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Research Article

Inequalities for Single Crystal Tube Growth by Edge-Defined Film-Fed Growth Technique

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The axi-symmetric Young-Laplace differential equation is analyzed. Solutions of this equation can describe the outer or inner free surface of a static meniscus (the static liquid bridge free surface between the shaper and the crystal surface) occurring in single crystal tube growth. The analysis concerns the dependence of solutions of the equation on a parameter p which represents the controllable part of the pressure difference across the free surface. Inequalities are established for p which are necessary or sufficient conditions for the existence of solutions which represent a stable and convex outer or inner free surfaces of a static meniscus. The analysis is numerically illustrated for the static menisci occurring in silicon tube growth by edge-defined film-fed growth (EFGs) technique. This kind of inequalities permits the adequate choice of the process parameter p. With this aim this study was undertaken.

1. Introduction

The first successful Si tube growth was reported in [1]. Also a theory of tube growth by E.F.G. process is developed there to show the dependence of the tube wall thickness on the growth variables. The theory concerns the calculation of the shape of the liquid-vapor interface (i.e., the free surface of the meniscus) and of the heat flow in the system. The inner and the outer free surface shapes of the meniscus (Figure 1) were calculated from Young-Laplace capillary equation, in which the pressure difference Δp across a point on the free surface is considered to be $\Delta p = \rho \cdot g \cdot H_{\text{eff}} = \text{constant}$, where H_{eff} represents the effective height of the growth interface (Figure 1). The above approximation of Δp is valid when $H_{\text{eff}} \gg h$, where h is the height of the growth interface above the shaper top. Another approximation used in [1] is that the outer and inner free surface shapes are approximated by circular segments. With these relatively tight tolerances concerning the menisci in conjunction with the heat flow calculation in the system, the predictive model developed in [1] has been shown to be useful tool understanding the feasible limits of the wall thickness control. A more accurate predictive model would require an increase of the acceptable tolerance range introduced by approximation.

The growth process was scaled up by Kaljes et al. in [2] to grow 15 cm diameter silicon tubes. It has been realized that theoretical investigations are necessary for the improvement of the technology. Since the growth system consists of a small die type (1 mm width) and a thin tube (order of 200 μm wall thickness), the width of the melt/solid interface and the meniscus are accordingly very small. Therefore, it is essential to obtain accurate solution for the free surface of the meniscus, the temperature, and the liquid-crystal interface position in this tinny region.

For single crystal tube growth by edge-defined film-fed growth (E.F.G.) technique, in hydrostatic approximation the free surface of a static meniscus is described by the Young-Laplace capillary equation [3]:

$$\gamma \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \rho \cdot g \cdot z - p. \tag{1.1}$$

Here γ is the melt surface tension, ρ denotes the melt density, g is the gravity acceleration, $1/R_1$, $1/R_2$ denote the main normal curvatures of the free surface at a point M of the free surface, z is the coordinate of M with respect to the Oz axis, directed vertically upwards, and p is the pressure difference across the free surface. For the outer free surface, $p = p_e = p_m - p_g^e - \rho \cdot g \cdot H$ and for the inner free surface, $p = p_i = p_m - p_g^i - \rho \cdot g \cdot H$.

Here p_m denotes the hydrodynamic pressure in the meniscus melt, p_g^e, p_g^i denote the pressure of the gas flow introduced in the furnace in order to release the heat from the outer and inner walls of the tube, respectively, and H denotes the melt column height between the horizontal crucible melt level and the shaper top level. When the shaper top level is above the crucible melt level, then $H > 0$, and when the crucible melt level is above the shaper top level, then $H < 0$ (see Figure 1).

To calculate the outer and inner free surface shapes of the static meniscus, it is convenient to employ the Young-Laplace (1.1) in its differential form. This form of the (1.1) can be obtained as a necessary condition for the minimum of the free energy of the melt column [3].

For the growth of a single crystal tube of inner radius $r_i \in (R_{gi}, (R_{gi} + R_{ge})/2)$ and outer radius $r_e \in ((R_{gi} + R_{ge})/2, R_{ge})$ the axi-symmetric differential equation of the outer free surface is given by

$$z'' = \frac{\rho \cdot g \cdot z - p_e}{\gamma} \left[1 + (z')^2 \right]^{3/2} - \frac{1}{r} \cdot \left[1 + (z')^2 \right] \cdot z' \quad \text{for } r \in [r_e, R_{ge}], \tag{1.2}$$

which is the Euler equation for the energy functional

$$I_e(z) = \int_{r_e}^{R_{ge}} \left\{ \gamma \cdot \left[1 + (z')^2 \right]^{1/2} + \frac{1}{2} \cdot \rho \cdot g \cdot z^2 - p_e \cdot z \right\} \cdot r \cdot dr, \qquad z(r_e) = h_e > 0, \quad z(R_{ge}) = 0. \tag{1.3}$$

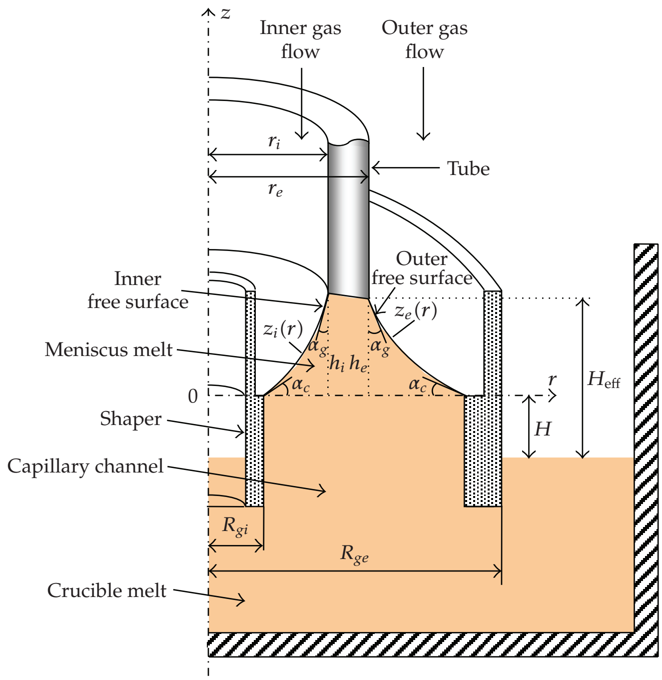

Figure 1: Axisymmetric meniscus geometry in the tube growth by E.F.G. method.

The axi-symmetric differential equation of the inner free surface is given by

$$z'' = \frac{\rho \cdot g \cdot z - p_i}{\gamma}\left[1 + (z')^2\right]^{3/2} - \frac{1}{r} \cdot \left[1 + (z')^2\right] \cdot z' \quad \text{for } r \in \left[R_{gi}, r_i\right], \tag{1.4}$$

which is the Euler equation for the energy functional

$$I_i(z) = \int_{R_{gi}}^{r_i} \left\{\gamma \cdot \left[1 + (z')^2\right]^{1/2} + \frac{1}{2} \cdot \rho \cdot g \cdot z^2 - p_i \cdot z\right\} \cdot r \cdot dr,$$
$$z(R_{gi}) = 0, \quad z(r_i) = h_i > 0. \tag{1.5}$$

The state of the arts at the time 1993-1994, concerning the dependence of the shape of the meniscus free surface on the pressure difference p for small and large bond numbers, in the case of the growth of single crystal rods by E.F.G. technique, are summarized in [4]. According to [4], for the general differential equation (1.2), (1.4) describing the free surface of a liquid meniscus, there are no complete analysis and solution. For the general equation only numerical integrations were carried out for a number of process parameter values that were of practical interest at the moment.

Later, in 2001, Rossolenko shows in [5] that the hydrodynamic factor is too small to be considered in the automated single crystal tube growth. Finally, in [6] the authors present theoretical and numerical study of meniscus dynamics under axi-symmetric and asymmetric configurations. In [6] the meniscus free surface is approximated by an arc of constant curvature, and a meniscus dynamics model is developed to consider meniscus shape and its dynamics, heat and mass transfer around the die-top and meniscus. Analysis reveals the correlations among tube wall thickness, effective melt height, pull-rate, die-top temperature, and crystal environmental temperature.

In the present paper the shape of the inner and outer free surfaces of the static meniscus is analyzed as function of p, the controllable part of the pressure difference across the free surface, and the static stability of the free surfaces is investigated. The novelty with respect to the considerations presented in literature consists in the fact that the free surface is not approximated as in [1, 6], by an arc with constant curvature, and the pressure of the gas flow introduced in the furnace for releasing the heat from the tube wall is taken in consideration. The setting of the thermal conditions is not considered in this paper.

2. Meniscus Outer Free Surface Shape Analysis in the Case of Tube Growth

Consider the differential equation

$$z'' = \frac{\rho \cdot g \cdot z - p_e}{\gamma}\left[1 + (z')^2\right]^{3/2} - \frac{1}{r} \cdot \left[1 + (z')^2\right] \cdot z' \quad \text{for } \frac{R_{gi} + R_{ge}}{2} \le r \le R_{ge}, \tag{2.1}$$

and α_c, α_g such that $0 < \alpha_c < \pi/2 - \alpha_g$, $\alpha_g \in (0, \pi/2)$.

Definition 2.1. A solution $z = z(x)$ of the (2.1) describes the outer free surface of a static meniscus on the interval $[r_e, R_{ge}]$ $((R_{gi} + R_{ge})/2 < r_e < R_{ge})$ if possesses the following properties:

(a) $z'(r_e) = -\tan(\pi/2 - \alpha_g)$,

(b) $z'(R_{ge}) = -\tan\alpha_c$, and

(c) $z(R_{ge}) = 0$ and $z(r)$ is strictly decreasing on $[r_e, R_{ge}]$.

The described outer free surface is convex on $[r_e, R_{ge}]$ if in addition the following inequality holds:

(d) $z''(r) > 0\ \forall r \in [r_e, R_{ge}]$.

