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Research Article

Stability of Mixed Type Cubic and Quartic Functional Equations in Random Normed Spaces

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We obtain the stability result for the following functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary t-norms $f(x+2y)+f(x-2y)=4[f(x+y)+f(x-y)]-24f(y)-6f(x)+3f(2y)$.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1,\cdot)$ be a group and let $(G_2,*,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\le\delta \tag{1.1}$$

for all $x,y\in E$, and for some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\|f(x)-T(x)\|\le\delta \tag{1.2}$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. In 1978, Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [5–12]).

Jun and Kim [13] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3). The function $f(x) = x^3$ satisfies the functional equation (1.3), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exits a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables.

Park and Bea [14] introduced the following quartic functional equation:

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] + 24f(y) - 6f(x). \tag{1.4}$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.4) if and only if there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \to Y$ such that $f(x) = Q(x,x,x,x)$ for all x (see also [15–18]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.4), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

In the sequel we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [19–21]. Throughout this paper, Δ^+ is the space of distribution functions that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$, such that F is leftcontinuous and nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{1.5}$$

Definition 1.1 (see [20]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm). Recall (see [22, 23]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^{n} x_i$ is defined recurrently by $T_{i=1}^{1} x_i = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$. $T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$. It is known [23] that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty. \tag{1.6}$$

Definition 1.2 (see [21]). A random normed space (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$;

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed spaces $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}, \tag{1.7}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 1.3. Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) A RN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4 (see [20]). *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

The generalized Hyers-Ulam-Rassias stability of different functional equations in random normed spaces has been recently studied in [24–29]. Recently, Eshaghi Gordji et al. [30] established the stability of mixed type cubic and quartic functional equations (see also [31]). In this paper we deal with the following functional equation:

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y)) - 24f(y) - 6f(x) + 3f(2y) \tag{1.8}$$

on random normed spaces. It is easy to see that the function $f(x) = ax^4 + bx^3 + c$ is a solution of the functional equation (1.8). In the present paper we establish the stability of the functional equation (1.8) in random normed spaces.

2. Main Results

From now on, we suppose that X is a real linear space, (Y, μ, T) is a complete RN-space, and $f : X \to Y$ is a function with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x, y)$ denoted by $\rho_{x,y}$) with the property

$$\mu_{f(x+2y)+f(x-2y)-4[f(x+y)+f(x-y)]+24f(y)+6f(x)-3f(2y)}(t) \geq \rho_{x,y}(t) \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$.

Theorem 2.1. *Let f be odd and let*

$$\lim_{n\to\infty} T_{i=1}^{\infty}\left(\rho_{0,2^{n+i-1}x}\left(2^{3n+2i}t\right)\right) = 1 = \lim_{n\to\infty} \rho_{2^n x,2^n y}\left(2^{3n}t\right) \tag{2.2}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{C(x)-f(x)}(t) \geq T_{i=1}^{\infty}\left(\rho_{0,2^{i-1}x}\left(2^{2i}t\right)\right), \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Setting $x = 0$ in (2.1), we get

$$\mu_{3f(2y)-24f(y)}(t) \geq \rho_{0,y}(t) \tag{2.4}$$

for all $y \in X$. If we replace y in (2.4) by x and divide both sides of (2.4) by 3, we get

$$\mu_{f(2x)-8f(x)}(t) \geq \rho_{0,x}(3t) \geq \rho_{0,x}(t) \tag{2.5}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{f(2x)/2^3-f(x)}(t) \geq \rho_{0,x}\left(2^3 t\right) \tag{2.6}$$

for all $x \in X$ and all $t > 0$. Therefore,

$$\mu_{f(2^{k+1}x)/2^{3(k+1)}-f(2^k x)/2^{3k}}(t) \geq \rho_{0,2^k x}\left(2^{3(k+1)}t\right) \tag{2.7}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Therefore we have

$$\mu_{f(2^{k+1}x)/2^{3(k+1)}-f(2^k x)/2^{3k}}\left(\frac{t}{2^{k+1}}\right) \geq \rho_{0,2^k x}\left(2^{2(k+1)}t\right) \tag{2.8}$$

