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Research Article

Generalizations of Shafer-Fink-Type Inequalities for the Arc Sine Function

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We give some generalizations of Shafer-Fink inequalities, and prove these inequalities by using a basic differential method and l'Hospital's rule for monotonicity.

1. Introduction

Shafer (see Mitrinovic and Vasic [1, page 247]) gives us a result as follows.

Theorem 1.1. *Let $x > 0$. Then*

$$\arcsin x > \frac{6\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}}. \tag{1.1}$$

The theorem is generalized by Fink [2] as follows.

Theorem 1.2. *Let $0 \le x \le 1$. Then*

$$\frac{3x}{2+\sqrt{1-x^2}} \le \arcsin x \le \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{1.2}$$

Furthermore, 3 and π are the best constants in (1.2).

In [3], Zhu presents an upper bound for $\arcsin x$ and proves the following result.

Theorem 1.3. *Let* $0 \leq x \leq 1$. *Then*

$$\frac{3x}{2+\sqrt{1-x^2}} \leq \frac{6\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} \leq \arcsin x \leq \frac{\pi\left(\sqrt{2}+1/2\right)\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} \leq \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{1.3}$$

Furthermore, 3 and π, *6 and* $\pi(\sqrt{2}+1/2)$ *are the best constants in* (1.3).

Malesevic [4–6] obtains the following inequality by using λ-method and computer separately.

Theorem 1.4. *Let* $0 \leq x \leq 1$. *Then*

$$\arcsin x \leq \frac{\left(\pi\left(2-\sqrt{2}\right)\right)/\left(\pi-2\sqrt{2}\right)\left(\sqrt{1+x}-\sqrt{1-x}\right)}{\left(\sqrt{2}(4-\pi)\right)/\left(\pi-2\sqrt{2}\right)+\sqrt{1+x}+\sqrt{1-x}} \leq \frac{\pi/(\pi-2)x}{(2/(\pi-2))+\sqrt{1-x^2}}. \tag{1.4}$$

Zhu [7, 8] offers some new simple proofs of inequality (1.4) by L'Hospital's rule for monotonicity.

In this paper, we give some generalizations of these above results and obtain two new Shafer-Fink type double inequalities as follows.

Theorem 1.5. *Let* $0 \leq x \leq 1$, *and* $a, b_1, b_2 > 0$. *If*

$$\begin{aligned}(a,b_1,b_2) \in & \left\{a \geq 3, b_1 \geq a-1, b_2 \leq \frac{2a}{\pi}\right\} \\ & \bigcup\left\{3 > a > \frac{\pi}{\pi-2}, b_2 \leq \frac{2a}{\pi}, b_1 \geq \frac{a\sin t_a}{t_a} - \cos t_a\right\} \\ & \bigcup\left\{\frac{\pi}{\pi-2} \geq a > \frac{\pi^2}{4}, b_2 \leq a-1, b_1 \geq \frac{a\sin t_a}{t_a} - \cos t_a\right\} \\ & \bigcup\left\{\frac{\pi^2}{4} \geq a > 1, b_1 \geq \frac{2a}{\pi}, b_2 \leq a-1\right\},\end{aligned} \tag{1.5}$$

then

$$\frac{ax}{b_1+\sqrt{1-x^2}} \leq \arcsin x \leq \frac{ax}{b_2+\sqrt{1-x^2}} \tag{1.6}$$

holds, where t_a *is a point in* $(0, \pi/2]$ *and satisfies* $a(t_a\cos t_a - \sin t_a) + t_a^2\sin t_a = 0$.

Theorem 1.6. *Let $0 \le x \le 1$, and $c, d_1, d_2 > 0$. If*

$$(c, d_1, d_2) \in \left\{c \ge 6, d_1 \ge c-2, d_2 \le \sqrt{2}\left(\frac{2c}{\pi}-1\right)\right\}$$
$$\bigcup\left\{6 > c > \frac{\pi\left(2-\sqrt{2}\right)}{\pi-2\sqrt{2}}, d_2 \le \sqrt{2}\left(\frac{2c}{\pi}-1\right), d_1 \ge \frac{c\sin t_c}{t_c} - 2\cos t_c\right\}$$
$$\bigcup\left\{\frac{\pi\left(2-\sqrt{2}\right)}{\pi-2\sqrt{2}} \ge c > \frac{\pi^2}{8-2\pi}, d_2 \le c-2, d_1 \ge \frac{c\sin t_c}{t_c} - 2\cos t_c\right\}$$
$$\bigcup\left\{\frac{\pi^2}{8-2\pi} \ge c > 2, d_1 \ge \frac{\sqrt{2}}{2}\left(\frac{4c}{\pi}-2\right), d_2 \le c-2\right\}, \tag{1.7}$$

then

$$\frac{c\left(\sqrt{1+x}-\sqrt{1-x}\right)}{d_1+\sqrt{1+x}+\sqrt{1-x}} \le \arcsin x \le \frac{c\left(\sqrt{1+x}-\sqrt{1-x}\right)}{d_2+\sqrt{1+x}+\sqrt{1-x}} \tag{1.8}$$

holds, where t_c is a point in $(0, \pi/4]$ and satisfies $c(t_c\cos t_c - \sin t_c) + 2t_c^2 \sin t_c = 0$.

