Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 763252, 7 pages
doi:10.1155/2009/763252

*Research Article*

# Inequalities for Generalized Logarithmic Means

**Yu-Ming Chu[1] and Wei-Feng Xia[2]**

[1] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*
[2] *School of Teacher Education, Huzhou Teachers College, Huzhou 313000, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

Received 2 June 2009; Accepted 10 December 2009

Recommended by Wing-Sum Cheung

For $p \in \mathbb{R}$, the generalized logarithmic mean $L_p$ of two positive numbers $a$ and $b$ is defined as $L_p(a,b) = a$, for $a = b$, $L_P(a,b) = [(b^{p+1} - a^{p+1})/(p+1)(b-a)]^{1/p}$, for $a \neq b$, $p \neq -1$, $p \neq 0$, $L_P(a,b) = (b-a)/(\log b - \log a)$, for $a \neq b$, $p = -1$, and $L_P(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, for $a \neq b$, $p = 0$. In this paper, we prove that $G(a,b) + H(a,b) \geqslant 2L_{-7/2}(a,b)$, $A(a,b) + H(a,b) \geqslant 2L_{-2}(a,b)$, and $L_{-5}(a,b) \geqslant H(a,b)$ for all $a, b > 0$, and the constants $-7/2, -2$, and $-5$ cannot be improved for the corresponding inequalities. Here $A(a,b) = (a+b)/2 = L_1(a,b)$, $G(a,b) = \sqrt{ab} = L_{-2}(a,b)$, and $H(a,b) = 2ab/(a+b)$ denote the arithmetic, geometric, and harmonic means of $a$ and $b$, respectively.



## 1. Introduction

For $p \in \mathbb{R}$, the generalized logarithmic mean $L_p(a,b)$ and power mean $M_p(a,b)$ of two positive numbers $a$ and $b$ are defined as

$$L_p(a,b) = \begin{cases} a, & a = b, \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & a \neq b,\ p \neq -1,\ p \neq 0, \\ \dfrac{b-a}{\log b - \log a}, & a \neq b,\ p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b,\ p = 0, \end{cases} \tag{1.1}$$

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.2}$$

It is well known that $L_p(a,b)$ and $M_p(a,b)$ are continuous and increasing with respect to $p \in \mathbb{R}$ for fixed $a$ and $b$. Let $A(a,b) = (a+b)/2$, $I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $L(a,b) = (b-a)/(\log b - \log a)$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the arithmetic, identric, logarithmic, geometric, and harmonic means of $a$ and $b$, respectively. Then

$$\begin{aligned} \min\{a,b\} &\leqslant H(a,b) = M_{-1}(a,b) \leqslant G(a,b) = M_0(a,b) = L_{-2}(a,b) \leqslant L(a,b) \\ &= L_{-1}(a,b) \leqslant I(a,b) = L_0(a,b) \leqslant A(a,b) = L_1(a,b) \\ &= M_1(a,b) \leqslant \max\{a,b\}. \end{aligned} \tag{1.3}$$

In [1], the following results are established: (1) $p \geqslant 1/3$ implies that $L(a,b) \leqslant M_p(a,b)$; (2) $p \leqslant 0$ implies that $L(a,b) \geqslant M_p(a,b)$; (3) $p < 1/3$ implies that there exist $a,b > 0$ such that $L(a,b) > M_p(a,b)$; (4) $p > 0$ implies that there exist $a,b > 0$ such that $L(a,b) < M_p(a,b)$. Hence the question was answered: what are the least value $q$ and the greatest value $p$ such that the inequality $M_p(a,b) \leqslant L(a,b) \leqslant M_q(a,b)$ holds for all $a,b > 0$?

Stolarsky [2] proved that $I(a,b) = L_0(a,b) \geqslant M_{2/3}(a,b)$, with equality if and only if $a = b$.

In [3], Pittenger proved that

$$M_{p_1}(a,b) \leqslant L_p(a,b) \leqslant M_{p_2}(a,b) \tag{1.4}$$

for all $a,b > 0$, where

$$\begin{aligned} p_1 &= \begin{cases} \min\left\{\dfrac{p+2}{3}, \dfrac{p\log 2}{\log(p+1)}\right\}, & p > -1,\ p \neq 0, \\ \dfrac{2}{3}, & p = 0, \\ \min\left\{\dfrac{p+2}{3}, 0\right\}, & p \leqslant -1, \end{cases} \\ p_2 &= \begin{cases} \max\left\{\dfrac{p+2}{3}, \dfrac{p\log 2}{\log(p+1)}\right\}, & p > -1,\ p \neq 0, \\ \log 2, & p = 0, \\ \max\left\{\dfrac{p+2}{3}, 0\right\}, & p \leqslant -1. \end{cases} \end{aligned} \tag{1.5}$$

Here $p_1$, $p_2$ are sharp and equality holds only if $a = b$ or $p = 1, -2$ or $-1/2$. The case $p = -1$ reduces to Lin's results [1]. Other generalizations of Lin's results were given by Imoru [4].

