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Research Article

Superstability for Generalized Module Left Derivations and Generalized Module Derivations on a Banach Module (I)

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We discuss the superstability of generalized module left derivations and generalized module derivations on a Banach module. Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-module, $f : X \to X$ and $g : \mathcal{A} \to \mathcal{A}$. The mappings $\Delta^1_{f,g}$, $\Delta^2_{f,g}$, $\Delta^3_{f,g}$, and $\Delta^4_{f,g}$ are defined and it is proved that if $\|\Delta^1_{f,g}(x,y,z,w)\|$ (resp., $\|\Delta^3_{f,g}(x,y,z,w,\alpha,\beta)\|$) is dominated by $\varphi(x,y,z,w)$, then f is a generalized (resp., linear) module-$\mathcal{A}$ left derivation and g is a (resp., linear) module-X left derivation. It is also shown that if $\|\Delta^2_{f,g}(x,y,z,w)\|$ (resp., $\|\Delta^4_{f,g}(x,y,z,w,\alpha,\beta)\|$) is dominated by $\varphi(x,y,z,w)$, then f is a generalized (resp., linear) module-$\mathcal{A}$ derivation and g is a (resp., linear) module-X derivation.

1. Introduction

The study of stability problems had been formulated by Ulam in [1] during a talk in 1940: under what condition does there exist a homomorphism near an approximate homomorphism? In the following year 1941, Hyers in [2] has answered affirmatively the question of Ulam for Banach spaces, which states that if $\varepsilon > 0$ and $f : X \to Y$ is a map with X, a normed space, Y, a Banach space, such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \tag{1.1}$$

for all x, y in X, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon, \tag{1.2}$$

for all x in X. In addition, if the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x in X, then the mapping T is real linear. This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation $f(x+y) = f(x) + f(y)$. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki in [3] and for approximate linear mappings was presented by Rassias in [4] by considering the case when the left-hand side of (1.1) is controlled by a sum of powers of norms. The stability result concerning derivations between operator algebras was first obtained by Šemrl in [5], Badora in [6] gave a generalization of Bourgin's result [7]. He also discussed the Hyers-Ulam stability and the Bourgin-type superstability of derivations in [8].

Singer and Wermer in [9] obtained a fundamental result which started investigation into the ranges of linear derivations on Banach algebras. The result, which is called the Singer-Wermer theorem, states that any continuous linear derivation on a commutative Banach algebra maps into the Jacobson radical. They also made a very insightful conjecture, namely, that the assumption of continuity is unnecessary. This was known as the Singer- Wermer conjecture and was proved in 1988 by Thomas in [10]. The Singer-Wermer conjecture implies that any linear derivation on a commutative semisimple Banach algebra is identically zero [11]. After then, Hatori and Wada in [12] proved that the zero operator is the only derivation on a commutative semisimple Banach algebra with the maximal ideal space without isolated points. Based on these facts and a private communication with Watanabe [13], Miura et al. proved the Hyers-Ulam-Rassias stability and Bourgin-type superstability of derivations on Banach algebras in [13]. Various stability results on derivations and left derivations can be found in [14–20]. More results on stability and superstability of homomorphisms, special functionals, and equations can be found in [21–30].

Recently, Kang and Chang in [31] discussed the superstability of generalized left derivations and generalized derivations. Indeed, these superstabilities are the so-called "Hyers-Ulam superstabilities." In the present paper, we will discuss the superstability of generalized module left derivations and generalized module derivations on a Banach module.

To give our results, let us give some notations. Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbb{F}$ and X an $\mathcal{A}$-bimodule.

Definition 1.1. A mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be *module-Xadditive* if

$$xd(a+b) = xd(a) + xd(b), \quad \forall a, b \in \mathcal{A},\ x \in X. \tag{1.3}$$

A module-X additive mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a *module-X left derivation* (resp., *module-X derivation*) if the functional equation

$$xd(ab) = axd(b) + bxd(a), \quad \forall a, b \in \mathcal{A},\ x \in X \tag{1.4}$$

respectively,

$$xd(ab) = axd(b) + d(a)xb, \quad \forall a, b \in \mathcal{A},\ x \in X. \tag{1.5}$$

holds.

