Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 640152, 7 pages
doi:10.1155/2009/640152

Research Article

On the Identities of Symmetry for the Generalized Bernoulli Polynomials Attached to χ of Higher Order

Taekyun Kim,[1] Lee-Chae Jang,[2] Young-Hee Kim,[1] and Kyung-Won Hwang[3]

[1] *Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, South Korea*
[2] *Department of Mathematics and Computer Science, Konkook University, Chungju 139-701, South Korea*
[3] *Department of General Education, Kookmin University, Seoul 136-702, South Korea*

Correspondence should be addressed to Taekyun Kim, tkkim@kw.ac.kr

Received 5 June 2009; Accepted 5 August 2009

Recommended by Vijay Gupta

We give some interesting relationships between the power sums and the generalized Bernoulli numbers attached to χ of higher order using multivariate p-adic invariant integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$ (see [1–24]). Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. Let d be a fixed positive integer. For $n \in \mathbb{N}$, let

$$\begin{gathered} X = X_d = \lim_{\overleftarrow{N}} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} \left(a + dp\mathbb{Z}_p\right), \\ a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \left(\bmod dp^N\right)\right\}, \end{gathered} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For $f \in \mathrm{UD}(X)$, the p-adic invariant integral on X is defined as

$$I(f) = \int_X f(x)dx = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{x=0}^{dp^N-1} f(x) \tag{1.2}$$

(see [11–19]). From (1.2), we note that

$$I(f_1) = I(f) + f'(0), \tag{1.3}$$

where $f'(0) = (df(x)/dx)|_{x=0}$ and $f_1(x) = f(x+1)$. Let $f_n(x) = f(x+n)$ $(n \in \mathbb{N})$. Then we can derive the following equation from (1.3):

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \tag{1.4}$$

(see [1–11]). Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$. Then the generalized Bernoulli polynomials attached to χ are defined as

$$\sum_{a=0}^{d-1} \frac{\chi(a)e^{at}t}{e^{dt}-1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!}, \tag{1.5}$$

and the generalized Bernoulli numbers attached to χ, $B_{n,\chi}$, are defined as $B_{n,\chi} = B_{n,\chi}(0)$ (see [1–20, 25]). The purpose of this paper is to derive some identities of symmetry for the generalized Bernoulli polynomials attached to χ of higher order.

2. Symmetric Properties for the Generalized Bernoulli Polynomials of Higher Order

Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$. Then we note that

$$\int_X \chi(x)e^{xt}dx = \frac{t\sum_{i=0}^{d-1}\chi(i)e^{it}}{e^{dt}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!}, \tag{2.1}$$

where $B_{n,\chi}$ are the nth generalized Bernoulli numbers attached to χ (see [7, 9, 15, 25]). Now we also see that the generalized Bernoulli polynomials attached to χ are given by

$$\int_X \chi(y)e^{(x+y)t}dy = \frac{t\sum_{i=0}^{d-1}\chi(i)e^{it}}{e^{dt}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\int_X \chi(x)x^n dx = B_{n,\chi} \tag{2.3}$$

(see [15, 25]), and

$$\int_X \chi(y)(x+y)^n dy = B_{n,\chi}(x) \tag{2.4}$$

(see [1–19, 25]). For $n \in \mathbb{N}$, we obtain that

$$\int_X f(x+n)dx = \int_X f(x)dx + \sum_{i=0}^{n-1} f'(i), \tag{2.5}$$

where $f'(i) = (df(x)/dx)|_{x=i}$. Thus, we have

$$\frac{1}{t}\left(\int_X \chi(x)e^{(nd+x)t}dx - \int_X \chi(x)e^{xt}dx\right) = \frac{nd\int_X \chi(x)e^{xt}dx}{\int_X e^{ndxt}dx} = \frac{e^{ndt}-1}{e^{dt}-1}\left(\sum_{i=0}^{d-1}\chi(i)e^{it}\right). \tag{2.6}$$

Then

$$\frac{1}{t}\left(\int_X \chi(x)e^{(nd+x)t}dx - \int_X \chi(x)e^{xt}dx\right) = \sum_{l=0}^{nd-1}\chi(l)e^{lt} = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{nd-1}\chi(l)l^k\right)\frac{t^k}{k!}. \tag{2.7}$$