for all $x \in X, t > 0$ and all $k \in \mathbb{N}$. As $1 > 1/2 + 1/2^2 + \cdots + 1/2^n$, by the triangle inequality it follows

$$\begin{aligned}\mu_{f(2^n x)/2^{3n} - f(x)}(t) &\geq T_{k=0}^{n-1}\left(\mu_{f(2^{k+1}x)/2^{3(k+1)} - f(2^k x)/2^{3k}}\left(\frac{t}{2^{k+1}}\right)\right) \geq T_{k=0}^{n-1}\left(\rho_{0,2^k x}\left(2^{2(k+1)}t\right)\right) \\ &= T_{i=1}^{n}\left(\rho_{0,2^{i-1}x}\left(2^{2i}t\right)\right)\end{aligned} \tag{2.9}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{f(2^n x)/2^{3n}\}$, we replace x with $2^m x$ in (2.9) to find that

$$\mu_{f(2^{n+m}x)/2^{3(n+m)} - f(2^m x)/2^{3m}}(t) \geq T_{i=1}^{n}\left(\rho_{0,2^{i+m-1}x}\left(2^{2i+3m}t\right)\right). \tag{2.10}$$

Since the right-hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{f(2^n x)/2^{3n}\}$ is a Cauchy sequence. Therefore, we may define $C(x) = \lim_{n\to\infty}(f(2^n x)/2^{3n})$ for all $x \in X$. Now, we show that C is a cubic map. Replacing x, y with $2^n x$ and $2^n y$ respectively in (2.1), it follows that

$$\begin{aligned}&\mu_{\frac{f(2^n x+2^{n+1}y)}{2^{3n}} + \frac{f(2^n x-2^{n+1}y)}{2^{3n}} - 4\left[\frac{f(2^n x+2^n y)}{2^{3n}} + \frac{f(2^n x-2^n y)}{2^{3n}}\right] + 24\frac{f(2^n y)}{2^{3n}} + 6\frac{f(2^n x)}{2^{3n}} - 3\frac{f(2^{n+1}y)}{2^3 n}}(t) \\ &\quad\geq \rho_{2^n x, 2^n y}\left(2^{3n}t\right).\end{aligned} \tag{2.11}$$

Taking the limit as $n \to \infty$, we find that C satisfies (1.8) for all $x, y \in X$. Therefore the mapping $C : X \to Y$ is cubic.

To prove (2.3), take the limit as $n \to \infty$ in (2.9). Finally, to prove the uniqueness of the cubic function C subject to (2.3), let us assume that there exists a cubic function C' which satisfies (2.3). Since $C(2^n x) = 2^{3n}C(x)$ and $C'(2^n x) = 2^{3n}C'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (2.3) it follows that

$$\begin{aligned}\mu_{C(x)-C'(x)}(2t) &= \mu_{C(2^n x)-C'(2^n x)}\left(2^{3n+1}t\right) \\ &\geq T\left(\mu_{C(2^n x)-f(2^n x)}\left(2^{3n}t\right), \mu_{f(2^n x)-C'(2^n x)}\left(2^{3n}t\right)\right) \\ &\geq T\left(T_{i=1}^{\infty}\left(\rho_{0,2^{i+n-1}x}\left(2^{2i+3n}t\right)\right), T_{i=1}^{\infty}\left(\rho_{0,2^{i+n-1}x}\left(2^{2i+3n}t\right)\right)\right)\end{aligned} \tag{2.12}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in above inequality, we find that $C = C'$. □

Theorem 2.2. *Let f be even and let*

$$\lim_{n\to\infty} T_{i=1}^{\infty}\left(\rho_{0,2^{n+i-1}x}\left(2^{4n+3i}t\right)\right) = 1 = \lim_{n\to\infty}\rho_{2^n x, 2^n y}\left(2^{4n}t\right) \tag{2.13}$$

for all $x, y \in X$ *and all* $t > 0$*, then there exists a unique quartic mapping* $Q : X \to Y$ *such that*

$$\mu_{Q(x)-f(x)}(t) \geq T_{i=1}^{\infty}\left(\rho_{0,2^{i-1}x}\left(2^{3i}t\right)\right), \tag{2.14}$$

for all $x \in X$ *and all* $t > 0$.