Theorem 2.2. *If there exists a solution of* (2.1)*, which describes a convex outer free surface of a static meniscus on the closed interval* $[r_e, R_{ge}]$*, then the following inequalities hold:*

$$\begin{aligned} &-\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge} - r_e} \cdot \cos\alpha_c + \frac{\gamma}{R_{ge}} \cdot \sin\alpha_c \\ &\quad \le p_e \le -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge} - r_e} \cdot \sin\alpha_g + \rho \cdot g \cdot \left[R_{ge} - r_e\right] \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) + \frac{\gamma}{r_e} \cdot \cos\alpha_g. \end{aligned} \tag{2.2}$$

Proof. Let $z(r)$ be a solution of (2.1), which describes a convex outer free surface of a static meniscus on the closed interval $[r_e, R_{ge}]$ and $\alpha(r) = -\arctan z'(r)$. The function $\alpha(r)$ verifies the equation

$$\alpha'(r) = \frac{p_e - \rho \cdot g \cdot z(r)}{\gamma} \cdot \frac{1}{\cos \alpha(r)} - \frac{1}{r} \cdot \tan \alpha(r) \tag{2.3}$$

and the boundary conditions

$$\alpha(r_e) = \frac{\pi}{2} - \alpha_g, \qquad \alpha(R_{ge}) = \alpha_c. \tag{2.4}$$

Hence, there exists $r' \in (r_e, R_{ge})$ such that the following equality holds:

$$p_e = -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge} - r_e} \cdot \cos \alpha(r') + \rho \cdot g \cdot z(r') + \frac{\gamma}{r'} \cdot \sin \alpha(r'). \tag{2.5}$$

Since $z''(r) > 0$ on $[r_e, R_{ge}]$, $z'(r)$ is strictly increasing and $\alpha(r) = -\arctan z'(r)$ is strictly decreasing on $[r_e, R_{ge}]$, therefore the values of $\alpha(r)$ are in the range $[\alpha_c, \pi/2 - \alpha_g]$ and the following inequalities hold:

$$\begin{gathered} \alpha_c \le \alpha(r') \le \frac{\pi}{2} - \alpha_g, \\ \sin \alpha_g \le \cos \alpha(r') \le \cos \alpha_c, \\ \sin \alpha_c \le \sin \alpha(r') \le \cos \alpha_g, \\ -\tan\left(\frac{\pi}{2} - \alpha_g\right) \le z'(r) \le -\tan \alpha_c, \\ \rho \cdot g \cdot (R_{ge} - r) \cdot \tan \alpha_c \le \rho \cdot g \cdot z(r) \le \rho \cdot g \cdot (R_{ge} - r) \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right). \end{gathered} \tag{2.6}$$

Equality (2.5) and inequalities (2.6) imply inequalities (2.2). □

Corollary 2.3. *If $r_e = R_{ge}/n$ with $1 < n < 2 \cdot R_{ge}/(R_{gi} + R_{ge})$, then inequalities* (2.2) *become*

$$\begin{aligned} &-\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge}} \cdot \frac{n}{n-1} \cdot \cos \alpha_c + \frac{\gamma}{R_{ge}} \cdot \sin \alpha_c \\ &\quad \le p_e \le -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge}} \cdot \frac{n}{n-1} \cdot \sin \alpha_g + \frac{\rho \cdot g \cdot R_{ge} \cdot (n-1)}{n} \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) \\ &\qquad + \frac{\gamma}{R_{ge}} \cdot n \cdot \cos \alpha_g. \end{aligned} \tag{2.7}$$

Corollary 2.4. *If $n \to 2 \cdot R_{ge}/(R_{gi} + R_{ge})$, then $r_e \to (R_{gi} + R_{ge})/2$ and (2.7) becomes*

$$-2 \cdot \gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge} - R_{gi}} \cdot \cos\alpha_c + \frac{\gamma}{R_{ge}} \cdot \sin\alpha_c$$
$$\leq p_e \leq -2 \cdot \gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge} - R_{gi}} \cdot \sin\alpha_g + \frac{\rho \cdot g \cdot (R_{ge} - R_{gi})}{2} \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) \tag{2.8}$$
$$+ \frac{2 \cdot \gamma}{R_{gi} + R_{ge}} \cdot \cos\alpha_g.$$

If $n \to 1$, then $r_e \to R_{ge}$ and $p_e \to -\infty$.

Theorem 2.5. *Let n be such that $1 < n < 2 \cdot R_{ge}/(R_{gi} + R_{ge})$. If $\mathrm{p_e}$ satisfies the inequality*

$$p_e < -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge}} \cdot \frac{n}{n-1} \cdot \cos\alpha_c + \frac{\gamma}{R_{ge}} \cdot \sin\alpha_c, \tag{2.9}$$

then there exists $\mathrm{r_e}$ in the closed interval $[R_{ge}/n, R_{ge}]$ such that the solution of the initial value problem

$$z'' = \frac{\rho \cdot g \cdot z - p_e}{\gamma} \cdot \left[1 + (z')^2\right]^{3/2} - \frac{1}{r} \cdot \left[1 + (z')^2\right] \cdot z' \quad \text{for } \frac{R_{gi} + R_{ge}}{2} < r \leq R_{ge}$$
$$z(R_{ge}) = 0, \quad z'(R_{ge}) = -\tan\alpha_c, \quad 0 < \alpha_c < \frac{\pi}{2} - \alpha_g, \ \alpha_g \in \left(0, \frac{\pi}{2}\right) \tag{2.10}$$

on the interval $[r_e, R_{ge}]$ describes the convex outer free surface of a static meniscus.

Proof. Consider the solution $z(r)$ of the initial value problem (2.10). Denote by I the maximal interval on which the function $z(r)$ exists and by $\alpha(r)$ the function $\alpha(r) = -\arctan z'(r)$ defined on I. Remark that for $\alpha(r)$ the equality (2.3) holds.

Since

$$z''(R_{ge}) > 0, \quad z'(R_{ge}) = -\tan\alpha_c < 0, \quad z'(R_{ge}) > -\tan\left(\frac{\pi}{2} - \alpha_g\right), \tag{2.11}$$

there exists $r' \in I$, $0 < r' < R_{ge}$ such that for any $r \in [r', R_{ge}]$ the following inequalities hold:

$$z''(r) > 0, \quad z'(r) \leq -\tan\alpha_c, \quad z'(r) \geq -\tan\left(\frac{\pi}{2} - \alpha_g\right). \tag{2.12}$$

Let r_* be defined by

$$r_* = \inf\{r' \in I \mid 0 < r' < R_{ge} \text{ such that for any } r \in [r', R_{ge}] \text{ inequalities (2.12) hold}\}. \tag{2.13}$$

It is clear that $r_* \geq 0$ and for any $r \in (r_*, R_{ge}]$ inequalities (2.12) hold.

From (2.12) and (2.13) it follows that $z'(r)$ is strictly increasing and bounded on $(r_*, R_{ge}]$. Therefore $z'(r_* + 0) = \lim_{r \to r_*, r > r_*} z'(r)$ exists and satisfies

$$-\tan\left(\frac{\pi}{2} - \alpha_g\right) \le z'(r_* + 0) \le -\tan\alpha_c. \tag{2.14}$$

Moreover, since $z(r)$ is strictly decreasing and possesses bounded derivative on $(r_*, R_{ge}]$, $z(r_* + 0) = \lim_{r \to r_*, r > r_*} z(r)$ exists too, it is finite, and satisfies

$$0 < \left(R_{ge} - r_*\right) \cdot \tan\alpha_c \le z(r_* + 0) \le \left(R_{ge} - r_*\right) \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) < +\infty. \tag{2.15}$$

We will show now that $r_* > R_{ge}/n$ and $z'(r_* + 0) = -\tan(\pi/2 - \alpha_g)$. In order to show that $r_* > R_{ge}/n$ we assume the contrary, that is, that $r_* \le R_{ge}/n$. Under this hypothesis we have

$$\begin{aligned}
\alpha(r_* + 0) - \alpha\left(R_{ge}\right) &= -\alpha'\left(r'\right) \cdot \left[R_{ge} - r_*\right] \\
&= \left[-\frac{p_e}{\gamma} + \frac{\rho \cdot g \cdot z(r')}{\gamma} + \frac{\sin\alpha(r')}{r'}\right] \cdot \frac{R_{ge} - r_*}{\cos\alpha(r')} \\
&> \left[\frac{\pi/2 - \left(\alpha_c + \alpha_g\right)}{R_{ge}} \cdot \frac{n}{n-1} \cdot \cos\alpha_c - \frac{1}{R_{ge}} \cdot \sin\alpha_c + \frac{\rho \cdot g \cdot z(r')}{\gamma} + \frac{\sin\alpha(r')}{r'}\right] \\
&\quad \cdot \frac{R_{ge} - r_*}{\cos\alpha(r')} \\
&> \frac{\pi}{2} - \left(\alpha_c + \alpha_g\right)
\end{aligned} \tag{2.16}$$

for some $r' \in (r_*, R_{ge})$. Hence $\alpha(r_* + 0) > \pi/2 - \alpha_g$. This last inequality is impossible, since according to the inequality (2.14), we have $\alpha(r_* + 0) \le \pi/2 - \alpha_g$. Therefore, r_*, defined by (2.14), satisfies $r_* > R_{ge}/n$.

In order to show that $z'(r_* + 0) = -\tan(\pi/2 - \alpha_g)$ we remark that from the definition (2.14) of r_* it follows that at least one of the following three equalities holds:

$$z'(r_* + 0) = -\tan\alpha_c, \qquad z'(r_* + 0) = -\tan\left(\frac{\pi}{2} - \alpha_g\right), \qquad z''(r_* + 0) = 0. \tag{2.17}$$

Since $z'(r_* + 0) < z'(r) \le -\tan\alpha_c$ for any $r \in (r_*, R_{ge}]$ it follows that the equality $z'(r_* + 0) = -\tan\alpha_c$ is impossible. Hence, we obtain that at r_* at least one of the following two equalities holds: $z''(r_* + 0) = 0$, $z'(r_* + 0) = -\tan(\pi/2 - \alpha_g)$. We show now that the equality $z''(r_* + 0) = 0$ is impossible. For that we assume the contrary, that is, $z''(r_* + 0) = 0$. Under this hypothesis, from (2.12) we have:

$$\begin{aligned} p_e &= g\cdot\rho\cdot z(r_*+0)+\frac{\gamma}{r_*}\cdot\sin\alpha(r_*+0)\\ &> g\cdot\rho\cdot\left(R_{ge}-r_*\right)\cdot\tan\alpha_c+\frac{\gamma}{R_{ge}}\cdot\sin\alpha_c\\ &> \frac{\gamma}{R_{ge}}\cdot\sin\alpha_c-\gamma\cdot\frac{\pi/2-\left(\alpha_c+\alpha_g\right)}{R_{ge}}\cdot\frac{n}{n-1}\cdot\cos\alpha_c\\ &> p_e \end{aligned} \tag{2.18}$$

what is impossible.

In this way we obtain that the equality $z'(r_*+0)=-\tan(\pi/2-\alpha_g)$ holds.