2. One Lemma: L'Hospital's Rule for Monotonicity

Lemma 2.1 (see [9–15]). *Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions which are differentiable and $g' \ne 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions $(f(x) - f(b))/(g(x)-g(b))$ and $(f(x)-f(a))/(g(x)-g(a))$ are also increasing (or decreasing) on (a,b).*

3. Proofs of Theorems 1.5 and 1.6

(A) We first process the proof of Theorem 1.5.

Let $x = \sin t$ for $x \in (0,1]$, in which case the proof of Theorem 1.5 can be completed when proving that the double inequality

$$\frac{b_1}{a} \ge \frac{\sin t}{t} - \frac{\cos t}{a} \ge \frac{b_2}{a} \tag{3.1}$$

holds for $t \in (0, \pi/2]$.

Let $F(t) = (\sin t/t) - (\cos t/a)$, we have

$$F'(t) = \frac{t\cos t - \sin t}{t^2} + \frac{\sin t}{a} = \sin t\left(\frac{t\cos t - \sin t}{t^2 \sin t} + \frac{1}{a}\right) =: \sin t\left[H(t) + \frac{1}{a}\right], \tag{3.2}$$

where $H(t) = (t\cos t - \sin t)/(t^2\sin t) =: f_1(t)/g_1(t)$ and $f_1(t) = t\cos t - \sin t$, $g_1(t) = t^2 \sin t$, $f_1(0) = 0$, $g_1(0) = 0$.

Since $f_1'(t)/g_1'(t) = (-t\sin t)/(2t\sin t + t^2\cos t) = -(1/(2 + (t/\tan t))$ decreases on $(0,\pi/2]$, we obtain that $H(t)$ decreases on $(0,\pi/2]$ by using Lemma 2.1. At the same time, $H(0+0) = -1/3$, $H(\pi/2) = -4/\pi^2$, and $F(0+0) = 1-(1/a)$, $F(\pi/2) = 2/\pi$.

There are four cases to consider.

Case 1 *($a \geq 3$)*

Since $F'(t) \leq 0$, $F(t)$ decreases on $(0,\pi/2]$, and $\inf_{x\in(0,\pi/2]} F(t) = 2/\pi$, $\sup_{x\in(0,\pi/2]} F(t) = 1 - 1/a$. So when $b_1 \geq a-1$ and $b_2 \leq 2a/\pi$, (3.1) and (1.6) hold.

Case 2 *($3 > a > \pi/(\pi-2)$)*

At this moment, there exists a number $t_a \in (0,\pi/2]$ such that $a(t_a\cos t_a - \sin t_a) + t_a^2\sin t_a = 0$, $F'(t)$ is positive on $(0,t_a]$ and negative on $(t_a,\pi/2]$. That is, $F(t)$ firstly increases on $(0,t_a]$ then decreases on $(t_a,\pi/2]$, and $\inf_{x\in(0,\pi/2]} F(t) = 2/\pi$, $\sup_{x\in(0,\pi/2]} F(t) = F(t_a)$. So when $b_2 \leq 2a/\pi$ and $b_1 \geq a\sin t_a/t_a - \cos t_a$, (3.1) and (1.6) hold.

Case 3 *($\pi/(\pi-2) \geq a > \pi^2/4$)*

Now, $F(t)$ also firstly increases on $(0,t_a]$ then decreases on $(t_a,2/\pi]$, and $\inf_{x\in(0,\pi/2]} F(t) = 1-1/a$, $\sup_{x\in(0,\pi/2]} F(t) = F(t_a)$. So when $b_2 \leq a-1$ and $b_1 \geq a\sin t_a/t_a - \cos t_a$, (3.1) and (1.6) hold too.

Case 4 ($\pi^2/4 \geq a > 1$)

Since $F'(t) \geq 0$, $F(t)$ increases on $(0,\pi/2]$, $\inf_{x\in(0,\pi/2]} F(t) = 1-1/a$, and $\sup_{x\in(0,\pi/2]} F(t) = 2/\pi$. So when $b_1 \geq 2a/\pi$ and $b_2 \leq a-1$, (3.1) and (1.6) hold.

(B) Now we consider proving Theorem 1.6.

In view of the fact that (1.8) holds for $x = 0$, we suppose that $0 < x \leq 1$ in the following.

First, let $\sqrt{1+x} = \sqrt{2}\cos\alpha$ and $\sqrt{1-x} = \sqrt{2}\sin\alpha$ for $x \in (0,1]$, we have $x = \cos 2\alpha$ and $\alpha \in [0,\pi/4)$. Second, let $\alpha + \pi/4 = \pi/2 - t$, then $t \in (0,\pi/4]$ and (1.8) is equivalent to

$$\frac{d_1}{c} \geq \frac{\sin t}{t} - \frac{2\cos t}{c} \geq \frac{d_2}{c}. \tag{3.3}$$

When letting $c = 2a$ and $d_i = 2b_i$ ($i = 1,2$), (3.3) becomes (3.1).