Let us define the p-adic function $T_k(\chi, n)$ as follows:

$$T_k(\chi, n) = \sum_{l=0}^{n}\chi(l)l^k \tag{2.8}$$

(see [25]). By (2.7) and (2.8), we see that

$$\frac{1}{t}\left(\int_X \chi(x)e^{(nd+x)t}dx - \int_X \chi(x)e^{xt}dx\right) = \sum_{k=0}^{\infty}T_k(\chi, nd-1)\frac{t^k}{k!} \tag{2.9}$$

(see [25]). Thus, we have

$$\int_X \chi(x)(nd+x)^k dx - \int_X \chi(x)x^k dx = kT_{k-1}(\chi, nd-1), \quad k, n, d \in \mathbb{N}. \tag{2.10}$$

This means that

$$B_{k,\chi}(nd) - B_{k,\chi} = kT_{k-1}(\chi, nd-1), \quad k, n, d \in \mathbb{N} \tag{2.11}$$

(see [25]).

The generalized Bernoulli polynomials attached to χ of order k, which is denoted by $B_{n,\chi}^{(k)}(x)$, are defined as

$$\left(\frac{t\sum_{i=0}^{d-1}\chi(i)e^{it}}{e^{dt}-1}\right)^{k}e^{xt}=\sum_{n=0}^{\infty}B_{n,\chi}^{(k)}(x)\frac{t^n}{n!}. \tag{2.12}$$

Then the values of $B_{n,\chi}^{(k)}(x)$ at $x=0$ are called the generalized Bernoulli numbers attached to χ of order k. When $k=1$, the polynomials of numbers are called the generalized Bernoulli polynomials or numbers attached to χ. Let $w_1, w_2 \in \mathbb{N}$. Then we set

$$\begin{aligned} &K(m,\chi;w_1,w_2)\\ &\quad=\frac{d\left(\int_{X^m}\prod_{i=1}^{m}\chi(x_i)e^{(\sum_{i=1}^{m}x_i+w_2x)w_1t}\prod_{i=1}^{m}dx_i\right)\left(\int_{X^m}\prod_{i=1}^{m}\chi(x_i)e^{(\sum_{i=1}^{m}x_i+w_1y)w_2t}\prod_{i=1}^{m}dx_i\right)}{\int_X e^{dw_1w_2xt}dx}, \end{aligned} \tag{2.13}$$

where

$$\int_{X^m}f(x_1,\ldots,x_m)dx_1\cdots dx_m=\int_X\cdots\int_X f(x_1,\ldots,x_m)dx_1\cdots dx_m. \tag{2.14}$$

In (2.13), we note that $K(m,\chi;w_1,w_2)$ is symmetric in w_1, w_2. From (2.13), we derive

$$\begin{aligned} &K(m,\chi;w_1,w_2)\\ &\quad=\left(\int_{X^m}\prod_{i=1}^{m}\chi(x_i)e^{(\sum_{i=1}^{m}x_i)w_1t}dx_1\cdots dx_m\right)e^{w_1w_2xt}\left(\frac{d\int_X\chi(x_m)e^{w_2x_mt}dx_m}{\int_X e^{dw_1w_2xt}dx}\right)\\ &\quad\times\left(\int_{X^{m-1}}\prod_{i=1}^{m-1}\chi(x_i)e^{(\sum_{i=1}^{m-1}x_i)w_2t}dx_1\cdots dx_{m-1}\right)e^{w_1w_2yt}. \end{aligned} \tag{2.15}$$

It is easy to see that

$$\begin{aligned} &\frac{w_1d\int_X\chi(x)e^{xt}dx}{\int_X e^{dw_1xt}dx}\\ &\quad=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{w_1d-1}\chi(i)i^k\right)\frac{t^k}{k!}=\sum_{k=0}^{\infty}T_k(\chi,w_1d-1)\frac{t^k}{k!},\\ &e^{w_1w_2xt}\int_{X^m}\prod_{i=1}^{m}\chi(x_i)e^{(\sum_{i=1}^{m}x_i)w_1t}dx_1\cdots dx_m\\ &\quad=e^{w_1w_2xt}\left(\frac{w_1t}{e^{dw_1t}-1}\sum_{a=0}^{d-1}\chi(a)e^{w_1at}\right)^m=\sum_{n=0}^{\infty}B_{n,\chi}^{(m)}(w_2x){w_1}^n\frac{t^n}{n!}. \end{aligned} \tag{2.16}$$