For $r_e = r_*$ the solution of the initial value problem (2.8) on the interval $[r_e, R_{ge}]$ describes a convex outer free surface of a static meniscus. □

Corollary 2.6. *If for p_e the following inequality holds:*

$$p_e < -2\cdot\gamma\cdot\frac{\pi/2-\left(\alpha_c+\alpha_g\right)}{R_{ge}-R_{gi}}\cdot\cos\alpha_c+\frac{\gamma}{R_{ge}}\cdot\sin\alpha_c, \tag{2.19}$$

then there exists r_e in the interval $((R_{gi}+R_{ge})/2, R_{ge})$ such that the solution of the initial value problem (2.10) *on the interval $[r_e, R_{ge}]$ describes a convex outer free surface of a static meniscus.*

Corollary 2.7. *If for $1 < n' < n < 2\cdot R_{ge}/(R_{gi}+R_{ge})$ the following inequalities hold:*

$$\begin{aligned} &-\gamma\cdot\frac{\pi/2-\left(\alpha_c+\alpha_g\right)}{R_{ge}}\cdot\frac{n'}{n'-1}\cdot\sin\alpha_g+\rho\cdot g\cdot R_{ge}\cdot\frac{n'-1}{n'}\cdot\tan\left(\frac{\pi}{2}-\alpha_g\right)+\frac{\gamma}{R_{ge}}\cdot n'\cos\alpha_g\\ &\quad< p_e < -\gamma\cdot\frac{\pi/2-\left(\alpha_c+\alpha_g\right)}{R_{ge}}\cdot\frac{n}{n-1}\cdot\cos\alpha_c+\frac{\gamma}{R_{ge}}\cdot\sin\alpha_c, \end{aligned} \tag{2.20}$$

then there exists r_e in the interval $[R_{ge}/n, R_{ge}/n']$ such that the solution of the initial value problem (2.10) *on the interval $[r_e, R_{ge}]$ describes a convex outer free surface of a static meniscus. The existence of r_e and the inequality $r_e \geq R_{ge}/n$ follows from Theorem 2.5. The inequality $r_e \leq R_{ge}/n'$ follows from Corollary 2.3.*

Remark 2.8. The solution of the initial value problem (2.10) is convex at R_{ge} (i.e., $z''(R_{ge})>0$) if and only if

$$p_e < \frac{\gamma}{R_{ge}}\cdot\sin\alpha_c. \tag{2.21}$$

That is because $z''(R_{ge}) > 0$ if and only if $\alpha'(R_{ge}) = -z''(R_{ge}) \cdot \cos^2\alpha_c < 0$, that is,

$$\frac{p_e}{\gamma} - \frac{\sin\alpha_c}{R_{ge}} < 0 \Longleftrightarrow p_e < \frac{\gamma}{R_{ge}} \cdot \sin\alpha_c. \tag{2.22}$$

Moreover, if $p_e < \gamma/R_{ge} \cdot \sin\alpha_c$, the solution $z(r)$ of the initial value problem (2.8) is convex everywhere (i.e., $z''(r) > 0$ for $r \in I$, $0 < r \le R_{ge}$). That is because the change of convexity implies the existence of $r' \in I$, $0 < r' < R_{ge}$ such that $\alpha(r') > \alpha_c$, $z(r') > 0$ and $p_e = \rho \cdot g \cdot z(r') + \gamma/r' \cdot \sin\alpha(r') > \gamma/R_{ge} \cdot \sin\alpha_c$, what is impossible.

Theorem 2.9. *If a solution* $z_1 = z_1(r)$ *of* (2.1) *describes a convex outer free surface of a static meniscus on the interval* $[r_e, R_{ge}]$ $((R_{gi} + R_{ge})/2 < r_e < R_{ge})$, *then it is a weak minimum for the energy functional of the melt column:*

$$\begin{gathered} I_e(z) = \int_{r_e}^{R_{ge}} \left\{\gamma \cdot \left[1 + (z')^2\right]^{1/2} + \frac{1}{2} \cdot \rho \cdot g \cdot z^2 - p_e \cdot z\right\} \cdot r \cdot dr \\ z(r_e) = z_1(r_e), \quad z(R_{ge}) = z_1(R_{ge}) = 0. \end{gathered} \tag{2.23}$$

Proof. Since (2.1) is the Euler equation for (2.23), it is sufficient to prove that the Legendre and Jacobi conditions are satisfied in this case.

Denote by $F(r, z, z')$ the function defined as

$$F(r, z, z') = r \cdot \left\{\frac{1}{2} \cdot \rho \cdot g \cdot z^2 - p_e \cdot z + \gamma \cdot \left[1 + (z')^2\right]^{1/2}\right\}. \tag{2.24}$$

It is easy to verify that we have

$$\frac{\partial^2 F}{\partial z'^2} = \frac{r \cdot \gamma}{\left[1 + (z')^2\right]^{3/2}} > 0. \tag{2.25}$$

Hence, the Legendre condition is satisfied.

The Jacobi equation

$$\left[\frac{\partial^2 F}{\partial z^2} - \frac{d}{dr}\left(\frac{\partial^2 F}{\partial z\, \partial z'}\right)\right] \cdot \eta - \frac{d}{dr}\left[\frac{\partial^2 F}{\partial z'^2} \cdot \eta'\right] = 0 \tag{2.26}$$

in this case is given by

$$\frac{d}{dr}\left(\frac{r \cdot \gamma}{\left[1 + (z')^2\right]^{3/2}} \cdot \eta'\right) - \rho \cdot g \cdot r \cdot \eta = 0. \tag{2.27}$$

For (2.27) the following inequalities hold:

$$\frac{r\cdot\gamma}{\left[1+(z')^2\right]^{3/2}}\geq r\cdot\gamma\cdot\cos^3\left(\frac{\pi}{2}-\alpha_g\right)=r\cdot\gamma\cdot\sin^3\alpha_g,\quad -\rho\cdot g\cdot r\leq 0. \tag{2.28}$$

Hence, the equation

$$\left(\eta'\cdot r\cdot\gamma\cdot\sin^3\alpha_g\right)'=0 \tag{2.29}$$

is a Sturm type upper bound for (2.27) [7].

Since every nonzero solution of (2.29) vanishes at most once on the interval $[r_e, R_{ge}]$, the solution $\eta(r)$ of the initial value problem

$$\frac{d}{dr}\left(\frac{r\cdot\gamma}{\left[1+(z')^2\right]^{3/2}}\cdot\eta'\right)-\rho\cdot g\cdot r\cdot\eta=0,$$
$$\eta(r_e)=0,\quad \eta'(r_e)=1 \tag{2.30}$$

has only one zero on the interval $[r_e, R_{ge}]$ [7]. Hence the Jacobi condition is satisfied. □

Definition 2.10. A solution $z = z(r)$ of (2.1) which describes the outer free surface of a static meniscus is said to be stable if it is a weak minimum of the energy functional of the melt column.

Remark 2.11. Theorem 2.9 shows that if $z = z(r)$ describes a convex outer free surface of a static meniscus on the interval $[r_e, R_{ge}]$, then it is stable.

Theorem 2.12. *If the solution $z = z(r)$ of the initial value problem* (2.10) *is concave (i.e., $z''(r) < 0$) on the interval $[r_e, R_{ge}]$ ($(R_{gi} + R_{ge})/2 < r_e < R_{ge}$), then it does not describe the outer free surface of a static meniscus on $[r_e, R_{ge}]$.*

Proof. $z''(r) < 0$ on $[r_e, R_{ge}]$ implies that $z'(r)$ is strictly decreasing on $[r_e, R_{ge}]$. Hence $z'(r_e) > z'(R_{ge}) = -\tan\alpha_c > -\tan(\pi/2 - \alpha_g)$. □

Theorem 2.13. *If $p_e > \gamma/R_{ge}\cdot\sin\alpha_c$ and there exists $r_e \in ((R_{gi}+R_{ge})/2, R_{ge})$ such that the solution of the initial value problem* (2.10) *is the outer free surface of a static meniscus on $[r_e, R_{ge}]$, then for p_e the following inequalities hold:*

$$\frac{\gamma}{R_{ge}}\cdot\sin\alpha_c<p_e\leq\rho\cdot g\cdot\left(R_{ge}-r_e\right)\cdot\tan\left(\frac{\pi}{2}-\alpha_g\right)+\frac{\gamma}{r_e}\cdot\cos\alpha_g. \tag{2.31}$$

Proof. Denote by $z(r)$ the solution of the initial value problem (2.10) which is assumed to represent the outer free surface of a static meniscus on the closed interval $[r_e, R_{ge}]$. Let $\alpha(r)$ be defined as in Theorem 2.2. for $r \in [r_e, R_{ge}]$. Since $p_e > \gamma/R_{ge}\cdot\sin\alpha_c$, we have $z''(R_{ge}) = -(1/\cos^2\alpha_c)\cdot\alpha'(R_{ge}) < 0$. Hence $\alpha'(R_{ge}) > 0$ and therefore $\alpha(r) < \alpha(R_{ge}) = \alpha_c$ for $r < R_{ge}$, r

close to R_{ge}. Taking into account the fact that $\alpha(r_e) = \pi/2 - \alpha_e > \alpha_c$ it follows that there exists $r' \in (r_e, R_{ge})$ such that $\alpha'(r') = 0$.

Therefore $p_e = \rho \cdot g \cdot z(r') + \gamma/r' \cdot \sin\alpha(r')$. Since $0 \le \alpha(r') \le \pi/2 - \alpha_g$ and $r_e < r'$, the following inequality holds: $\gamma/r' \cdot \sin\ \alpha(r') < \gamma/r_e \cdot \cos\alpha_g$. On the other hand $z(r') < z(r_e) \le (R_{ge} - r_e) \cdot \tan(\pi/2 - \alpha_g)$. Using the above evaluations we obtain inequalities (2.31). □

Remark 2.14. If r_e appearing in Theorem 2.13 is represented as $r_e = R_{ge}/n$, $1 < n < 2 \cdot R_{ge}/(R_{gi} + R_{ge})$, then inequality (2.31) becomes

$$\frac{\gamma}{R_{ge}} \cdot \sin\alpha_c < p_e \le \rho \cdot g \cdot \frac{n-1}{n} \cdot R_{ge} \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) + \frac{\gamma}{R_{ge}} \cdot n \cdot \cos\alpha_g. \tag{2.32}$$

For $n \to 2 \cdot R_{ge}/(R_{gi} + R_{ge})$ inequality (2.32) becomes

$$\frac{\gamma}{R_{ge}} \cdot \sin\alpha_c < p_e \le \rho \cdot g \cdot \frac{R_{ge} - R_{gi}}{2 \cdot R_{ge}} \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) + \frac{2 \cdot \gamma}{(R_{gi} + R_{ge})} \cdot \cos\alpha_g \tag{2.33}$$

Theorem 2.15. *Let n be $1 < n < 2 \cdot R_{ge}/(R_{gi} + R_{ge})$. If for p_e the following inequality holds:*

$$p_e > \rho \cdot g \cdot R_{ge} \cdot \frac{n-1}{n} \cdot \tan\alpha_c + \frac{n}{R_{ge}} \cdot \gamma, \tag{2.34}$$

then the solution $z(r)$ of the initial value problem (2.10) *is concave on the interval $I \cap [R_{ge}/n, R_{ge}]$ where I is the maximal interval of the existence of $z(r)$.*

Proof. Consider $\alpha(r) = -\arctan z'(r)$ and remark that for $r \in I \cap [R_{ge}/n, R_{ge}]$ the following relations hold:

$$\begin{aligned}
\alpha'(r) &= \frac{1}{\cos\alpha(r)} \cdot \left[\frac{p_e}{\gamma} - \frac{\rho \cdot g \cdot z(r)}{\gamma} - \frac{\sin\alpha(r)}{r}\right] \\
&\ge \frac{1}{\cos\alpha(r)} \cdot \left[\frac{\rho \cdot g \cdot R_{ge}(n-1)}{\gamma \cdot n} \cdot \tan\alpha_c + \frac{n}{R_{ge}} - \frac{\rho \cdot g \cdot R_{ge}(n-1)}{\gamma \cdot n} \cdot \tan\alpha_c - \frac{n}{R_{ge}}\right] \\
&\ge 0.
\end{aligned} \tag{2.35}$$

Hence: $z''(r) = -(1/\cos^2\alpha(r)) \cdot \alpha'(r) \le 0$ for $r \in I \cap [R_{ge}/n, R_{ge}]$. □

3. Meniscus Inner Free Surface Shape Analysis in the Case of Tube Growth

Consider now the differential equation

$$z'' = \frac{\rho \cdot g \cdot z - p_i}{\gamma}\left[1 + (z')^2\right]^{3/2} - \frac{1}{r} \cdot \left[1 + (z')^2\right] \cdot z' \quad \text{for } r \in \left[R_{gi}, \frac{R_{gi} + R_{ge}}{2}\right], \tag{3.1}$$

and α_c, α_g such that $0 < \alpha_c < \pi/2 - \alpha_g$, $\alpha_g \in (0, \pi/2)$.