Proof. By putting $x = 0$ in (2.1), we obtain

$$\mu_{f(2y)-16f(y)}(t) \geq \rho_{0,y}(t) \tag{2.15}$$

for all $y \in X$. Replacing y in (2.15) by x to get

$$\mu_{f(2x)-16f(x)}(t) \geq \rho_{0,x}(t) \tag{2.16}$$

for all $x \in X$ and all $t > 0$. Hence,

$$\mu_{f(2x)/2^4-f(x)}(t) \geq \rho_{0,x}\left(2^4 t\right) \tag{2.17}$$

for all $x \in X$ and all $t > 0$. Therefore,

$$\mu_{f(2^{k+1}x)/2^{4(k+1)}-f(2^k x)/2^{4k}}(t) \geq \rho_{0,2^k x}\left(2^{4(k+1)}t\right) \tag{2.18}$$

for all $x \in X$ and all $k \in \mathbb{N}$. So we have

$$\mu_{f(2^{k+1}x)/2^{4(k+1)}-f(2^k x)/2^{4k}}\left(\frac{t}{2^{k+1}}\right) \geq \rho_{0,2^k x}\left(2^{3(k+1)}t\right) \tag{2.19}$$

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. As $1 > 1/2 + 1/2^2 + \cdots + 1/2^n$, by the triangle inequality it follows that

$$\begin{aligned}\mu_{f(2^n x)/2^{4n}-f(x)}(t) &\geq T_{k=0}^{n-1}\left(\mu_{f(2^{k+1}x)/2^{4(k+1)}-f(2^k x)/2^{4k}}\left(\frac{t}{2^{k+1}}\right)\right) \geq T_{k=0}^{n-1}\left(\rho_{0,2^k x}\left(2^{3(k+1)}t\right)\right)\\ &= T_{i=1}^{n}\left(\rho_{0,2^{i-1}x}\left(2^{3i}t\right)\right)\end{aligned} \tag{2.20}$$

for all $x \in X$ and $t > 0$. We replace x with $2^m x$ in (2.20) to obtain

$$\mu_{f(2^{n+m}x)/2^{4(n+m)}-f(2^m x)/2^{4m}}(t) \geq T_{i=1}^{n}\left(\rho_{0,2^{i+m-1}x}\left(2^{3i+4m}t\right)\right). \tag{2.21}$$

Since the right-hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{f(2^n x)/2^{4n}\}$ is a Cauchy sequence. Therefore, we may define $Q(x) = \lim_{n\to\infty}(f(2^n x)/2^{4n})$

for all $x \in X$. Now, we show that Q is a quartic map. Replacing x, y with $2^n x$ and $2^n y$ respectively, in (2.1), it follows that

$$\begin{aligned}
&\mu_{\frac{f(2^n x+2^{n+1}y)}{2^{4n}}+\frac{f(2^n x-2^{n+1}y)}{2^{4n}}-4[\frac{f(2^n x+2^n y)}{2^{4n}}+\frac{f(2^n x-2^n y)}{2^{4n}}]+24\frac{f(2^n y)}{2^{4n}}+6\frac{f(2^n x)}{2^{4n}}-3\frac{f(2^{n+1}y)}{2^{4}n}}(t) \\
&\quad \geq \rho_{2^n x, 2^n y}\left(2^{4n} t\right).
\end{aligned} \tag{2.22}$$

Taking the limit as $n \to \infty$, we find that Q satisfies (1.8) for all $x, y \in X$. Hence, the mapping $Q : X \to Y$ is quartic.

To prove (2.14), take the limit as $n \to \infty$ in (2.20). Finally, to prove the uniqueness property of Q subject to (2.14), let us assume that there exists a quartic function Q' which satisfies (2.14). Since $Q(2^n x) = 2^{4n} Q(x)$ and $Q'(2^n x) = 2^{4n} Q'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (2.14) it follows that

$$\begin{aligned}
\mu_{Q(x)-Q'(x)}(2t) &= \mu_{Q(2^n x)-Q'(2^n x)}\left(2^{4n+1} t\right) \\
&\geq T\left(\mu_{Q(2^n x)-f(2^n x)}\left(2^{4n} t\right), \mu_{f(2^n x)-Q'(2^n x)}\left(2^{4n} t\right)\right) \\
&\geq T\left(T_{i=1}^{\infty}\left(\rho_{0,2^{i+n-1}x}\left(2^{3i+4n} t\right)\right), T_{i=1}^{\infty}\left(\rho_{0,2^{i+n-1}x}\left(2^{3i+4n} t\right)\right)\right)
\end{aligned} \tag{2.23}$$

for all $x \in X$ and all $t > 0$. Taking the limit as $n \to \infty$, we find that $Q = Q'$. □