From (2.16), we note that

$$
\begin{aligned}
&K(m,\chi;w_1,w_2)\\
&\quad=\left(\sum_{l=0}^{\infty}B_{l,\chi}^{(m)}(w_2x)\frac{{w_1}^lt^l}{l!}\right)\left(\sum_{k=0}^{\infty}T_k(\chi,w_1d-1)\frac{{w_2}^kt^k}{k!}\right)\left(\sum_{i=0}^{\infty}B_{i,\chi}^{(m-1)}(w_1y)\frac{{w_2}^it^i}{i!}\right)\left(\frac{1}{w_1}\right)\\
&\quad=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\binom{n}{j}w_2^jw_1^{n-j-1}B_{n-j,\chi}^{(m)}(w_2x)\sum_{k=0}^{j}T_k(\chi,w_1d-1)\binom{j}{k}B_{j-k,\chi}^{(m-1)}(w_1y)\right]\frac{t^n}{n!}.
\end{aligned}
\tag{2.17}
$$

By the symmetry of $K(m,\chi;w_1,w_2)$ in w_1 and w_2, we see that

$$
\begin{aligned}
&K(m,\chi;w_1,w_2)\\
&\quad=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\binom{n}{j}w_1^jw_2^{n-j-1}B_{n-j,\chi}^{(m)}(w_1x)\sum_{k=0}^{j}T_k(\chi,w_2d-1)\binom{j}{k}B_{j-k,\chi}^{(m-1)}(w_2y)\right]\frac{t^n}{n!}.
\end{aligned}
\tag{2.18}
$$

By comparing the coefficients on the both sides of (2.17) and (2.18), we see the following theorem.

Theorem 2.1. *For $d,w_1,w_2\in\mathbb{N}$, $n\geq 0$, $m\geq 1$, one has*

$$
\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j}w_2^jw_1^{n-j-1}B_{n-j,\chi}^{(m)}(w_2x)\sum_{k=0}^{j}T_k(\chi,w_1d-1)\binom{j}{k}B_{j-k,\chi}^{(m-1)}(w_1y)\\
&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^jw_2^{n-j-1}B_{n-j,\chi}^{(m)}(w_1x)\sum_{k=0}^{j}T_k(\chi,w_2d-1)\binom{j}{k}B_{j-k,\chi}^{(m-1)}(w_2y).
\end{aligned}
\tag{2.19}
$$

Remark 2.2. Let $y=0$ and $m=1$ in (1.4). Then we have

$$
\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j}w_2^jw_1^{n-j-1}B_{n-j,\chi}(w_2x)T_j(\chi,w_1d-1)\\
&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^jw_2^{n-j-1}B_{n-j,\chi}(w_1x)T_j(\chi,w_2d-1)
\end{aligned}
\tag{2.20}
$$

(see [25]).

We also calculate that

$$K(m,\chi;w_1,w_2) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k,\chi}^{(m-1)}(w_1 y)\sum_{i=0}^{dw_1-1} B_{k,\chi}^{(m)}\left(w_2 x + \frac{w_2}{w_1} i\right)\right]\frac{t^n}{n!}. \tag{2.21}$$

From the symmetric property of $K(m,\chi;w_1,w_2)$ in w_1 and w_2, we derive

$$K(m,\chi;w_1,w_2) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k,\chi}^{(m-1)}(w_2 y)\sum_{i=0}^{dw_2-1} B_{k,\chi}^{(m)}\left(w_1 x + \frac{w_1}{w_2} i\right)\right]\frac{t^n}{n!}. \tag{2.22}$$

By comparing the coefficients on the both sides of (2.21) and (2.22), we obtain the following theorem.

Theorem 2.3. *For $w_1, w_2 \in \mathbb{N}$, $n \in \mathbb{Z}$, $m \in \mathbb{N}$, one has*

$$\sum_{k=0}^{n}\binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k,\chi}^{(m-1)}(w_1 y)\sum_{i=0}^{dw_1-1} B_{k,\chi}^{(m)}\left(w_2 x + \frac{w_2}{w_1} i\right) = \sum_{k=0}^{n}\binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k,\chi}^{(m-1)}(w_2 y)\sum_{i=0}^{dw_2-1} B_{k,\chi}^{(m)}\left(w_1 x + \frac{w_1}{w_2} i\right). \tag{2.23}$$

Remark 2.4. Let $y = 0$ and $m = 1$ in (2.23). We have

$$w_1^{n-1}\sum_{i=0}^{dw_1-1} B_{n,\chi}\left(w_2 x + \frac{w_2}{w_1} i\right) = w_2^{n-1}\sum_{i=0}^{dw_2-1} B_{n,\chi}\left(w_1 x + \frac{w_1}{w_2} i\right) \tag{2.24}$$

(see [25]).