Definition 1.2. A mapping $f : X \to X$ is said to be *module-$\mathcal{A}$ additive* if

$$af(x_1 + x_2) = af(x_1) + af(x_2), \quad \forall x_1, x_2 \in X,\ a \in \mathcal{A}. \tag{1.6}$$

A module-$\mathcal{A}$ additive mapping $f : X \to X$ is called a *generalized module-$\mathcal{A}$ left derivation* (resp., *generalized module-$\mathcal{A}$ derivation*) if there exists a module-X left derivation (resp., module-X derivation) $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$af(bx) = abf(x) + ax\delta(b), \quad \forall x \in X,\ a, b \in \mathcal{A} \tag{1.7}$$

respectively,

$$af(bx) = abf(x) + a\delta(b)x, \quad \forall x \in X,\ a, b \in \mathcal{A}. \tag{1.8}$$

In addition, if the mappings f and δ are all linear, then the mapping f is called a *linear generalized module-$\mathcal{A}$ left derivation* (resp., *linear generalized module-$\mathcal{A}$ derivation*).

Remark 1.3. Let $\mathcal{A} = X$ and $\mathcal{A}$ be one of the following cases: (a) a unital algebra; (b) a Banach algebra with an approximate unit; (c) a C*-algebra. Then module-$\mathcal{A}$ left derivations, module-$\mathcal{A}$ derivations, generalized module-$\mathcal{A}$ left derivations, and generalized module-$\mathcal{A}$ derivations on $\mathcal{A}$ become left derivations, derivations, generalized left derivations, and generalized derivations on $\mathcal{A}$ discussed in [31].

2. Main Results

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra, X a Banach $\mathcal{A}$-bimodule, k and l integers greater than 1, and $\varphi : X \times X \times \mathcal{A} \times X \to [0, \infty)$ satisfy the following conditions:*

(a) $\lim_{n\to\infty} k^{-n}[\varphi(k^n x, k^n y, 0, 0) + \varphi(0, 0, k^n z, w)] = 0$, *for all* $x, y, w \in X, z \in \mathcal{A}$,

(b) $\lim_{n\to\infty} k^{-2n}\varphi(0, 0, k^n z, k^n w) = 0$, *for all* $z \in \mathcal{A}, w \in X$,

(c) $\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} k^{-n+1}\varphi(k^n x, 0, 0, 0) < \infty$ $(\forall x \in X)$.

Suppose that $f : X \to X$ and $g : \mathcal{A} \to \mathcal{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim_{n\to\infty}(1/k^n)g(k^n z)$ exists for all $z \in \mathcal{A}$ and

$$\left\| \Delta^1_{f,g}(x, y, z, w) \right\| \le \varphi(x, y, z, w) \tag{2.1}$$

for all $x, y, w \in X$ and $z \in \mathcal{A}$, where

$$\Delta^1_{f,g}(x, y, z, w) = f\left(\frac{x}{k} + \frac{y}{l} + zw\right) + f\left(\frac{x}{k} - \frac{y}{l} + zw\right) - \frac{2f(x)}{k} - 2zf(w) - 2wg(z). \tag{2.2}$$

Then f is a generalized module-$\mathcal{A}$ left derivation and g is a module-X left derivation.