Definition 3.1. A solution $z = z(x)$ of (3.1) describes the inner free surface of a static meniscus on the interval $[R_{gi}, r_i]$ $(R_{gi} < r_i < (R_{gi} + R_{ge})/2)$ if possesses the following properties:

(a) $z'(R_{gi}) = \tan \alpha_c$,

(b) $z'(r_i) = \tan(\pi/2 - \alpha_g)$, and

(c) $z(R_{gi}) = 0$ and $z(r)$ is strictly increasing on $[R_{gi}, r_i]$.

The described inner free surface is convex on $[R_{gi}, r_i]$ if in addition the following inequality holds:

(d) $z''(r) > 0\ \forall r \in [\mathrm{R_{gi}}, r_i]$.

Theorem 3.2. *If there exists a solution of* (3.1)*, which describes a convex inner free surface of a static meniscus on the closed interval* $[R_{gi}, r_i]$*, then the following inequalities hold:*

$$-\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{r_i - R_{gi}} \cdot \cos \alpha_c - \frac{\gamma}{R_{gi}} \cdot \cos \alpha_g \leq p_i \leq -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{r_i - R_{gi}} \cdot \sin \alpha_g + \rho \cdot g \cdot [r_i - R_{gi}] \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) - \frac{\gamma}{r_i} \cdot \sin \alpha_c. \tag{3.2}$$

Proof. Let $z(r)$ be a solution of (3.1), which describes a convex inner free surface of a static meniscus on the closed interval $[R_{gi}, r_i]$ and $\alpha(r) = \arctan z'(r)$. The function $\alpha(r)$ verifies the equation

$$\alpha'(r) = \frac{\rho \cdot g \cdot z(r) - p_i}{\gamma} \cdot \frac{1}{\cos \alpha(r)} - \frac{1}{r} \cdot \tan \alpha(r) \tag{3.3}$$

and the boundary conditions

$$\alpha(R_{gi}) = \alpha_c, \qquad \alpha(r_i) = \frac{\pi}{2} - \alpha_g. \tag{3.4}$$

Hence, there exists $r' \in (R_{gi}, r_i)$ such that the following equality holds:

$$p_i = -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{r_i - R_{gi}} \cdot \cos \alpha(r') + \rho \cdot g \cdot z(r') - \frac{\gamma}{r'} \cdot \sin \alpha(r'). \tag{3.5}$$

Since $z''(r) > 0$ on $[R_{gi}, r_i]$, $z'(r)$ is strictly increasing and $\alpha(r) = \arctan z'(r)$ is strictly increasing on $[R_{gi}, r_i]$, therefore the following inequalities hold:

$$\begin{gathered}
\alpha_c \leq \alpha(r') \leq \frac{\pi}{2} - \alpha_g, \\
\sin \alpha_g \leq \cos \alpha(r') \leq \cos \alpha_c, \\
\sin \alpha_c \leq \sin \alpha(r') \leq \cos \alpha_g, \\
\rho \cdot g \cdot (r' - R_{gi}) \cdot \tan \alpha_c \leq \rho \cdot g \cdot z(r') \leq \rho \cdot g \cdot (r' - R_{gi}) \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right).
\end{gathered} \tag{3.6}$$

Equality (3.5) and inequalities (3.6) imply inequalities (3.2). □

Corollary 3.3. *If $r_i = m \cdot R_{gi}$ with $1 < m < (R_{gi} + R_{ge})/2 \cdot R_{gi}$, then inequalities* (3.2) *become*

$$
\begin{aligned}
&-\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{(m-1) \cdot R_{gi}} \cdot \cos\alpha_c - \frac{\gamma}{R_{gi}} \cdot \cos\alpha_g \\
&\quad \le p_i \le -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{(m-1) \cdot R_{gi}} \cdot \sin\alpha_g + \rho \cdot g \cdot R_{gi} \cdot (m-1) \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) - \frac{\gamma}{m \cdot R_{gi}} \cdot \sin\alpha_c.
\end{aligned}
\tag{3.7}
$$

Corollary 3.4. *If $m \to (R_{gi} + R_{ge})/2 \cdot R_{ge}$, then $r_i \to (R_{gi} + R_{ge})/2$ and* (3.7) *becomes*

$$
\begin{aligned}
&-2 \cdot \gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge} - R_{gi}} \cdot \cos\alpha_c - \frac{\gamma}{R_{gi}} \cdot \cos\alpha_g \\
&\quad \le p_i \le -2 \cdot \gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge} - R_{gi}} \cdot \sin\alpha_g + \frac{\rho \cdot g \cdot (R_{ge} - R_{gi})}{2} \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) \\
&\quad - \frac{2 \cdot \gamma}{R_{gi} + R_{ge}} \cdot \sin\alpha_c.
\end{aligned}
\tag{3.8}
$$

If $m \to 1$, then $r_i \to R_{gi}$ and $p_i \to -\infty$.

Theorem 3.5. *Let m be such that $1 < m < (R_{gi} + R_{ge})/2 \cdot R_{gi}$. If p_i satisfies the inequality*

$$
p_i < -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{(m-1) \cdot R_{gi}} \cdot \cos\alpha_c + \frac{\gamma}{R_{gi}} \cdot \cos\alpha_g, \tag{3.9}
$$

then there exists r_i in the closed interval $[R_{gi}, m \cdot R_{gi}]$ such that the solution of the initial value problem

$$
\begin{aligned}
&z'' = \frac{\rho \cdot g \cdot z - p_i}{\gamma} \cdot \left[1 + (z')^2\right]^{3/2} - \frac{1}{r} \cdot \left[1 + (z')^2\right] \cdot z' \quad \text{for } R_{gi} < r \le \frac{R_{gi} + R_{ge}}{2}, \\
&z(R_{gi}) = 0, \quad z'(R_{gi}) = \tan\alpha_c, \quad 0 < \alpha_c < \frac{\pi}{2} - \alpha_g,\ \alpha_g \in \left(0, \frac{\pi}{2}\right)
\end{aligned}
\tag{3.10}
$$

on the interval $[R_{gi}, r_i]$ describes the convex inner free surface of a static meniscus.

Proof. Consider the solution $z(r)$ of the initial value problem (3.10). Denote by I the maximal interval on which the function $z(r)$ exists and by $\alpha(r)$ the function $\alpha(r) = \arctan z'(r)$ defined on I. Remark that for $\alpha(r)$ the following equality holds:

$$
\alpha'(r) = \frac{1}{\cos\alpha(r)} \cdot \left[\frac{\rho \cdot g \cdot z(r) - p_i}{\gamma} - \frac{\sin\alpha(r)}{r}\right]. \tag{3.11}
$$

Since

$$z''(R_{gi}) > 0, \quad z'(R_{gi}) = \tan\alpha_c > 0, \quad z'(R_{gi}) < \tan\left(\frac{\pi}{2} - \alpha_g\right), \tag{3.12}$$

there exists $r' \in I$, $R_{gi} < r' < (R_{gi} + R_{ge})/2$ such that for any $r \in [r', R_{gi}]$ the following inequalities hold:

$$z''(r) > 0, \quad z'(r) > 0, \quad z'(r) < \tan\left(\frac{\pi}{2} - \alpha_g\right). \tag{3.13}$$

Let r^* be defined by

$$r^* = \sup\left\{ r' \in I \mid R_{gi} < r' < \frac{R_{gi}+R_{ge}}{2} \text{ such that for any } r \in \left[R_{gi}, r'\right] \text{ inequalities (3.13) hold} \right\}. \tag{3.14}$$

It is clear that $r^* \leq (R_{gi} + R_{ge})/2$ and for any $r \in [R_{gi}, r^*)$ inequalities (3.13) hold. Moreover, $z'(r^* - 0) = \lim_{r\to r^*, r<r^*} z'(r)$ exists and satisfies, $z'(r^* - 0) > 0$ and $z'(r^* - 0) \leq \tan(\pi/2 - \alpha_g)$. Hence $z(r^* - 0) = \lim_{r\to r^*, r<r^*} z(r)$ is finite, it is strictly positive, and for every $r \in [R_{gi}, r^*)$ the following inequalities hold:

$$\begin{gathered} \tan\alpha_c \leq z'(r^* - 0) \leq \tan\left(\frac{\pi}{2} - \alpha_g\right), \\ \left(r^* - R_{gi}\right) \cdot \tan\alpha_c \leq z(r^* - 0) \leq \left(r^* - R_{gi}\right) \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right). \end{gathered} \tag{3.15}$$

We will show now that $r^* \leq m \cdot R_{gi}$ and $z'(r^* - 0) = \tan(\pi/2 - \alpha_g)$.

In order to show that $r^* \leq m \cdot R_{gi}$, we assume the contrary, that is, $r^* > m \cdot R_{gi}$. Under this hypothesis we have

$$\begin{aligned} &\alpha(r^* - 0) - \alpha\left(R_{gi}\right) \\ &\quad = \alpha'\left(r'\right) \cdot \left(r^* - R_{gi}\right) \\ &\quad = \frac{1}{\cos\alpha(r')} \cdot \left[-\frac{p_i}{\gamma} + \frac{\rho \cdot g \cdot z(r')}{\gamma} - \frac{\sin\alpha(r')}{r'}\right] \cdot \left(r^* - R_{gi}\right) \\ &\quad > \frac{r^* - R_{gi}}{\cos\alpha(r')} \cdot \left[\frac{\pi/2 - \left(\alpha_c + \alpha_g\right)}{R_{gi}} \cdot \frac{1}{m-1} \cdot \cos\alpha_c + \frac{1}{R_{gi}} \cdot \cos\alpha_g + \frac{\rho \cdot g \cdot z(r')}{\gamma} - \frac{\sin\alpha(r')}{r'}\right] \\ &\quad > \frac{\pi}{2} - \left(\alpha_c + \alpha_g\right) \end{aligned} \tag{3.16}$$

for some $r' \in (R_{gi}, r^*)$. Hence $\alpha(r^* - 0) > \pi/2 - \alpha_g$ and it follows that there exists r_1 such that $R_{gi} < r_1 < r^*$ and $\alpha(r_1) = \pi/2 - \alpha_g$. This last inequality is impossible according to the definition of r^*.