Qi and Guo [5] established that

$$\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} < \frac{I(a,b)}{I(a,b+\delta)} \tag{1.6}$$

for all $b > a > 0$, $\delta > 0$ and $r \in \mathbb{R}$. The upper bound in (1.6) is the best possible.

In [6], Chu et al. established the following result:

$$(b - L(a,b))\Psi(b) + (L(a,b) - a)\Psi(a) > (b-a)\Psi\left(\sqrt{ab}\right) \tag{1.7}$$

for all $b > a \geqslant 2$, where the $\Psi$ function is the logarithmic derivative of the gamma function.

Recently, some monotonicity results of the ratio between generalized logarithmic means were established in [7–9].

The purpose of this paper is to answer the following questions: what are the greatest values $p$ and $q$, and the least value $r$ such that $G(a,b)+H(a,b) \geqslant 2L_p(a,b)$, $A(a,b)+H(a,b) \geqslant 2L_q(a,b)$, and $H(a,b) \leqslant L_r(a,b)$ for all $a, b > 0$?

## 2. Main Results

**Theorem 2.1.** *$G(a,b) + H(a,b) \geqslant 2L_{-7/2}(a,b)$ for all $a, b > 0$, with inequality if and only if $a = b$, and the constant $-7/2$ cannot be improved.*

*Proof.* If $a = b$, then from (1.1) we clearly see that $G(a,b) + H(a,b) = 2L_{-7/2}(a,b) = 2a$. Next, we assume that $a \neq b$ and $t = \sqrt{a/b}$, and then elementary computations yield

$$L_{-7/2}(a,b) = b\left[\frac{(5/2)t^5(t+1)}{t^4+t^3+t^2+t+1}\right]^{2/7},$$

$$G(a,b) + H(a,b) = b\frac{t(t+1)^2}{t^2+1},$$

$$[G(a,b)+H(a,b)]^7 - [2L_{-7/2}(a,b)]^7$$
$$= \frac{b^7t^7(t+1)^2}{(t^2+1)^7\left(t^4+t^3+t^2+t+1\right)^2}\left[(t+1)^{12}\left(t^4+t^3+t^2+t+1\right)^2 - 800t^3\left(t^2+1\right)^7\right]$$

$$
\begin{aligned}
&= \frac{b^7 t^7 (t+1)^2}{(t^2+1)^7 \left(t^4+t^3+t^2+t+1\right)^2} \\
&\quad \times \Big(t^{20} + 14t^{19} + 93t^{18} - 408t^{17} + 1186t^{16} - 2830t^{15} + 5254t^{14} - 8402t^{13} + 11597t^{12} - 13974t^{11} \\
&\qquad + 14938t^{10} - 13974t^{9} + 11597t^{8} - 8402t^{7} + 5254t^{6} - 2830t^{5} + 1186t^{4} - 408t^{3} + 93t^{2} + 14t + 1\Big) \\
&= \frac{b^7 t^7 (t+1)^2 (t-1)^4}{(t^2+1)^7 \left(t^4+t^3+t^2+t+1\right)^2} \\
&\quad \times \Big(t^{16} + 18t^{15} + 159t^{14} + 124t^{13} + 799t^{12} + 240t^{11} + 1757t^{10} + 258t^{9} + 2248t^{8} \\
&\qquad + 258t^{7} + 1757t^{6} + 240t^{5} + 799t^{4} + 124t^{3} + 159t^{2} + 18t + 1\Big) > 0.
\end{aligned}
\tag{2.1}
$$

To prove that $-7/2$ is the largest number for which the inequality holds, we take $0 < \varepsilon < 1$ and $0 < x < 1$, and we see that

$$
L_{-7/2+\varepsilon}\left((1+x)^2, 1\right) = \left[\frac{(5-2\varepsilon)x(1+x/2)(1+x)^{5-2\varepsilon}}{(1+x)^{5-2\varepsilon}-1}\right]^{1/(7/2-\varepsilon)},
$$

$$
G\left((1+x)^2, 1\right) + H\left((1+x)^2, 1\right) = \frac{2(1+x)(1+x/2)^2}{1+x+x^2/2},
\tag{2.2}
$$

$$
\begin{aligned}
&\left[2L_{-7/2+\varepsilon}\left((1+x)^2, 1\right)\right]^{7/2-\varepsilon} - \left[G\left((1+x)^2, 1\right) + H\left((1+x)^2, 1\right)\right]^{7/2-\varepsilon} \\
&\quad = \frac{2^{7/2-\varepsilon}(1+x/2)(1+x)^{7/2-\varepsilon}}{\left[(1+x)^{5-2\varepsilon}-1\right](1+x+x^2/2)^{7/2-\varepsilon}} f(x),
\end{aligned}
\tag{2.3}
$$

where $f(x) = (5-2\varepsilon)x(1+x)^{3/2-\varepsilon}(1+x+x^2/2)^{7/2-\varepsilon} - (1+x/2)^{6-2\varepsilon}[(1+x)^{5-2\varepsilon}-1]$.