Proof. By taking $w = z = 0$, we see from (2.1) that

$$\left\| f\left(\frac{x}{k}+\frac{y}{l}\right)+f\left(\frac{x}{k}-\frac{y}{l}\right)-\frac{2f(x)}{k}\right\| \le \varphi(x,y,0,0) \tag{2.3}$$

for all $x, y \in X$. Letting $y = 0$ and replacing x by kx in (2.3) yield that

$$\left\| f(x)-\frac{f(kx)}{k}\right\| \le \frac{1}{2}\varphi(kx,0,0,0) \tag{2.4}$$

for all $x \in X$. From [32, Theorem 1] (analogously as in [33, the proof of Theorem 1] or [34]), one can easily deduce that the limit $d(x) = \lim_{n\to\infty} f(k^n x)/k^n$ exists for every $x \in X$, $f(0) = d(0) = 0$ and

$$\|f(x)-d(x)\| \le \frac{1}{2}\widetilde{\varphi}(x), \quad \forall x \in X. \tag{2.5}$$

Next, we show that the mapping d is additive. To do this, let us replace x, y by $k^n x, k^n y$ in (2.3), respectively. Then

$$\left\| \frac{1}{k^n} f\left(\frac{k^n x}{k}+\frac{k^n y}{l}\right)+\frac{1}{k^n} f\left(\frac{k^n x}{k}-\frac{k^n y}{l}\right)-\frac{1}{k}\cdot\frac{2f(k^n x)}{k^n}\right\| \le k^{-n}\varphi(k^n x,k^n y,0,0) \tag{2.6}$$

for all $x, y \in X$. If we let $n \to \infty$ in the above inequality, then the condition (a) yields that

$$d\left(\frac{x}{k}+\frac{y}{l}\right)+d\left(\frac{x}{k}-\frac{y}{l}\right)=\frac{2}{k}d(x) \tag{2.7}$$

for all $x, y \in X$. Since $d(0) = 0$, taking $y = 0$ and $y = (l/k)x$, respectively, we see that $d(x/k) = d(x)/k$ and $d(2x) = 2d(x)$ for all $x \in X$. Now, for all $u, v \in X$, put $x = (k/2)(u + v)$, $y = (l/2)(u - v)$. Then by (2.7), we get that

$$d(u)+d(v)=d\left(\frac{x}{k}+\frac{y}{l}\right)+d\left(\frac{x}{k}-\frac{y}{l}\right)=\frac{2}{k}d(x)=\frac{2}{k}d\left(\frac{k}{2}(u+v)\right)=d(u+v). \tag{2.8}$$

This shows that d is additive.

Now, we are going to prove that f is a generalized module-$\mathcal{A}$ left derivation. Letting $x = y = 0$ in (2.1) gives that

$$\|f(zw)+f(zw)-2zf(w)-2wg(z)\| \le \varphi(0,0,z,w), \tag{2.9}$$

that is,

$$\|f(zw)-zf(w)-wg(z)\| \le \frac{1}{2}\varphi(0,0,z,w) \tag{2.10}$$

for all $z \in \mathcal{A}$ and $w \in X$. By replacing z, w with $k^n z, k^n w$ in (2.10), respectively, we deduce that

$$\left\| \frac{1}{k^{2n}} f\left(k^{2n} zw\right) - z\frac{1}{k^n} f(k^n w) - w\frac{1}{k^n} g(k^n z) \right\| \leq \frac{1}{2} k^{-2n} \varphi(0, 0, k^n z, k^n w) \tag{2.11}$$

for all $z \in \mathcal{A}$ and $w \in X$. Letting $n \to \infty$, the condition (b) yields that

$$d(zw) = zd(w) + w\delta(z) \tag{2.12}$$

for all $z \in \mathcal{A}$ and $w \in X$. Since d is additive, δ is module-X additive. Put $\Delta(z, w) = f(zw) - zf(w) - wg(z)$. Then by (2.10) we see from the condition (a) that

$$k^{-n} \|\Delta(k^n z, w)\| \leq \frac{1}{2} k^{-n} \varphi(0, 0, k^n z, w) \longrightarrow 0 \quad (n \to \infty) \tag{2.13}$$

