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Research Article

A Generalized Wirtinger's Inequality with Applications to a Class of Ordinary Differential Equations

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We first prove a generalized Wirtinger's inequality. Then, applying the inequality, we study estimates for lower bounds of periods of periodic solutions for a class of delay differential equations $\dot{x}(t) = -\sum_{k=1}^{n} f(x(t-kr))$, and $\dot{x}(t) = -\sum_{k=1}^{n} g(t, x(t-ks))$, where $x \in \mathbb{R}^p$, $f \in C(\mathbb{R}^p, \mathbb{R}^p)$, and $g \in C(\mathbb{R} \times \mathbb{R}^p, \mathbb{R}^p)$ and $r > 0$, $s > 0$ are two given constants. Under some suitable conditions on f and g, lower bounds of periods of periodic solutions for the equations aforementioned are obtained.

1. Introduction and Statement of Main Results

In the present paper, we are concerned with a generalized Wirtinger's inequality and estimates for lower bounds of periods of periodic solutions for the following autonomous delay differential equation:

$$\dot{x}(t) = -\sum_{k=1}^{n} f(x(t-kr)), \tag{1.1}$$

and the following nonautonomous delay differential equation

$$\dot{x}(t) = -\sum_{k=1}^{n} g(t, x(t-ks)), \tag{1.2}$$

where $x \in \mathbb{R}^p$, $f \in C(\mathbb{R}^p, \mathbb{R}^p)$, and $g \in C(\mathbb{R} \times \mathbb{R}^p, \mathbb{R}^p)$, and $r > 0, s > 0$ are two given constants.

For the special case that $n = 1$ and $p = 1$, various problems on the solutions of (1.1), such as the existence of periodic solutions, bifurcations of periodic solutions, and stability of solutions, have been studied by many authors since 1970s of the last century, and a lot of remarkable results have been achieved. We refer to [1–6] for reference.

The delay equation (1.1) with more than one delay and $p = 1$ is also considered by a lot of researchers (see [7–13]). Most of the work contained in literature on (1.1) is the existence and multiplicity of periodic solutions. However, except the questions of the existence of periodic solutions with prescribed periods, little information was given on the periods of periodic solutions. Moreover, few work on the nonautonomous delay differential equation (1.2) has been done to the best of the author knowledge. Motivated by these cases, as a part of this paper, we study the estimates of periods of periodic solutions for the differential delay equation (1.1) and the nonautonomous equation (1.2). We first give a generalized Wirtinger's inequality. Then we turn to consider the problems on (1.1) and (1.2) by using the inequality.

In order to state our main results, we make the following definitions.

Definition 1.1. For a positive constant κ, $f(x) \in C(\mathbb{R}^p, \mathbb{R}^p)$ is called κ-Lipschitz continuous, if for all $x, y \in \mathbb{R}^p$,

$$\left|f(x) - f(y)\right| \le \kappa|x - y|, \tag{1.3}$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^p$.

Definition 1.2. For a positive constant κ, $g(t, x) \in C(\mathbb{R} \times \mathbb{R}^p, \mathbb{R}^p)$ is called κ-Lipschitz continuous uniformly in t, if for all $x, y \in \mathbb{R}^p$, and any $t \in \mathbb{R}$,

$$\left|g(t, x) - g(t, y)\right| \le \kappa|x - y|. \tag{1.4}$$

Then our main results read as follows.

Theorem 1.3. *Let x be a nontrivial T-periodic solution of the autonomous delay differential equation* (1.1) *with the second derivative. Suppose that the function $f : \mathbb{R}^p \to \mathbb{R}^p$ is κ-Lipschitz continuous. Then one has $T \ge 2\pi/n\kappa$.*

Theorem 1.4. *Let x be a nontrivial T-periodic solution of the nonautonomous delay differential equation* (1.2) *with the second derivative. Suppose that the function $g \in C(\mathbb{R} \times \mathbb{R}^p, \mathbb{R}^p)$ is T-periodic with respect to t and κ-Lipschitz continuous uniformly in t. If the following limit*

$$\lim_{u \to 0} \frac{\left|g(t + u, x) - g(t, x)\right|}{|u|} = h(t, x) \tag{1.5}$$

exists for all t and x and $h(t, x)$ is uniformly bounded, then one has $T \ge 2\pi/n\kappa$.

2. Proof of the Main Results

We will apply Wirtinger's inequality to prove the two theorems. Firstly, let us recall some notation concerning the Sobolev space. It is well known that $H_T^1(\mathbb{R}, \mathbb{R}^p)$ is a Hilbert space consisting of the T-periodic functions x on $\mathbb{R}$ which together with weak derivatives belong to $L^2(0,T;\mathbb{R}^p)$. For all $x,y \in L^2(0,T;\mathbb{R}^p)$, let $\langle x,y\rangle = \int_0^T (x,y)dt$ and $\|x\| = \sqrt{\langle x,x\rangle}$ denote the inner product and the norm in $L^2(0,T;\mathbb{R}^p)$, respectively, where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^p$. Then according to [14], we give Wirtinger's inequality and its proof.