Therefore, r^* defined by (3.14) satisfies $r^* < m \cdot R_{gi}$.

In order to show that $z'(r^* - 0) = \tan(\pi/2 - \alpha_g)$ we remark that from the definition (3.14) of r^* it follows that at least one of the following three equalities holds:

$$z'(r^* - 0) = \tan \alpha_c, \quad z'(r^* - 0) = \tan\left(\frac{\pi}{2} - \alpha_g\right), \quad z''(r^* - 0) = 0. \tag{3.17}$$

Since $z'(r^* - 0) > z'(r) > \tan \alpha_c$ for $r \in (R_{gi}, r^*)$, it follows that the equality $z'(r^* - 0) = \tan \alpha_c$ is impossible.

Hence we obtain that in r^* at least one of the following two equalities holds:

$$z''(r^* - 0) = 0, \quad z'(r^* - 0) = \tan\left(\frac{\pi}{2} - \alpha_g\right). \tag{3.18}$$

We show now that the equality $z''(r^* - 0) = 0$ is impossible. For this we assume the contrary, that is, $z''(r^* - 0) = 0$. Under this hypothesis, from (3.11), we have

$$\begin{aligned} p_i &= \rho \cdot g \cdot z(r^* - 0) - \frac{\gamma}{r^*} \cdot \sin\ \alpha(r^* - 0) \\ &> -\frac{\gamma}{R_{gi}} \cdot \cos \alpha_g \\ &> -\frac{\gamma}{R_{gi}} \cdot \cos \alpha_g - \gamma \cdot \frac{1}{m - 1} \cdot \frac{[\pi/2 - (\alpha_c + \alpha_g)]}{R_{gi}} \cdot \cos \alpha_c \\ &> p_i, \end{aligned} \tag{3.19}$$

what is impossible.

In this way we obtain that the equality $z''(r^* - 0) = \tan(\pi/2 - \alpha_g)$ holds.

For $r_i = r^*$ the solution of the initial value problem (3.10) on the interval $[R_{gi}, r_i]$ describes a convex inner free surface of a static meniscus. □

Corollary 3.6. *If for p_i the following inequality holds*

$$p_e < -2 \cdot \gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{R_{ge} - R_{gi}} \cdot \cos \alpha_c - \frac{\gamma}{R_{gi}} \cdot \cos \alpha_g, \tag{3.20}$$

then there exists r_i in the interval $(R_{gi}, (R_{gi} + R_{ge})/2)$ such that the solution of the initial value problem (3.10) *on the interval $[R_{gi}, r_i]$ describes a convex inner free surface of a static meniscus.*

Corollary 3.7. *If for $1 < m' < m < (R_{gi} + R_{ge})/2 \cdot R_{gi}$ the following inequalities hold:*

$$\begin{aligned} &-\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{(m' - 1) \cdot R_{gi}} \cdot \sin \alpha_g + \rho \cdot g \cdot R_{gi} \cdot (m' - 1) \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) - \frac{\gamma}{m' \cdot R_{gi}} \cdot \sin \alpha_c \\ &\quad < p_i < -\gamma \cdot \frac{\pi/2 - (\alpha_c + \alpha_g)}{(m - 1) \cdot R_{gi}} \cdot \cos \alpha_c - \frac{\gamma}{R_{gi}} \cdot \cos \alpha_g, \end{aligned} \tag{3.21}$$

then there exists r_i in the interval $[m' \cdot R_{gi}, m \cdot R_{gi}]$ such that the solution of the initial value problem (3.10) *on the interval $[R_{gi}, r_i]$ describes a convex inner free surface of a static meniscus.*

The existence of r_i and the inequality $r_i \leq m \cdot R_{gi}$ follows from Theorem 3.5. The inequality $r_i \geq m' \cdot R_{gi}$ follows from the Corollary 3.3.

Theorem 3.8. *If a solution $z_1 = z_1(r)$ of* (3.1) *describes a convex inner free surface of a static meniscus on the interval $[R_{gi}, r_i]$ $(r_i \in (R_{gi}, (R_{gi} + R_{ge})/2))$, then it is a weak minimum for the energy functional of the melt column:*

$$I_i(z) = \int_{R_{gi}}^{r_i} \left\{ \gamma \cdot \left[1 + (z')^2\right]^{1/2} + \frac{1}{2} \cdot \rho \cdot g \cdot z^2 - p \cdot z \right\} \cdot r \cdot dr,$$
$$z(R_{gi}) = z_1(R_{gi}) = 0, \quad z(r_i) = z_1(r_i). \tag{3.22}$$

Proof. It is similar to the proof of Theorem 2.9. □

Definition 3.9. A solution $z = z(r)$ of (3.1) which describes the inner free surface of a static meniscus is said to be stable if it is a weak minimum of the energy functional of the melt column.

Remark 3.10. Theorem 3.8 shows that if $z = z(r)$ describes a convex inner free surface of a static meniscus on the interval $[R_{gi}, r_i]$, then it is stable.

Remark 3.11. The solution of the initial value problem (3.10) is convex at R_{gi} (i.e., $z''(R_{gi}) > 0$) if and only if

$$p_i < -\frac{\gamma}{R_{gi}} \cdot \sin \alpha_c. \tag{3.23}$$

Theorem 3.12. *If $z(r)$ represents the inner free surface of a static meniscus on the closed interval $[R_{gi}, r_i]$ $(r_i < (R_{gi} + R_{ge})/2)$ which possesses the following properties:*

(a) *$z(r)$ is convex at R_{gi}, and*

(b) *the shape of $z(r)$ changes once on the interval (R_{gi}, r_i), that is, there exists a point $r' \in (R_{gi}, r_i)$ such that $z''(r) > 0$ for $r \in [R_{gi}, r_i)$, $z''(r') = 0$ and $z''(r) < 0$ for $r \in (r', r_i]$,*

then there exists $r_i^1 \in (R_{gi}, r_i)$ such that $z'(r_i^1) = \tan(\pi/2 - \alpha_g)$ and for p_i the following inequality holds:

$$-\frac{\gamma}{R_{gi}} < p_i < -\frac{\gamma}{R_{gi}} \cdot \sin \alpha_c. \tag{3.24}$$

Proof. Since $\alpha(r) = \arctan z'(r)$ increases on $[R_{gi}, r')$ and decreases on $(r', r_i]$, and $\alpha(r_i) = \pi/2 - \alpha_g > \alpha_c$, there exists $r_i^1 \in (R_{gi}, r_i)$ such that $\alpha(r_i^1) = \pi/2 - \alpha_g$. The maximum value $\alpha(r')$ is less than $\pi/2$. From (3.11) we have

$$p_i = \rho \cdot g \cdot z(r') - \frac{\gamma}{r'} \cdot \sin \alpha(r'), \tag{3.25}$$

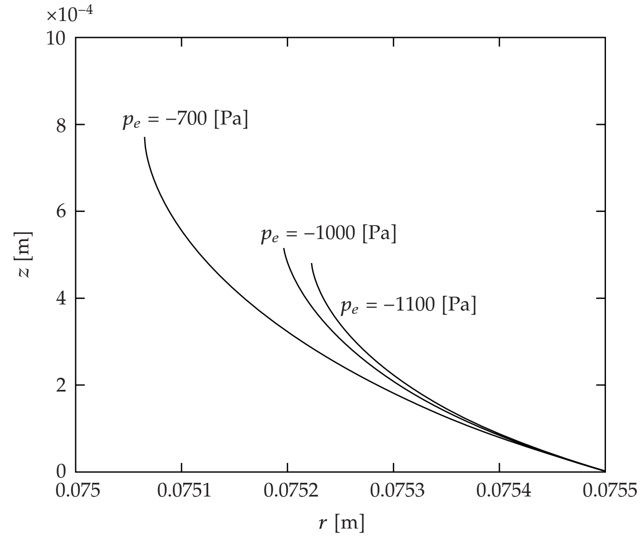

Figure 2: z versus r for $p_e = -1100; -1000; -700$ [Pa].

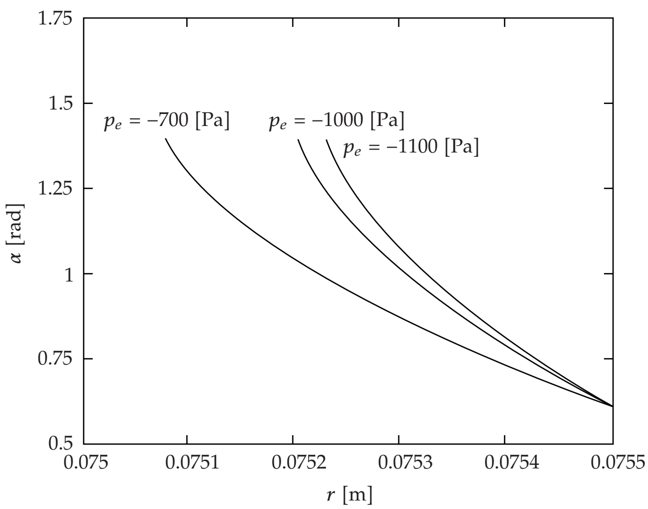

Figure 3: α versus r for $p_e = -1100; -1000; -700$ [Pa].

and therefore

$$-\frac{\gamma}{R_{gi}} < p_i < -\frac{\gamma}{R_{gi}} \cdot \sin \alpha_c. \tag{3.26}$$

□

Remark 3.13. If the solution $z(r)$ of the initial value problem (3.10) is concave (i.e., $z''(r) < 0$) on the interval $[R_{gi}, r_i]$ ($r_i \in (R_{gi}, (R_{gi} + R_{ge})/2)$), then it does not describe the inner free surface of a static meniscus on $[R_{gi}, r_i]$.