for all $z \in \mathcal{A}$ and $w \in X$. Hence

$$\begin{aligned} d(zw) &= \lim_{n\to\infty} \frac{f(k^n z \cdot w)}{k^n} \\ &= \lim_{n\to\infty} \left(\frac{k^n z f(w) + wg(k^n z) + \Delta(k^n z, w)}{k^n} \right) \\ &= zf(w) + w\delta(z) \end{aligned} \tag{2.14}$$

for all $z \in \mathcal{A}$ and $w \in X$. It follows from (2.12) that $zf(w) = zd(w)$ for all $z \in \mathcal{A}$ and $w \in X$, and then $d(w) = f(w)$ for all $w \in X$. Since d is additive, f is module-$\mathcal{A}$ additive. So, for all $a, b \in \mathcal{A}$ and $x \in X$ by (2.12)

$$\begin{aligned} af(bx) &= ad(bx) = abf(x) + ax\delta(b), \\ x\delta(ab) &= d(abx) - abf(x) \\ &= af(bx) + bx\delta(a) - abf(x) \\ &= a\big(d(bx) - bf(x)\big) + bx\delta(a) \\ &= ax\delta(b) + bx\delta(a). \end{aligned} \tag{2.15}$$

This shows that δ is a module-X left derivation on $\mathcal{A}$ and then f is a generalized module-$\mathcal{A}$ left derivation on X.

Lastly, we prove that g is a module-X left derivation on $\mathcal{A}$. To do this, we compute from (2.10) that

$$\left\| \frac{f(k^n zw)}{k^n} - z\frac{f(k^n w)}{k^n} - wg(z) \right\| \leq \frac{1}{2} k^{-n} \varphi(0, 0, z, k^n w) \tag{2.16}$$

for all $z \in \mathcal{A}, w \in X$. By letting $n \to \infty$, we get from the condition (a) that

$$d(zw) = zd(w) + wg(z) \tag{2.17}$$

for all $z \in \mathcal{A}, w \in X$. Now, (2.12) implies that $wg(z) = w\delta(z)$ for all $z \in \mathcal{A}$ and all $w \in X$. Hence, g is a module-X left derivation on $\mathcal{A}$. This completes the proof. □

Remark 2.2. It is easy to check that the functional $\varphi(x,y,z,w) = \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$ satisfies the conditions (a), (b), and (c) in Theorem 2.1, where $\varepsilon \geq 0$, $p,q,s,t \in [0,1)$. Especially, if $\mathcal{A}$ has a unit and $f,g : \mathcal{A} \to \mathcal{A}$ are mappings with $f(0) = 0$ such that $\|\Delta^1_{f,g}(x,y,z,w)\| \leq \varepsilon$ for all $x,y,w,z \in \mathcal{A}$, then f is a generalized left derivation and g is a left derivation.

Remark 2.3. In Theorem 2.1, if the condition (2.1) is replaced with

$$\left\|\Delta^2_{f,g}(x,y,z,w)\right\| \leq \varphi(x,y,z,w) \tag{2.18}$$

for all $x,y,w \in X$ and $z \in \mathcal{A}$ where

$$\Delta^2_{f,g}(x,y,z,w) = f\left(\frac{x}{k} + \frac{y}{l} + zw\right) + f\left(\frac{x}{k} - \frac{y}{l} + zw\right) - \frac{2f(x)}{k} - 2zf(w) - 2g(z)w, \tag{2.19}$$

then f is a generalized module-$\mathcal{A}$ derivation and g is a module-X derivation. Especially, if $\mathcal{A}$ has a unit and $f,g : \mathcal{A} \to \mathcal{A}$ are mappings with $f(0) = 0$ such that $\|\Delta^2_{f,g}(x,y,z,w)\| \leq \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$ for all $x,y,w,z \in \mathcal{A}$ and some constants $p,q,s,t \in [0,1)$, then f is a generalized derivation and g is a derivation.