Lemma 2.1. *If $x \in H_T^1$ and $\int_0^T x(t)dt = 0$, then*

$$\int_0^T |x(t)|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot{x}(t)|^2 dt. \tag{2.1}$$

Proof. By the assumptions, x has the following Fourier expansion:

$$x(t) = \sum_{m=-\infty,\, m\neq 0}^{+\infty} x_m \exp\left(\frac{2i\pi mt}{T}\right). \tag{2.2}$$

Then Parseval equality yields that

$$\begin{aligned}\int_0^T |\dot{x}(t)|^2 dt &= \sum_{m=-\infty,\, m\neq 0}^{+\infty} T\left(4\pi^2 m^2/T^2\right)|x_m|^2 \\ &\ge \frac{4\pi^2}{T^2}\sum_{m=-\infty,\, m\neq 0}^{+\infty} T|x_m|^2 \\ &= \frac{4\pi^2}{T^2}\int_0^T |x(t)|^2 dt.\end{aligned} \tag{2.3}$$

This completes the proof. □

Now, we generalize Wirtinger's inequality to a more general form which includes (2.1) as a special case. We prove the following lemma.

Lemma 2.2. *Suppose that $z \in H_T^1$ and $y \in L^2(0,T;\mathbb{R}^p)$ with $\int_0^T y(t)dt = 0$. Then*

$$|\langle z,y\rangle|^2 \le \frac{T^2}{4\pi^2}\|\dot{z}\|^2\|y\|^2. \tag{2.4}$$

Proof. Since $\int_0^T y(t)dt = 0$, by Lemma 2.1, we have

$$\int_0^T |y(t)|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot{y}(t)|^2 dt, \tag{2.5}$$

that is,

$$2\pi\|y\| \leq T\|\dot{y}\|. \tag{2.6}$$

Let c denote the average of $z \in L^2(0,T;\mathbb{R}^p)$, that is, $c = (1/T)\int_0^T z(t)dt$. This means that $\int_0^T (z(t) - c)dt = 0$. Hence, Schwarz inequality, together with (2.6) and $\int_0^T y(t)dt = 0$ implies that

$$\begin{aligned} |\langle z, y\rangle| &= |\langle z - c, y\rangle| \\ &\leq \|z - c\|\|y\| \\ &\leq \frac{T}{2\pi}\|\dot{z} - \dot{c}\|\|y\| \\ &= \frac{T}{2\pi}\|\dot{z}\|\|y\|. \end{aligned} \tag{2.7}$$

Then the proof is complete. □

Corollary 2.3. *Under the conditions of Lemma 2.1, the inequality* (2.4) *implies Wirtinger's inequality* (2.1)*.*

Proof. If $x \in H_T^1$ and $\int_0^T x(t)dt = 0$, then (2.1) follows (2.4) on taking $z = x = y$. □

We call (2.4) a generalized Wirtinger's inequality. For other study of Wirtinger's inequality, one may see [15] and the references therein. Now, we are ready to prove our main results. We first give the proof of Theorem 1.3.

Proof of Theorem 1.3. From (1.1) and Definition 1.1, for all $t, u \in \mathbb{R}$, one has

$$\begin{aligned} |\dot{x}(t+u) - \dot{x}(t)| &= \left|\sum_{k=1}^{n} f(x(t-kr+u)) - f(x(t-kr))\right| \\ &\leq \sum_{k=1}^{n} |f(x(t-kr+u)) - f(x(t-kr))| \\ &\leq \kappa\sum_{k=1}^{n} |x(t-kr+u) - x(t-kr)|. \end{aligned} \tag{2.8}$$

Hence, since x has the second derivative,

$$|\ddot{x}(t)| \leq \kappa(|\dot{x}(t-r)| + \cdots + |\dot{x}(t-nr)|). \tag{2.9}$$

Noting that $\dot{x}$ is also T-periodic, $\int_0^T |\dot{x}(t-k\tau)|^2 dt = \int_0^T |\dot{x}(t)|^2 dt$, for $k = 1, 2, \ldots, n$. Hence, by Hölder inequality, one has

$$\begin{aligned}
\int_0^T |\ddot{x}(t)|^2 dt &\le \kappa^2 \int_0^T \left(|\dot{x}(t-r)| + \cdots + |\dot{x}(t-nr)|\right)^2 dt \\
&= \kappa^2 \Bigg(\sum_{k=1}^{n} \int_0^T |\dot{x}(t-kr)|^2 dt + 2\sum_{k=2}^{n} \int_0^T |\dot{x}(t-r)||\dot{x}(t-kr)| dt \\
&\qquad + \cdots + \int_0^T |\dot{x}(t-(n-1)r)||\dot{x}(t-nr)| dt \Bigg) \\
&\le \kappa^2 \Bigg(\sum_{k=1}^{n} \int_0^T |\dot{x}(t-kr)|^2 dt + 2\sum_{k=2}^{n} \left(\int_0^T |\dot{x}(t-r)|^2 dt\right)^{1/2} \left(\int_0^T |\dot{x}(t-kr)|^2 dt\right)^{1/2} \\
&\qquad + \cdots + 2\left(\int_0^T |\dot{x}(t-(n-1)r)|^2 dt\right)^{1/2} \left(\int_0^T |\dot{x}(t-nr)|^2 dt\right)^{1/2} \Bigg) \\
&= \kappa^2 \left(n + 2(1 + 2 + \cdots + (n-1))\right) \int_0^T |\dot{x}(t)|^2 dt = n^2 \kappa^2 \int_0^T |\dot{x}(t)|^2 dt,
\end{aligned} \tag{2.10}$$

that is,

$$\|\ddot{x}\| \le n\kappa \|\dot{x}\| \Longrightarrow T\|\ddot{x}\| \le n\kappa T \|\dot{x}\|. \tag{2.11}$$