Making use of the Taylor expansion, we have

$$
\begin{aligned}
f(x) &= (5-2\varepsilon)x\left[1 + \frac{3-2\varepsilon}{2}x + \frac{(3-2\varepsilon)(1-2\varepsilon)}{8}x^2 - \frac{(3-2\varepsilon)(1-2\varepsilon)(1+2\varepsilon)}{48}x^3 + o\left(x^3\right)\right] \\
&\quad \times \left[1 + \frac{7-2\varepsilon}{2}x + \frac{(7-2\varepsilon)^2}{8}x^2 + \frac{(9-2\varepsilon)(7-2\varepsilon)(5-2\varepsilon)x^3}{48}x^3 + o\left(x^3\right)\right] \\
&\quad - \left[1 + (3-\varepsilon)x + \frac{(3-\varepsilon)(5-2\varepsilon)}{4}x^2 + \frac{(3-\varepsilon)(5-2\varepsilon)(2-\varepsilon)}{12}x^3 + o\left(x^3\right)\right] \\
&\quad \times (5-2\varepsilon)x\left[1 + (2-\varepsilon)x + \frac{(2-\varepsilon)(3-2\varepsilon)}{3}x^2 + \frac{(2-\varepsilon)(3-2\varepsilon)(1-\varepsilon)}{6}x^3 + o\left(x^3\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= (5-2\varepsilon)x\left[1+(5-2\varepsilon)x+\frac{47-38\varepsilon+8\varepsilon^2}{4}x^2+o\left(x^2\right)\right]\\
&\quad -(5-2\varepsilon)x\left[1+(5-2\varepsilon)x+\frac{141-121\varepsilon+26\varepsilon^2}{12}x^2+o\left(x^2\right)\right]\\
&= \frac{\varepsilon(7-2\varepsilon)(5-2\varepsilon)}{12}x^3+o\left(x^3\right).
\end{aligned}
\tag{2.4}
$$

Equations (2.3) and (2.4) imply that for any $0<\varepsilon<1$ there exists $0<\delta=\delta(\varepsilon)<1$, such that $2L_{-7/2+\varepsilon}((1+x)^2,1)>G((1+x)^2,1)+H((1+x)^2,1)$ for $x\in(0,\delta)$. □

**Theorem 2.2.** *$A(a,b)+H(a,b)\geqslant 2L_{-2}(a,b)$ for all $a,b>0$, with equality if and only if $a=b$, and the constant $-2$ cannot be improved.*

*Proof.* Simple computations yield

$$
A(a,b)+H(a,b)-2L_{-2}(a,b)=\frac{a+b}{2}+\frac{2ab}{a+b}-2\sqrt{ab}=\frac{\left(\sqrt{a}-\sqrt{b}\right)^4}{2(a+b)}\geqslant 0. \tag{2.5}
$$

Next we prove that $-2$ is the optimal value for which the inequality holds.

For $0<\varepsilon<1$ and $0<t<1$, elementary computations yield

$$
\begin{aligned}
L_{-2+\varepsilon}(1+t,1)&=\left[\frac{(1-\varepsilon)t(1+t)^{1-\varepsilon}}{(1+t)^{1-\varepsilon}-1}\right]^{1/(2-\varepsilon)},\\
A(1+t,1)+H(1+t,1)&=\frac{2\left(1+t+t^2/8\right)}{1+t/2},
\end{aligned}
\tag{2.6}
$$

$$
[2L_{-2+\varepsilon}(1+t,1)]^{2-\varepsilon}-[A(1+t,1)+H(1+t,1)]^{2-\varepsilon}=\frac{2^{2-\varepsilon}}{\left[(1+t)^{1-\varepsilon}-1\right](1+t/2)^{2-\varepsilon}}f(t), \tag{2.7}
$$

where $f(t)=(1-\varepsilon)t(1+t)^{1-\varepsilon}(1+t/2)^{2-\varepsilon}-[(1+t)^{1-\varepsilon}-1](1+t+t^2/8)^{2-\varepsilon}$.