Lemma 2.4. *Let X, Y be complex vector spaces. Then a mapping $f : X \to Y$ is linear if and only if*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \tag{2.20}$$

for all $x,y \in X$ and all $\alpha,\beta \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Proof. It suffices to prove the sufficiency. Suppose that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x,y \in X$ and all $\alpha,\beta \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then f is additive and $f(\alpha x) = \alpha f(x)$ for all $x \in X$ and all $\alpha \in \mathbb{T}$. Let α be any nonzero complex number. Take a positive integer n such that $|\alpha/n| < 2$. Take a real number θ such that $0 \leq a := e^{-i\theta}\alpha/n < 2$. Put $\beta = \arccos(a/2)$. Then $\alpha = n(e^{i(\beta+\theta)} + e^{-i(\beta-\theta)})$ and, therefore,

$$f(\alpha x) = nf\left(e^{i(\beta+\theta)}x\right) + nf\left(e^{-i(\beta-\theta)}x\right) = ne^{i(\beta+\theta)}f(x) + ne^{-i(\beta-\theta)}f(x) = \alpha f(x) \tag{2.21}$$

for all $x \in X$. This shows that f is linear. The proof is completed. □

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra, X a Banach $\mathcal{A}$-bimodule, k and l integers greater than* 1*, and* $\varphi : X \times X \times \mathcal{A} \times X \to [0,\infty)$ *satisfy the following conditions:*

(a) $\lim_{n\to\infty} k^{-n}[\varphi(k^n x, k^n y, 0, 0) + \varphi(0,0,k^n z, w)] = 0$, *for all* $x, y, w \in X$, $z \in \mathcal{A}$,

(b) $\lim_{n\to\infty} k^{-2n}\varphi(0,0,k^n z, k^n w) = 0$, *for all* $z \in \mathcal{A}$, $w \in X$.

(c) $\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} k^{-n+1}\varphi(k^n x, 0,0,0) < \infty$, *for all* $x \in X$.

Suppose that $f : X \to X$ *and* $g : \mathcal{A} \to \mathcal{A}$ *are mappings such that* $f(0) = 0$, $\delta(z) := \lim_{n\to\infty}(1/k^n)g(k^n z)$ *exists for all* $z \in \mathcal{A}$ *and*

$$\left\| \Delta^3_{f,g}(x,y,z,w,\alpha,\beta) \right\| \le \varphi(x,y,z,w) \tag{2.22}$$

for all $x, y, w \in X$, $z \in \mathcal{A}$ *and all* $\alpha, \beta \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, *where* $\Delta^3_{f,g}(x,y,z,w,\alpha,\beta)$ *stands for*

$$f\left(\frac{\alpha x}{k} + \frac{\beta y}{l} + zw\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l} + zw\right) - \frac{2\alpha f(x)}{k} - 2zf(w) - 2wg(z). \tag{2.23}$$

Then f is a linear generalized module-$\mathcal{A}$ left derivation and g is a linear module-X left derivation.

Proof. Clearly, the inequality (2.1) is satisfied. Hence, Theorem 2.1 and its proof show that f is a generalized left derivation and g is a left derivation on $\mathcal{A}$ with

$$f(x) = \lim_{n\to\infty}\frac{f(k^n x)}{k^n}, \qquad g(x) = f(x) - xf(e) \tag{2.24}$$

for every $x \in X$. Taking $z = w = 0$ in (2.22) yields that

$$\left\| f\left(\frac{\alpha x}{k} + \frac{\beta y}{l}\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l}\right) - \frac{2\alpha f(x)}{k} \right\| \le \varphi(x,y,0,0) \tag{2.25}$$

for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. If we replace x and y with $k^n x$ and $k^n y$ in (2.25), respectively, then we see that

$$\begin{aligned} &\left\| \frac{1}{k^n} f\left(\frac{\alpha k^n x}{k} + \frac{\beta k^n y}{l}\right) + \frac{1}{k^n} f\left(\frac{\alpha k^n x}{k} - \frac{\beta k^n y}{l}\right) - \frac{1}{k^n}\frac{2\alpha f(k^n x)}{k} \right\| \\ &\quad \le k^{-n}\varphi(k^n x, k^n y, 0, 0) \\ &\quad \longrightarrow 0 \end{aligned} \tag{2.26}$$