Let $F(t) = \sin t/t - \cos t/a$. At this moment, $H(t)$ decreases on $(0,\pi/4]$, $H(0+0) = -1/3$, $H(\pi/4) = -(1-\pi/4)16/\pi^2$, and $F(0+0) = 1-2/c$, $F(\pi/4) = \sqrt{2}(2/\pi - 1/c)$.

There are four cases to consider too.

Case 1 *($c \geq 6$)*

Since $F'(t) \leq 0$, $F(t)$ decreases on $(0,\pi/4]$, and $\inf_{x\in(0,\pi/4]} F(t) = \sqrt{2}(2/\pi - 1/c)$, $\sup_{x\in(0,\pi/4]} F(t) = 1-2/c$. If $d_1 \geq c-2$ and $d_2 \leq \sqrt{2}(2c/\pi - 1)$, then (3.1) holds on $(0,\pi/4]$ and (1.8) holds.

Case 2 ($6 > c > (\pi(2-\sqrt{2}))/(\pi - 2\sqrt{2})$)

At this moment, there exists a number $t_a \in (0, \pi/4]$ such that $a(t_c \cos t_c - \sin t_c) + 2t_c^2 \sin t_c = 0$, $F'(t)$ is positive on $(0, t_c]$ and negative on $(t_c, \pi/4]$. That is, $F(t)$ firstly increases on $(0, t_c]$ then decreases on $(t_c, \pi/4]$, and $\inf_{x\in(0,\pi/4]} F(t) = \sqrt{2}((2/\pi) - (1/c))$, $\sup_{x\in(0,\pi/4]} F(t) = F(t_c)$. If $d_2 \le \sqrt{2}((2c/\pi - 1))$ and $d_1 \ge (c \sin t_c / t_c) - 2\cos t_c$, then (3.1) holds on $(0, \pi/4]$ and (1.8) holds.

Case 3 ($(\pi(2-\sqrt{2}))/(\pi - 2\sqrt{2}) \ge c > \pi^2/(8-2\pi)$)

Now, $F(t)$ also firstly increases on $(0, t_c]$ then decreases on $(t_c, \pi/4]$, and $\inf_{x\in(0,\pi/4]} F(t) = 1 - 2/c$, $\sup_{x\in(0,\pi/4]} F(t) = F(t_c)$. If $d_2 \le c - 2$ and $d_1 \ge (c \sin t_c / t_c) - 2\cos t_c$, then (3.1) holds on $(0, \pi/4]$ and (1.8) holds too.

Case 4 ($\pi^2/(8-2\pi) \ge c > 2$)

Since $F'(t) \ge 0$, $F(t)$ increases on $(0, \pi/4]$, $\inf_{x\in(0,\pi/4]} F(t) = 1 - 2/c$, and $\sup_{x\in(0,\pi/4]} F(t) = \sqrt{2}(2/\pi - 1/c)$. If $d_1 \ge \sqrt{2}(2c/\pi - 1)$ and $d_2 \le c - 2$, then (3.1) holds on $(0, \pi/4]$ and (1.8) holds.

4. The Special Cases of Theorems 1.5 and 1.6

(1) Taking $a = 3$, $b_1 = a - 1 = 2$ in Theorem 1.5 and $c = 6$, $d_1 = c - 2 = 4$ in Theorem 1.6 leads to the inequality (1.1).

(2) Taking $a = \pi/(\pi - 2)$, $b_2 = a - 1 = 2/(\pi - 2)$ in Theorem 1.5 and $c = (\pi(2-\sqrt{2}))/(\pi - 2\sqrt{2})$, $d_2 = c - 2 = \sqrt{2}(4-\pi)/(\pi - 2\sqrt{2})$ in Theorem 1.6 leads to the inequality (1.4).

(3) Let $a = \pi^2/4$, $b_1 = (2/\pi)a = \pi/2$ in Theorem 1.5 and $c = \pi^2/2(4-\pi)$, $d_1 = (2\sqrt{2}/\pi)c - \sqrt{2} = 2\sqrt{2}(\pi - 2)/(4 - \pi)$ in Theorem 1.6, we have the following result.

Theorem 4.1. *Let* $0 \le x \le 1$. *Then*

$$\frac{(\pi^2/4)x}{\pi/2 + \sqrt{1-x^2}} \le \frac{(\pi^2/(8-2\pi))\left(\sqrt{1+x} - \sqrt{1-x}\right)}{2\sqrt{2}(\pi-2)/(4-\pi) + \sqrt{1+x} + \sqrt{1-x}} \le \arcsin x. \tag{4.1}$$

Furthermore, $\pi^2/4$ *and* $\pi/2$, $\pi^2/(8-2\pi)$ *and* $2\sqrt{2}(\pi-2)/(4-\pi)$ *are the best constants in* (4.1).

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