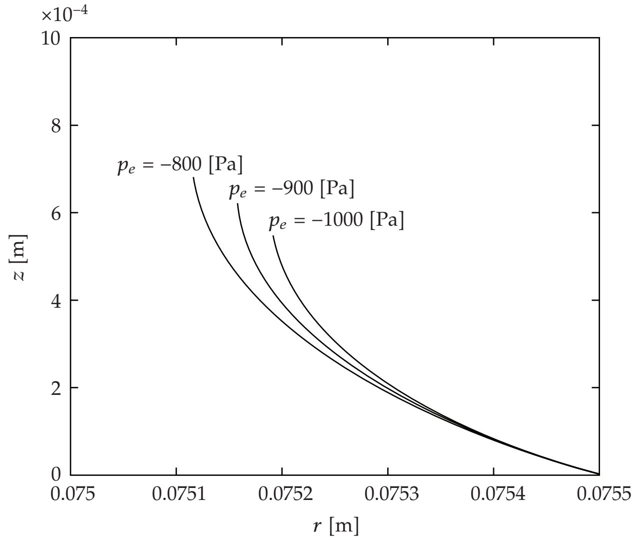

Figure 4: z versus r for $p_e = -1000$; -900; -800 [Pa].

Theorem 3.14. *Let m be $1 < m < (R_{gi} + R_{ge})/2 \cdot R_{gi}$. If for p_i the following inequality holds:*

$$p_i > \frac{\gamma}{R_{gi}} + \rho \cdot g \cdot (m-1) \cdot R_{gi} \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right), \tag{3.27}$$

then the solution $z(r)$ of the initial value problem (3.10) *is concave on the interval $I \cap [R_{gi}, m \cdot R_{gi}]$ where I is the maximal interval of the existence of $z(r)$.*

Proof. Consider $\alpha(r) = \arctan z'(r)$ and remark that for $r \in I \cap [R_{gi}, m \cdot R_{gi}]$ the following relations hold:

$$\begin{aligned}\alpha'(r) &= \frac{1}{\cos\alpha(r)} \cdot \left[\frac{\rho \cdot g \cdot z(r)}{\gamma} - \frac{p_i}{\gamma} - \frac{\sin\alpha}{r}\right] \\ &\le \frac{1}{\cos\alpha(r)} \cdot \left[\frac{\rho \cdot g \cdot (m-1) \cdot R_{gi}}{\gamma} \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) - \frac{n}{R_{gi}}\right. \\ &\qquad \left. - \frac{\rho \cdot g \cdot (m-1) \cdot R_{gi}}{\gamma} \cdot \tan\left(\frac{\pi}{2} - \alpha_g\right) - \frac{\sin\alpha}{r}\right] \\ &< 0.\end{aligned} \tag{3.28}$$

Hence $z''(r) = -1/(\cos^2\alpha(r)) \cdot \alpha'(r) < 0$ for $r \in I \cap [R_{gi}, m \cdot R_{ge}]$. □

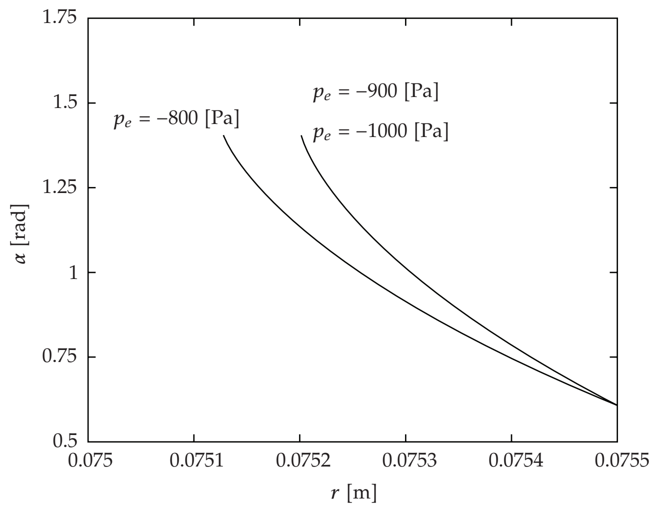

Figure 5: α versus r for $p_e = -1000; -900; -800$ [Pa].

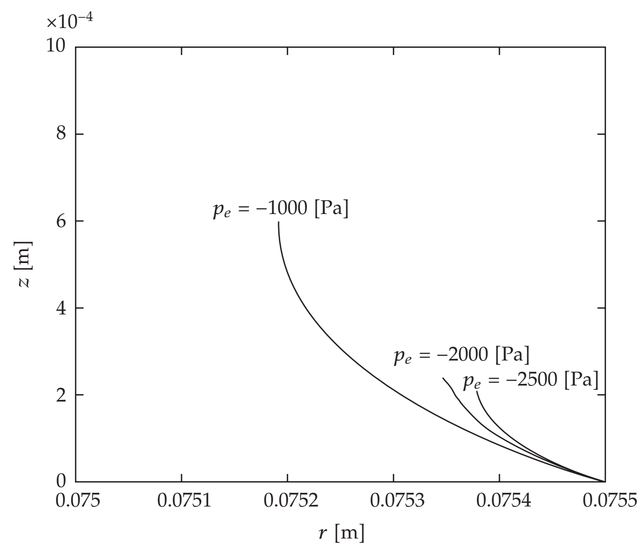

Figure 6: z versus r for $p_e = -2500; -2000; -1000$ [Pa].

4. Numerical Illustration

Numerical computations were performed for a Si cylindrical tube for the following data:

$$R_{ge} = 75.5 \cdot 10^{-3}\,[\mathrm{m}], \qquad R_{gi1} = 74.5 \cdot 10^{-3}\,[\mathrm{m}], \qquad \alpha_c = 0.523\,[\mathrm{rad}], \qquad \alpha_g = 0.192\,[\mathrm{rad}],$$
$$\rho = 2.5 \cdot 10^{3}\left[\mathrm{kg/m^3}\right], \qquad \gamma = 7.2 \cdot 10^{-1}\,[\mathrm{N/m}], \qquad g = 9.81\left[\mathrm{m/s^2}\right],$$
$$n = 1.006, \qquad n' = 1.0006, \qquad m = 1.0067, \qquad m' = 1.0006. \tag{4.1}$$

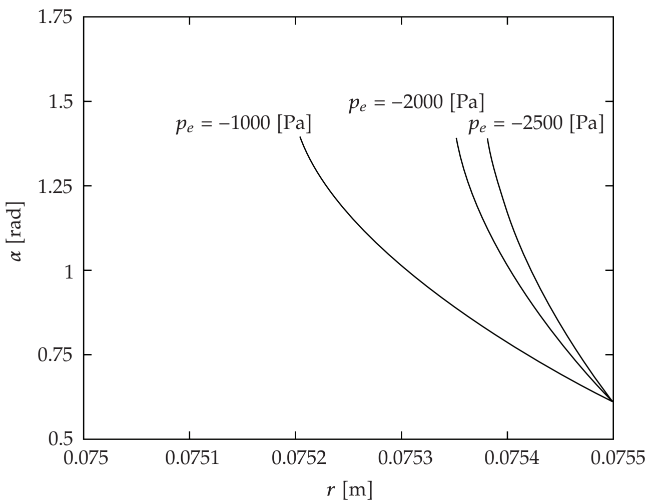

Figure 7: α versus r for $p_e = -2500; -2000; -1000$ [Pa].

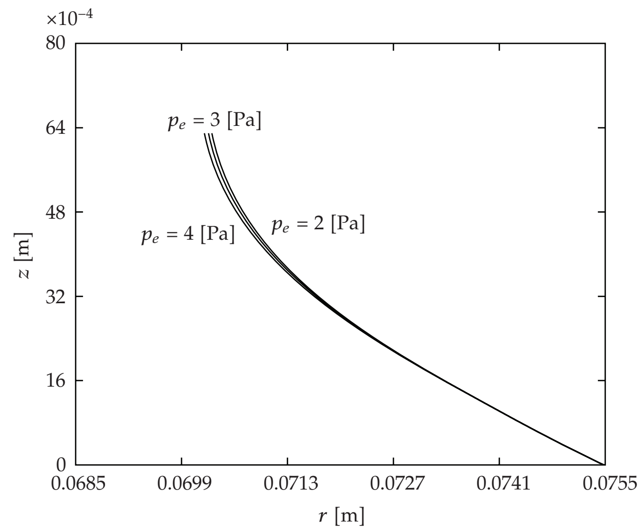

Figure 8: z versus r for $p_e = 2; 3; 4$ [Pa].

The objective was to verify if the necessary conditions are also sufficient, or if the sufficient conditions are also necessary. Moreover, the above data were used in experiments and the computed results can be tested against the experiments in order to evaluate the accuracy of the theoretical predictions. This test is not the subject of this paper.

For the considered numerical data, inequality (2.7) becomes

$$-1179.443\,[\text{Pa}] \le p_e \le -194.682\,[\text{Pa}]. \tag{4.2}$$

Integration of (2.10) shows that for $p_e = -1100; -1000$ [Pa] there exists $r_e \in (R_{ge}/n, R_{ge})$ such that the solution is a convex outer free surface of a static meniscus on $[r_e, R_{ge}]$, but for

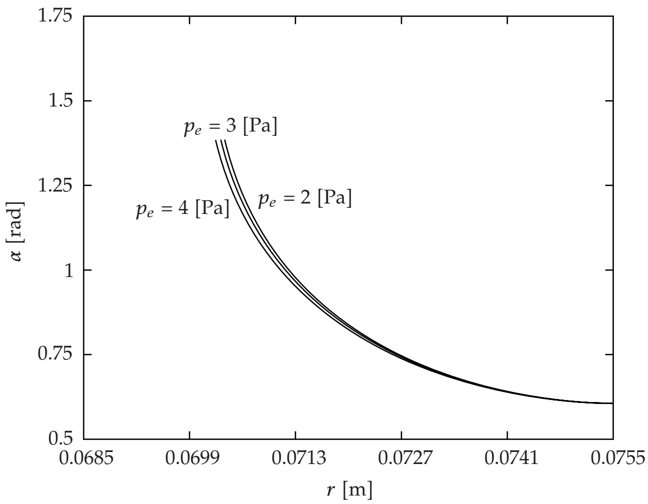

Figure 9: α versus r for $p_e = 2; 3; 4$ [Pa].

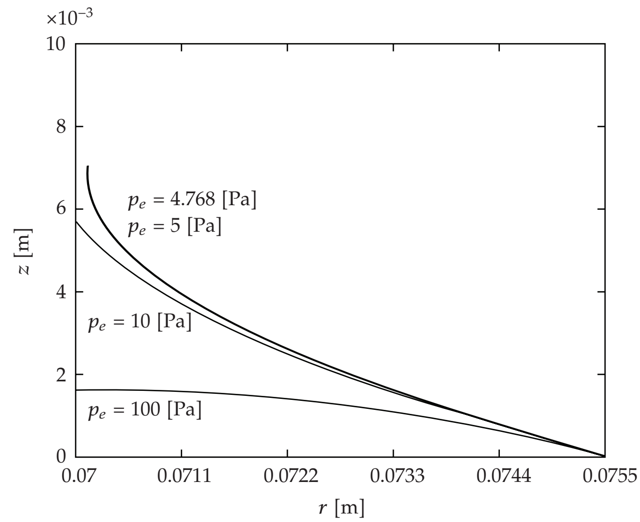

Figure 10: z versus r for $p_e = 4.768; 5; 10; 100$ [Pa].