Theorem 2.3. *Let*

$$\begin{aligned}
&\lim_{n\to\infty} T_{i=1}^{\infty}\left[T\left(\rho_{0,2^{n+i-1}x}\left(2^{2i+4n} t\right), \rho_{0,-2^{n+i-1}x}\left(2^{2i+4n} t\right)\right)\right] = 1 \\
&\quad = \lim_{n\to\infty} T_{i=1}^{\infty}\left[T\left(\rho_{0,2^{n+i-1}x}\left(2^{i+3n} t\right), \rho_{0,2^{n+i-1}x}\left(2^{i+3n} t\right)\right)\right], \\
&\lim_{n\to\infty} T\left(\rho_{2^n x, 2^n y}\left(2^{4n-1} t\right), \rho_{2^n x, 2^n y}\left(2^{4n-1} t\right)\right) = 1 \\
&\quad = \lim_{n\to\infty} T\left(\rho_{2^n x, 2^n y}\left(2^{3n-1} t\right), \rho_{2^n x, 2^n y}\left(2^{3n-1} t\right)\right)
\end{aligned} \tag{2.24}$$

for all $x, y \in X$ and all $t > 0$, then there exist a unique cubic mapping $C : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that

$$\begin{aligned}
\mu_{f(x)-C(x)-Q(x)}(t) \geq T\Big(&T_{i=1}^{\infty}\left[T\left(\rho_{0,2^{i-1}x}\left(2^{2i-1} t\right), \rho_{0,-2^{i-1}x}\left(2^{2i-1} t\right)\right)\right], \\
&T_{i=1}^{\infty}\left[T\left(\rho_{0,2^{i-1}x}\left(2^{i-1} t\right), \rho_{0,-2^{i-1}x}\left(2^{i-1} t\right)\right)\right]\Big)
\end{aligned} \tag{2.25}$$

for all $x \in X$ and all $t > 0$.

Proof. Let

$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x)\right] \tag{2.26}$$

for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\mu_{f_e(x+2y)+f_e(x-2y)-4[f_e(x+y)+f_e(x-y)]+24f_e(y)+6f_e(x)-3f_e(2y)}(t) \geq T\left(\rho_{x,y}\left(\frac{t}{2}\right), \rho_{-x,-y}\left(\frac{t}{2}\right)\right) \quad (2.27)$$

for all $x, y \in X$. Hence, in view of Theorem 2.1, there exists a unique quartic function $Q : X \to Y$ such that

$$\mu_{Q(x)-f_e(x)}(t) \geq T_{i=1}^{\infty}\left[T\left(\rho_{0,2^{i-1}x}\left(2^{2i}t\right), \rho_{0,-2^{i-1}x}\left(2^{2i}t\right)\right)\right]. \quad (2.28)$$

Let

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)] \quad (2.29)$$

for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and

$$\mu_{f_o(x+2y)+f_o(x-2y)-4[f_o(x+y)+f_o(x-y)]+24f_o(y)+6f_o(x)-3f_o(2y)}(t) \geq T\left(\rho_{x,y}\left(\frac{t}{2}\right), \rho_{-x,-y}\left(\frac{t}{2}\right)\right) \quad (2.30)$$

for all $x, y \in X$. From Theorem 2.2, it follows that there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{C(x)-f_o(x)}(t) \geq T_{i=1}^{\infty}\left[T\left(\rho_{0,2^{i-1}x}\left(2^{i}t\right), \rho_{0,-2^{i-1}x}\left(2^{i}t\right)\right)\right]. \quad (2.31)$$

Obviously, (2.25) follows from (2.28) and (2.31). □

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