as $n \to \infty$ for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. Hence,

$$f\left(\frac{\alpha x}{k} + \frac{\beta y}{l}\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l}\right) = \frac{2\alpha f(x)}{k} \tag{2.27}$$

for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. Since f is additive, taking $y = 0$ in (2.27) implies that

$$f(\alpha x) = \alpha f(x) \tag{2.28}$$

for all $x \in X$ and all $\alpha \in \mathbb{T}$. Lemma 2.4 yields that f is linear and so is g. This completes the proof. □

Remark 2.6. It is easy to check that the functional $\varphi(x, y, z, w) = \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$ satisfies the conditions (a), (b), and (c) in Theorem 2.5, where $\varepsilon \geq 0$, $p, q, s, t \in [0, 1)$ are constants. Especially, if $\mathcal{A}$ is a complex semiprime Banach algebra with unit and $f, g : \mathcal{A} \to \mathcal{A}$ are mappings with $f(0) = 0$ such that

$$\left\|\Delta^3_{f,g}(x, y, z, w, \alpha, \beta)\right\| \leq \varepsilon\left(\|x\|^p + \left\|y\right\|^q + \|z\|^s\|w\|^t\right) \tag{2.29}$$

for all $x, y, w, z \in \mathcal{A}, \alpha, \beta \in \mathbb{T}$. Then f is a linear generalized left derivation and g is a linear derivation which maps $\mathcal{A}$ into the intersection of the center $Z(\mathcal{A})$ and the Jacobson radical rad $(\mathcal{A})$ of $\mathcal{A}$.

Remark 2.7. In Theorem 2.5, if the condition (2.22) is replaced with

$$\left\|\Delta^4_{f,g}(x, y, z, w, \alpha, \beta)\right\| \leq \varphi(x, y, z, w) \tag{2.30}$$

for all $x, y, w \in X$, $z \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{T}$ where $\Delta^4_{f,g}(x, y, z, w, \alpha, \beta)$ stands for

$$f\left(\frac{\alpha x}{k} + \frac{\beta y}{l} + zw\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l} + zw\right) - \frac{2\alpha f(x)}{k} - 2zf(w) - 2g(z)w, \tag{2.31}$$

then f is a linear generalized module-$\mathcal{A}$ derivation on X and g is a linear module-X derivation on $\mathcal{A}$. Especially, if $\mathcal{A}$ is a unital commutative Banach algebra and $f, g : \mathcal{A} \to \mathcal{A}$ are mappings with $f(0) = 0$ such that $\|\Delta^4_{f,g}(x, y, z, w, \alpha, \beta)\| \leq \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$ for all $x, y, w, z \in \mathcal{A}$, all $\alpha, \beta \in \mathbb{T}$ and some constants $p, q, s, t \in [0, 1)$, then f is a linear generalized derivation and g is a linear derivation which maps $\mathcal{A}$ into the Jacobson radical $\mathrm{rad}(\mathcal{A})$ of $\mathcal{A}$.

Remark 2.8. The controlling function

$$\varphi(x, y, z, w) = \varepsilon\left(\|x\|^p + \left\|y\right\|^q + \|z\|^s\|w\|^t\right) \tag{2.32}$$

consists of the "mixed sum-product of powers of norms," introduced by Rassias (in 2007) [28] and applied afterwards by Ravi et al. (2007-2008) . Moreover, it is easy to check that the functional

$$\varphi(x, y, z, w) = P\|x\|^p + Q\left\|y\right\|^q + S\|z\|^s + T\|w\|^t \tag{2.33}$$

satisfies the conditions (a), (b), and (c) in Theorems 2.1 and 2.5, where $P, Q, T, S \in [0, \infty)$ and $p, q, s, t \in [0, 1)$ are all constants.

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