From (2.1) and $\int_0^T |\dot{x}(t)|^2 dt = 0$, we have

$$2\pi \|\dot{x}\| \le T\|\ddot{x}\|. \tag{2.12}$$

Combining (2.11) and (2.12), one has $T \ge 2\pi / n\kappa$. □

Now, we prove Theorem 1.4.

Proof of Theorem 1.4. From (1.2), Definition 1.2 and the assumptions of Theorem 1.4, for all $t, u \in \mathbb{R}$, one has

$$\begin{aligned}
|\dot{x}(t+u) - \dot{x}(t)| &= \left| \sum_{k=1}^{n} g(t+u, x(t-ks+u)) - g(t, x(t-ks)) \right| \\
&\le \sum_{k=1}^{n} \left| g(t+u, x(t-ks+u)) - g(t+u, x(t-ks)) \right| \\
&\quad + \sum_{k=1}^{n} \left| g(t+u, x(t-ks)) - g(t, x(t-ks)) \right| \\
&\le \kappa \sum_{k=1}^{n} |x(t+u-ks) - x(t-ks)| + \sum_{k=1}^{n} \left| g(t+u, x(t-ks)) - g(t, x(t-ks)) \right|.
\end{aligned} \tag{2.13}$$

Since $h(t,x)$ is nonnegative and uniformly bounded (for all t and x), there is $M \in \mathbb{R}^+$ such that $h(t,x) \le M$. Together with the fact that x has the second derivative, our estimates imply that

$$|\ddot{x}(t)| \le \kappa \sum_{k=1}^{n} |\dot{x}(t-ks)| + nh(t,x) \le \kappa \sum_{k=1}^{n} |\dot{x}(t-ks)| + nM. \tag{2.14}$$

As in the proof of Theorem 1.3, we get

$$\begin{aligned} \int_0^T |\ddot{x}(t)|^2 dt &\le \kappa^2 \int_0^T \left(\sum_{k=1}^{n} |\dot{x}(t-ks)| \right)^2 dt + 2\kappa n M \sum_{k=1}^{n} \int_0^T |\dot{x}(t-ks)| dt + n^2 M^2 T \\ &\le \kappa^2 n^2 \int_0^T |\dot{x}(t)|^2 dt + 2\kappa n^2 M \left(\int_0^T 1\, dt \right)^{1/2} \left(\int_0^T |\dot{x}(t)|^2 dt \right)^{1/2} + n^2 M^2 T \\ &= \kappa^2 n^2 \|\dot{x}\|^2 + 2\kappa n^2 M \sqrt{T} \|\dot{x}\| + n^2 M^2 T, \end{aligned} \tag{2.15}$$

that is,

$$T^2 \|\ddot{x}\|^2 \le T^2 \left(\kappa^2 n^2 \|\dot{x}\|^2 + 2\kappa n^2 M \sqrt{T} \|\dot{x}\| + n^2 M^2 T \right). \tag{2.16}$$

Thus, (2.1) together with (2.16) yields that

$$\varphi(\|\dot{x}\|) = \left(T^2 \kappa^2 n^2 - 4\pi^2 \right) \|\dot{x}\|^2 + 2T^2 \sqrt{T} \kappa n^2 M \|\dot{x}\| + T^3 n^2 M^2 \ge 0. \tag{2.17}$$

By an argument of Viete theorem with respect to the quadratic function $\varphi(\|\dot{x}\|)$, we have that

$$T^2 \kappa^2 n^2 - 4\pi^2 \ge 0 \Longrightarrow T \ge \frac{2\pi}{n\kappa}. \tag{2.18}$$

□

Remark 2.4. Roughly speaking, the period T can reach the lower bound $(2\pi)/(n\kappa)$. Let us take an example for (1.1). Take $p = 2$ and $n = 1$. For each $z \in \mathbb{R}^2 \cong \mathbb{C}$, we define a function f by

$$f(z) = -i \exp(-ir) z. \tag{2.19}$$

Then one can check easily that f is κ-Lipschitz continuous with $\kappa = 1$. Let $z(t) = \exp(-it)$. One has

$$\dot{z} = -i \exp(-it) = -i \exp\left(-i(t-r) \right) \exp(-ir) = -f\left(z(t-r) \right). \tag{2.20}$$

This means that $z(t) = \exp(-it)$ is a periodic solution of (1.2) with period $T = 2\pi$.

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