0) + \psi(z_0). \tag{5.4}$$

Proof. It follows from Proposition 3.12 that

$$\begin{aligned} &(F, G)(\overline{x}) - (y_0, z_0) \\ &\quad \subset d^{b(m)}_w(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(\overline{x} - x_0) + C \times D. \end{aligned} \tag{5.5}$$

Since $(\overline{x}, \overline{y})$ is a feasible solution of (SP), $G(\overline{x}) \bigcap (-D) \neq \emptyset$. Take $\overline{z} \in G(\overline{x}) \bigcap (-D)$. Then it follows from (5.3) that

$$\psi(\overline{z}) \leq 0. \tag{5.6}$$

From (5.1)–(5.6), we get

$$\phi(\overline{y}) \geq \phi(y_0) + \psi(z_0), \tag{5.7}$$

and the proof is complete. □

Theorem 5.2 (weak duality). *Let $(x_0, y_0) \in graph(F)$, $z_0 \in G(x_0) \bigcap (-D)$, and $(u_i, v_i, w_i + z_0) \in epi(F, G) - (x_0, y_0, z_0)$, $i = 1, 2, \ldots, m-1$ and the set $\{(y, z) \in Y \times Z \mid (x, y, z) \in G\text{-}T^{b(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \ldots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)$ fulfill the weak domination property for all $x \in \Omega$. Suppose that (F, G) is $C \times D$-convex like on a nonempty convext subset E. If $(\overline{x}, \overline{y})$ is a feasible solution of (SP) and $(x_0, y_0, z_0, \phi, \psi)$ is a feasible solution of $(WDSP)$, then*

$$\phi(\overline{y}) \geq \phi(y_0) + \psi(z_0). \tag{5.8}$$

Proof. By using similar proof method of Theorem 5.1 and Proposition 3.14, we have that the conclusion holds. □

Theorem 5.3 (strong duality). *If the assumptions in Theorem 4.7 are satisfied and $y_0 = 0_Y$, then there exist $\phi \in (C^* \setminus \{0_{Y^*}\})$ and $\psi \in D^*$ such that $(x_0, y_0, z_0, \phi, \psi)$ is an optimal solution of $(WDSP)$.*

Proof. It follows from the proof of Theorem 4.7 that there exist $\phi \in C^*$ and $\psi \in D^*$ such that $(x_0, y_0, z_0, \phi, \psi)$ is a feasible solution of $(WDSP)$ and $\psi(z_0) = 0$.

We now prove that $(x_0, y_0, z_0, \phi, \psi)$ is an optimal solution of $(WDSP)$.

Suppose that $(x_0, y_0, z_0, \phi, \psi)$ is not an optimal solution of $(WDSP)$. Then there exists a feasible solution $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\phi}, \tilde{\psi})$ such that

$$\Phi(x_0, y_0, z_0, \phi, \psi) < \Phi\left(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\phi}, \tilde{\psi}\right). \tag{5.9}$$

Therefore, it follows from $\psi(z_0) = 0$ that

$$\phi(y_0) < \tilde{\phi}(\tilde{y}) + \tilde{\psi}(\tilde{z}). \tag{5.10}$$

Since (x_0, y_0) is a weakly minimal solution of (SP), it follows from Theorem 5.2 that $\tilde{\phi}(\tilde{y}) + \tilde{\psi}(\tilde{z}) \leq \tilde{\phi}(y_0)$. From (5.10), we get $\phi(y_0) < \tilde{\phi}(y_0)$, this is impossible since $y_0 = 0_Y$. So $(x_0, y_0, z_0, \phi, \psi)$ is an optimal solution of $(WDSP)$. □

By using similar proof methods for Theorems 4.9 and 4.10, we get the following results.

Theorem 5.4 (converse duality). *Suppose that there exists a $(\phi, \psi) \in (C^* \setminus \{0_{Y^*}\}) \times D^*$ such that $(x_0, y_0, z_0, \phi, \psi)$ is an optimal solution of $(WDSP)$ and $\psi(z_0) \geq 0$. Moreover, the assumptions (i) and (ii) in Theorem 4.9 are satisfied. Then (x_0, y_0) is a weakly minimal solution of (SP).*

Theorem 5.5 (converse duality). *Suppose that there exists a $(\phi, \psi) \in (C^* \setminus \{0_{Y^*}\}) \times D^*$ such that $(x_0, y_0, z_0, \phi, \psi)$ is an optimal solution of $(WDSP)$ and $\psi(z_0) \geq 0$. Moreover, the assumptions (i) and (ii) in Theorem 4.10 are satisfied. Then (x_0, y_0) is a weakly minimal solution of (SP).*

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