$p_e = -700$ [Pa] there is no $r_e \in (R_{ge}/n, R_{ge})$ such that the solution is a convex outer free surface of a static meniscus on $[r_e, R_{ge}]$ (Figures 2 and 3). Hence, inequality (2.7) is not a sufficient condition.

For the same numerical data inequality (2.9) becomes $p_e < -1179.443$ [Pa]. We have already obtained that for $p_e = -1100; -1000$ [Pa] there exists r_e in the interval $[R_{ge}/n, R_{ge}]$ such that the solution of (2.10) describes a convex outer free surface of a static meniscus on the interval $[r_e, R_{ge}]$. Hence, inequality (2.9) is not a necessary condition.

For the same numerical data inequality (2.19) becomes $p_e < -1061.728$ [Pa]. Integration shows that for $p_e = -1000; -900; -800$ [Pa] there exists $r_e \in ((R_{gi} + R_{ge})/2, R_{ge})$, such that the solution of (2.10) describes a convex outer free surface of a static meniscus

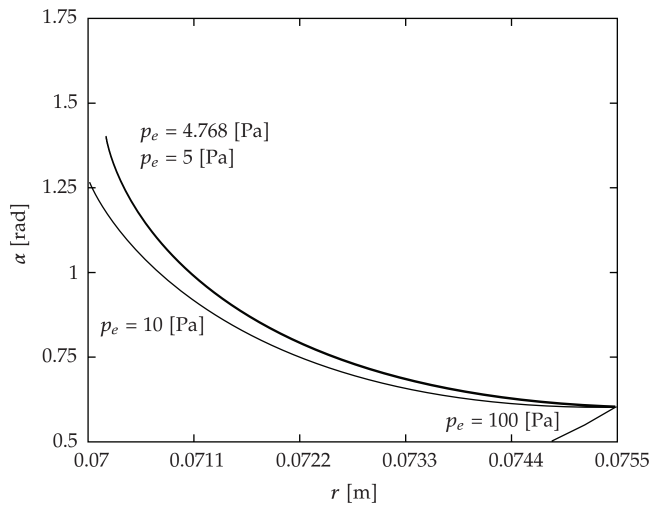

Figure 11: α versus r for $p_e = 4.768$; 5; 10; 100 [Pa].

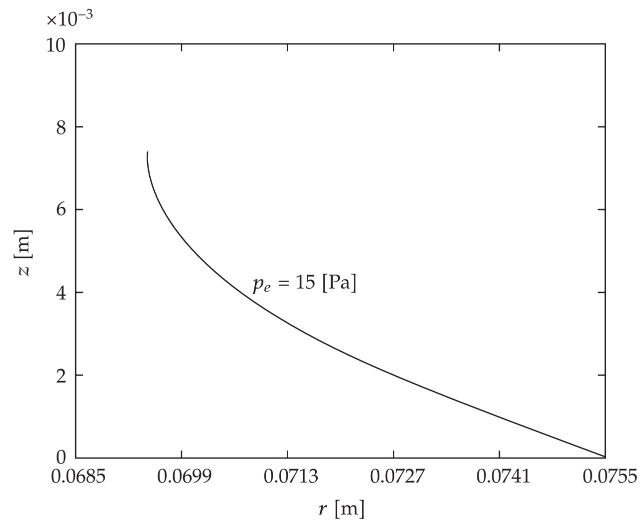

Figure 12: z versus r for $p_e = 15$ [Pa].

on the closed interval $[r_e, R_{ge}]$ (Figures 4 and 5). Hence, inequality (2.19) is not a necessary condition.

For the same numerical data inequality (2.20) becomes $-2580 < p_e < -1179.443$ [Pa]. Integration of (2.10) illustrates the above phenomenon for $p_e = -2000; -2500$ [Pa] (Figures 6 and 7) and also the fact that the condition is not necessary (see $p_e = -1000$ [Pa]).

For the same numerical data inequality (2.21) becomes $p_e < 4.768$ [Pa]. Integration of (2.10) illustrates the above phenomenon for $p_e = 2$; 3; 4 [Pa] (Figures 8 and 9).

For the considered numerical data inequality (2.32) becomes $4.768\,[\text{Pa}] < p_e < 66.23\,[\text{Pa}]$. Integration shows that for $p_e = 4.768$ [Pa] there is no $r_e \in (R_{ge}/n, R_{ge})$ such that the solution of (2.10) is a nonglobally convex outer free surface of a static meniscus on $[r_e, R_{ge}]$.

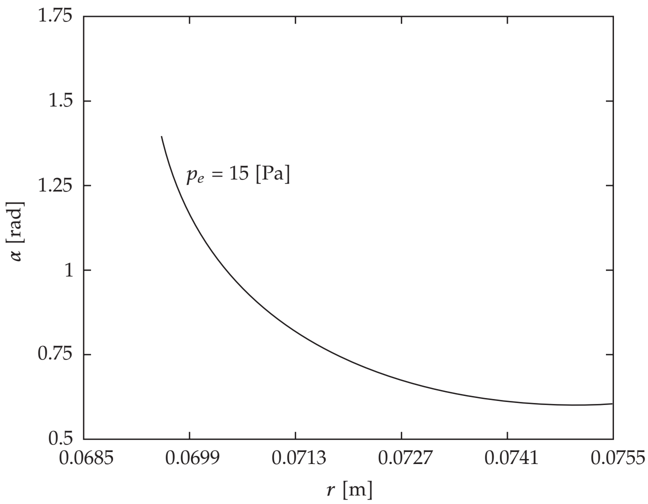

Figure 13: α versus r for $p_e = 15$ [Pa].

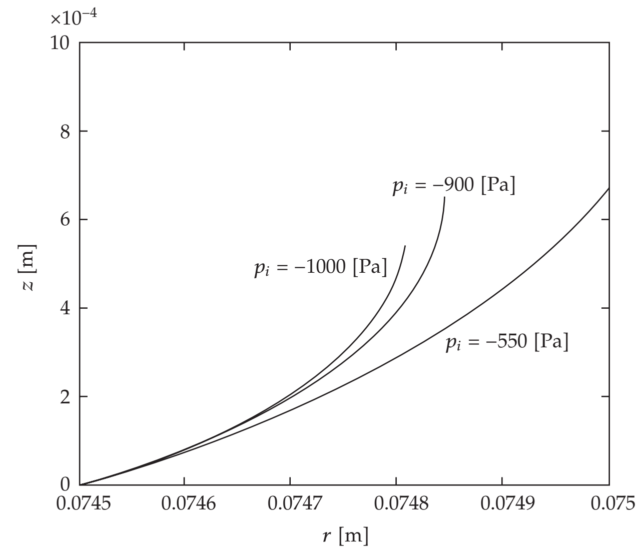

Figure 14: z versus r for $p_i = -1000; -900; -550$ [Pa].

Moreover, for $p_e = 5; 10; 100$ [Pa] it is not anymore the outer free surface of a static meniscus (Figures 10 and 11). Hence, inequality (2.32) is not a sufficient condition.

For the same numerical data inequality (2.33) becomes $4.768\,[\text{Pa}] \leq p_e \leq 72.5\,[\text{Pa}]$. We have already obtained that for $p_e = 5; 10$ [Pa] (Figure 10) the solution of (2.10) is not anymore the outer free surface of a static meniscus. Hence, inequality (2.33) is not a sufficient condition.

For the same numerical data inequality (2.34) becomes $p_e > 15.9696$ [Pa]. Integration of (2.10) for $p_e = 15$ [Pa] proves that the solution is globally concave on $[R_{ge}/n, R_{ge}]$ (Figures 12 and 13). Hence, inequality (2.34) is not a necessary condition.

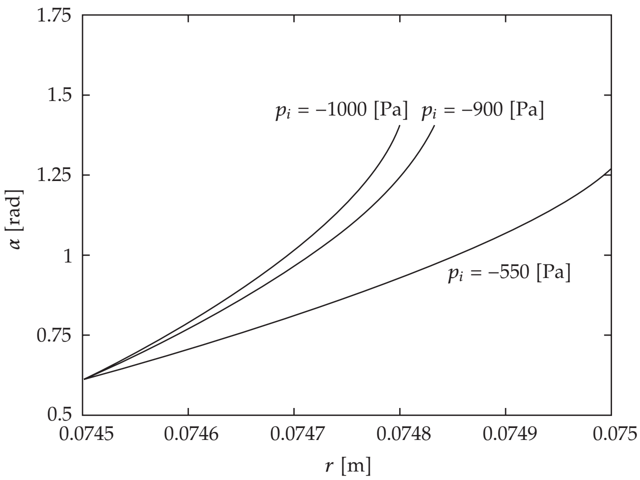

Figure 15: α versus r for $p_i = -1000; -900; -550$ [Pa].

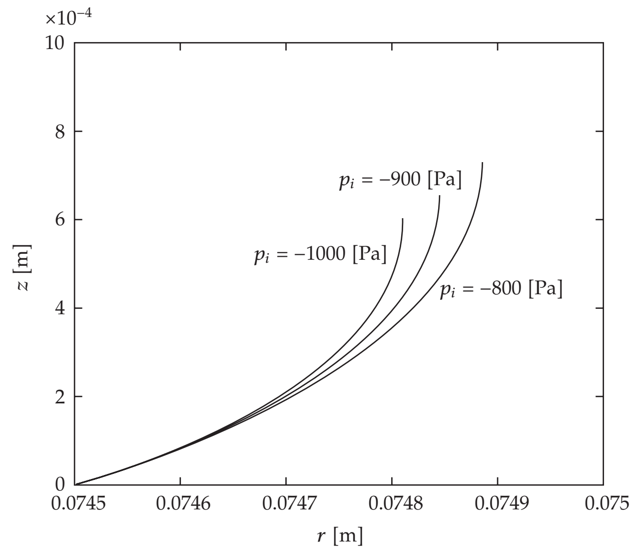

Figure 16: z versus r for $p_i = -1000; -900; -800$ [Pa].

For the considered numerical data inequality (3.7) becomes -1076.682 [Pa] $\leq p_i \leq -177.201$ [Pa]. Integration of (3.10) shows that for $p_i = -1000; -900$ [Pa] there exists $r_i \in (R_{gi}, m \cdot R_{gi})$ such that the solution is a convex inner free surface of a static meniscus on $[R_{gi}, r_i]$, but for $p_i = -550$ [Pa] there is no $r_i \in (R_{gi}, m \cdot R_{gi})$ such that the solution is a convex inner free surface of a static meniscus on $[R_{gi}, r_i]$ (Figures 14 and 15). Hence, inequality (3.7) is not a sufficient condition.

For the considered numerical data (3.9) becomes $p_i < -1076.682$ [Pa]. We have already obtained that for $p_i = -1000; -900$ [Pa] there exists $r_i \in (R_{gi}, m \cdot R_{gi})$ such that the solution of (3.10) describes a convex inner free surface of a static meniscus on $[R_{gi}, r_i]$. Hence, inequality (3.9) is not a necessary condition.