Using Taylor expansion we get

$$
\begin{aligned}
f(t)&=(1-\varepsilon)t\left[1+(1-\varepsilon)t-\frac{\varepsilon(1-\varepsilon)}{2}t^2+o\left(t^2\right)\right]\times\left[1+\frac{2-\varepsilon}{2}t+\frac{(2-\varepsilon)(1-\varepsilon)}{8}t^2+o\left(t^2\right)\right]\\
&\quad-(1-\varepsilon)t\left[1-\frac{\varepsilon}{2}t+\frac{\varepsilon(1+\varepsilon)}{6}t^2+o\left(t^2\right)\right]\times\left[1+(2-\varepsilon)t+\frac{(2-\varepsilon)(5-4\varepsilon)}{8}t^2+o\left(t^2\right)\right]
\end{aligned}
$$

$$= (1-\varepsilon)t\left[1+\frac{4-3\varepsilon}{2}t+\frac{10-19\varepsilon+9\varepsilon^2}{8}t^2+o\left(t^2\right)\right]$$
$$-(1-\varepsilon)t\left[1+\frac{4-3\varepsilon}{2}t+\frac{30-59\varepsilon+28\varepsilon^2}{24}t^2+o\left(t^2\right)\right]$$
$$=\frac{\varepsilon(1-\varepsilon)(2-\varepsilon)}{24}t^3+o\left(t^3\right). \tag{2.8}$$

Equations (2.7) and (2.8) imply that for any $0<\varepsilon<1$ there exists $0<\delta=\delta(\varepsilon)<1$, such that $2L_{-2+\varepsilon}(1+t,1)>A(1+t,1)+H(1+t,1)$ for $t\in(0,\delta)$. □

**Theorem 2.3.** *$H(a,b)\leqslant L_{-5}(a,b)$ for all $a,b>0$, with equality if and only if $a=b$, and the constant $-5$ cannot be improved.*

*Proof.* Form (1.1) we clearly see that $L_{-5}(a,b)=H(a,b)=a$ if $a=b$. If $a\neq b$, then simple computations yield

$$L_{-5}(a,b)=b\left[\frac{4(a/b)^4}{((a/b)^2+1)(a/b+1)}\right]^{1/5},$$
$$H(a,b)=b\frac{2\cdot a/b}{1+a/b}, \tag{2.9}$$
$$[L_{-5}(a,b)]^5-[H(a,b)]^5=\frac{4b^5(a/b)^4}{(1+a/b)^5\left(1+(a/b)^2\right)}\left(\frac{a}{b}-1\right)^4>0.$$

To show that $-5$ is the best possible constant for which the inequality holds, let $0<\varepsilon<1$ and $0<t<1$, and then

$$[H(1+t,1)]^{5+\varepsilon}-\left[L_{-(5+\varepsilon)}(1+t,1)\right]^{5+\varepsilon}=\frac{(1+t)^{4+\varepsilon}}{(1+t/2)^{5+\varepsilon}\left[(1+t)^{4+\varepsilon}-1\right]}f(t), \tag{2.10}$$

where $f(t)=(1+t)[(1+t)^{4+\varepsilon}-1]-(4+\varepsilon)t(1+t/2)^{5+\varepsilon}$.

Using Taylor expansion we have

$$f(t)=(1+t)\left[(4+\varepsilon)t+\frac{(4+\varepsilon)(3+\varepsilon)}{2}t^2+\frac{(4+\varepsilon)(3+\varepsilon)(2+\varepsilon)}{6}t^3+o\left(t^3\right)\right]$$
$$-(4+\varepsilon)t\left[1+\frac{5+\varepsilon}{2}t+\frac{(5+\varepsilon)(4+\varepsilon)}{8}t^2+o\left(t^2\right)\right] \tag{2.11}$$
$$=\frac{\varepsilon(4+\varepsilon)(5+\varepsilon)}{24}t^3+o\left(t^3\right).$$

Equations (2.10) and (2.11) imply that for any $0<\varepsilon<1$ there exists $0<\delta=\delta(\varepsilon)<1$, such that $H(1+t,1)>L_{-(5+\varepsilon)}(1+t,1)$ for $t\in(0,\delta)$. □

*Remark 2.4.* If $p \geqslant 5$, then

$$[L_{-p}(a,1)]^p - [H(a,1)]^p = \frac{1}{(a^{p-1}-1)(1+a)^p}\left[(p-1)(a-1)a^{p-1}(1+a)^p - 2^p a^p\left(a^{p-1}-1\right)\right],$$
$$\lim_{a\to+\infty}\left[(p-1)(a-1)a^{p-1}(1+a)^p - 2^p a^p\left(a^{p-1}-1\right)\right] = +\infty. \tag{2.12}$$

Therefore, we cannot get inequality $H(a,b) \geqslant L_p(a,b)$ for any $p \in \mathbb{R}$ and all $a,b > 0$.

*Remark 2.5.* It is easy to verify that $A(a,b) + G(a,b) = 2L_{-1/2}(a,b)$ for all $a,b > 0$.

## Acknowledgments

This research is partly supported by N S Foundation of China under Grant 60850005 and N S Foundation of Zhejiang Province under Grants D7080080 and Y7080185.

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