Acknowledgment

The present research has been conducted by the research Grant of the Kwangwoon University in 2009.

References

[1] M. Cenkci, "The p-adic generalized twisted (h, q)-Euler-l-function and its applications," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 1, pp. 37–47, 2007.

[2] M. Cenkci, Y. Simsek, and V. Kurt, "Multiple two-variable p-adic q-L-function and its behavior at $s = 0$," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 447–459, 2008.

[3] L.-C. Jang, S.-D. Kim, D.-W. Park, and Y.-S. Ro, "A note on Euler number and polynomials," *Journal of Inequalities and Applications*, vol. 2006, Article ID 34602, 5 pages, 2006.

[4] T. Kim, "q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.

[5] T. Kim, "A note on q-Volkenborn integration," *Proceedings of the Jangjeon Mathematical Society*, vol. 8, no. 1, pp. 13–17, 2005.

[6] T. Kim, "q-Euler numbers and polynomials associated with p-adic q-integrals," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 15–27, 2007.

[7] T. Kim, "A note on p-adic q-integral on $\mathbb{Z}_p$ associated with q-Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 133–137, 2007.

[8] T. Kim, "On the q-extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.

[9] T. Kim, "On p-adic q-l-functions and sums of powers," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 2, pp. 1472–1481, 2007.

[10] T. Kim, "On the multiple q-Genocchi and Euler numbers," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 481–486, 2008.

[11] T. Kim, "New approach to q-Euler, Genocchi numbers and their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 105–112, 2009.

[12] T. Kim, "On a q-analogue of the p-adic log gamma functions and related integrals," *Journal of Number Theory*, vol. 76, no. 2, pp. 320–329, 1999.

[13] T. Kim, "Sums of products of q-Euler numbers," to appear in *Journal of Computational Analysis and Applications*.

[14] T. Kim, J. Y. Choi, and J. Y. Sug, "Extended q-Euler numbers and polynomials associated with fermionic p-adic q-integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 14, no. 2, pp. 160–163, 2007.

[15] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[16] Y.-H. Kim, W. Kim, and L.-C. Jang, "On the q-extension of Apostol-Euler numbers and polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 296159, 10 pages, 2008.

[17] J. Pečarić and A. Vukelić, "General dual Euler-Simpson formulae," *Journal of Mathematical Inequalities*, vol. 2, no. 4, pp. 511–526, 2008.

[18] Y. Simsek, "Complete sum of products of (h, q)-extension of the Euler polynomials and numbers," http://arxiv.org/abs/0707.2849.

[19] Y. Simsek, "On p-adic twisted q-L-functions related to generalized twisted Bernoulli numbers," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 340–348, 2006.

[20] Y. Simsek, V. Kurt, and D. Kim, "New approach to the complete sum of products of the twisted (h, q)-Bernoulli numbers and polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 44–56, 2007.

[21] T. Kim, "Symmetry properties of higher order Bernoulli polynomials," (communicated).

[22] Z. Zhang and H. Yang, "Some closed formulas for generalized Bernoulli-Euler numbers and polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 11, no. 2, pp. 191–198, 2008.

[23] A. Tekcan, A. Özkoç, B. Gezer, and O. Bizim, "Some relations involving the sums of Fibonacci numbers," *Proceedings of the Jangjeon Mathematical Society*, vol. 11, no. 1, pp. 1–12, 2008.

[24] A. Yu. Khrennikov, "Generalized probabilities taking values in non-Archimedean fields and in topological groups," *Russian Journal of Mathematical Physics*, vol. 14, no. 2, pp. 142–159, 2007.

[25] T. Kim, S.-H. Rim, and B. Lee, "Some identities of symmetry for the generalized Bernoulli numbers and polynomials," *Abstract and Applied Analysis*, vol. 2009, Article ID 848943, 8 pages, 2009.