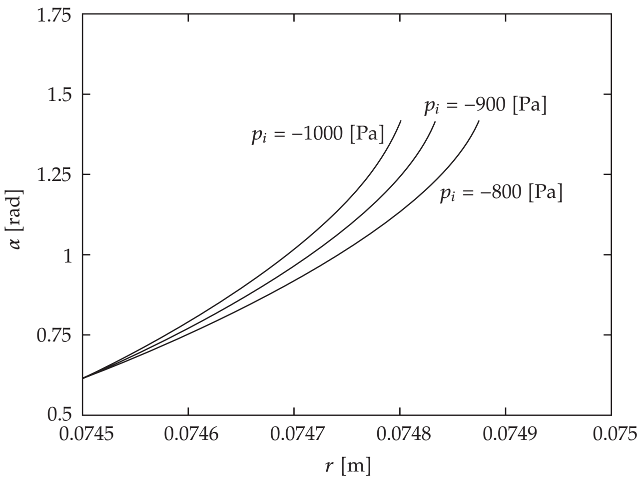

Figure 17: α versus r for $p_i = -1000; -900; -800$ [Pa].

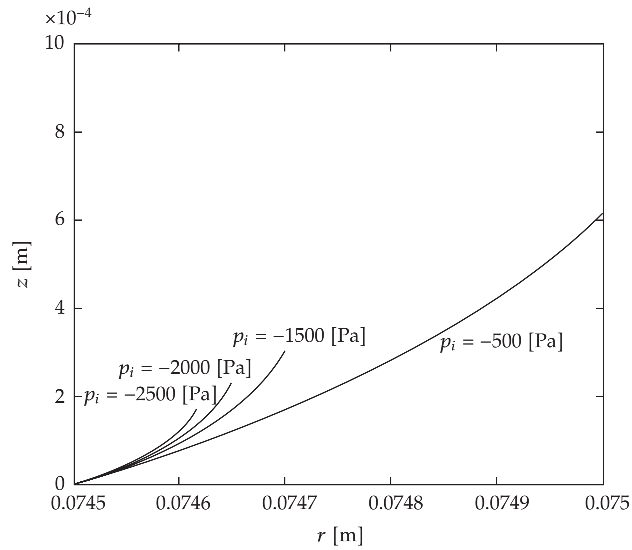

Figure 18: z versus r for $p_i = -2500; -2000; -1500; -500$ [Pa].

For the considered numerical data (3.20) becomes $p_i < -1075.98$ [Pa]. Integration of (3.10) shows that for $p_i = -1000; -900; -800$ [Pa] there exists $r_i \in (R_{gi}, (R_{gi} + R_{ge})/2)$ such that the solution describes a convex inner free surface of a static meniscus on $[R_{gi}, r_i]$ (Figures 16 and 17). Hence, inequality (3.20) is not a necessary condition.

For the considered numerical data (3.21) becomes -2613 [Pa] $< p_i < -1076.68$ [Pa]. Integration of (3.10) illustrates the above phenomenon for $p_i = -1500; -2000; -2500$ [Pa] and also the fact that the condition is not necessary (Figures 18 and 19) (see $p_i = -500$ [Pa]).

For the same numerical data inequality (3.23) becomes $p_i < -4.8322$ [Pa]. Integration of (3.10) illustrates the above phenomenon for $p_i = -7; -6; -5$ [Pa] (Figures 20 and 21).

For the considered numerical data inequality (3.24) becomes -9.664 [Pa] $< p_i < -4.827$ [Pa]. Integration shows that for $p_i = -5$ [Pa] there exists $r_i \in (R_{gi}, m \cdot R_{gi})$ such

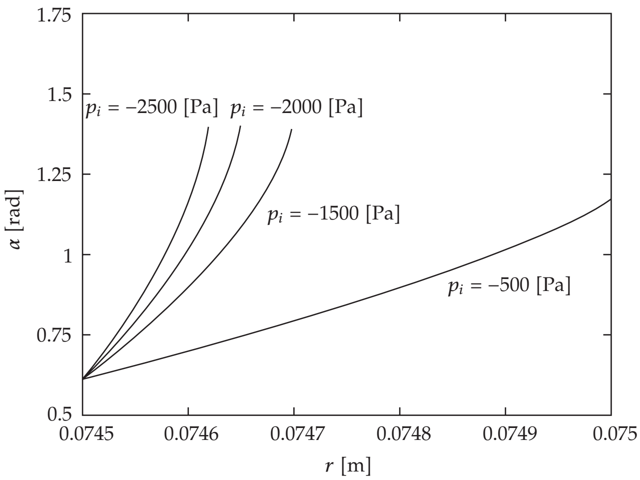

Figure 19: α versus r for $p_i = -2500; -2000; -1500; -500$ [Pa].

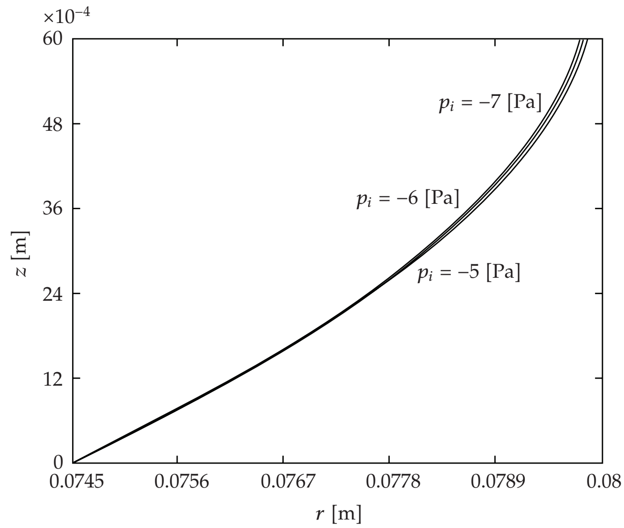

Figure 20: z versus r for $p_i = -7; -6; -5$ [Pa].

that the solution of (3.10) is a nonglobally convex inner free surface of a static meniscus on $[R_{gi}, r_i]$, but for $p_i = -6; -7$ [Pa] it is not anymore the inner free surface of a nonglobally static meniscus (Figures 20 and 21). Hence, (3.24) is not a sufficient condition.

For the same numerical data (3.27) becomes $p_i > 72.6422$ [Pa]. Integration of (3.8) for $p_i = 70$ [Pa] proves that the solution is globally concave on $[R_{gi}, m \cdot R_{gi}]$ (Figures 22 and 23). Hence, (3.27) is not a necessary condition.

5. Conclusions

(1) Inequalities (2.2) and (2.7) localize regions on the pressure axis where the outer pressure has to be taken in order to obtain convex outer free surface.

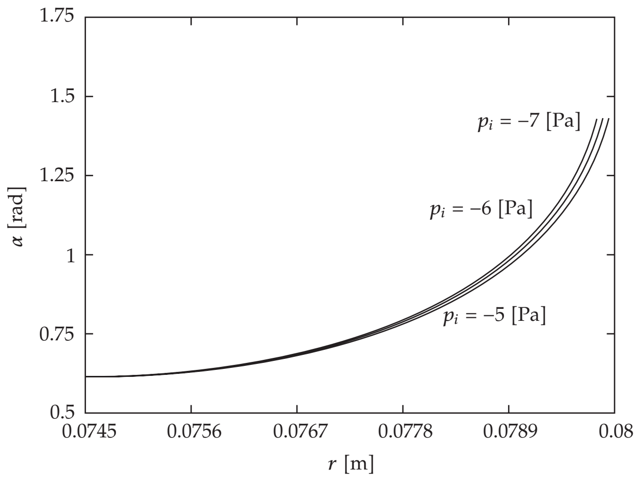

Figure 21: α versus r for $p_i = -7; -6; -5$ [Pa].

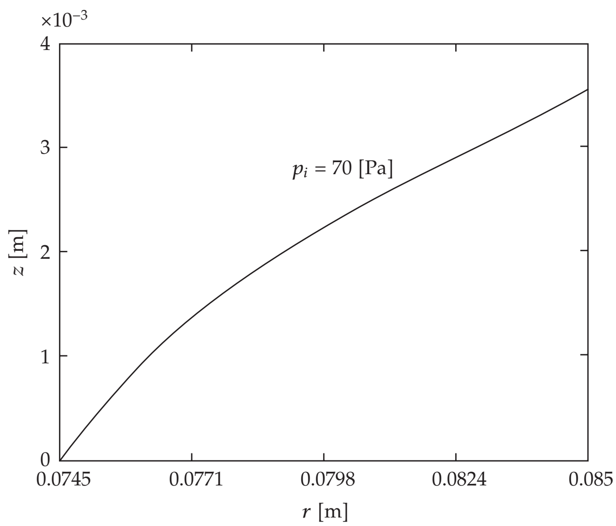

Figure 22: z versus r for $p_i = 70$ [Pa].

Inequalities (2.9), (2.19), and (2.20) localize regions on the pressure axis, having the property that if the outer pressure is taken in this region, then a convex outer free surface is obtained.

Inequalities (2.31) and (2.32) localize region on the pressure axis, where the outer pressure has to be taken in order to obtain a convex-concave outer free surface.

(2) Inequalities (3.2) and (3.7) localize regions on the pressure axis where the inner pressure has to be taken in order to obtain convex inner free surface.

Inequalities (3.5), (3.20), and (3.21) localize regions on the pressure axis, having the property that if the inner pressure is taken in this region, then a convex inner free surface is obtained.

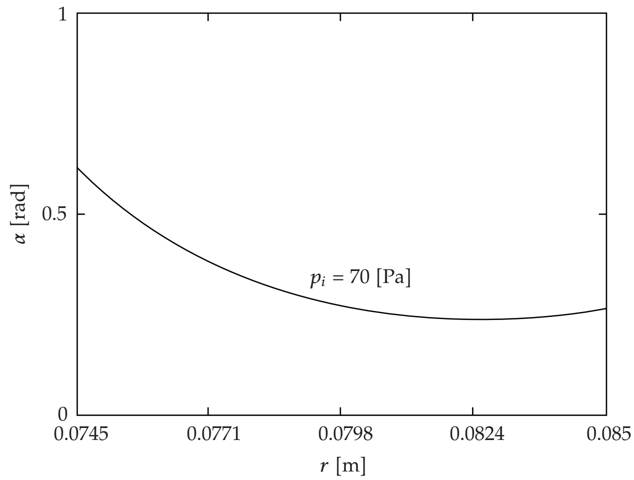

Figure 23: α versus r for $p_i = 70$ [Pa].

Inequality (3.24) localizes region on the pressure axis, where the inner pressure has to be taken in order to obtain a convex-concave inner free surface.

(3) By computation these values are found in a real case, and the "accuracy" (sufficiency or necessity) of the reported inequalities is discussed.

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