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Research Article

On Bounded Boundary and Bounded Radius Rotations

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We establish a relation between the functions of bounded boundary and bounded radius rotations by using three different techniques. A well-known result is observed as a special case from our main result. An interesting application of our work is also being investigated.

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. We say that $f \in A$ is subordinate to $g \in A$, written as $f \prec g$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ $(z \in E)$, such that $f(z) = g(w(z))$. In particular, when g is univalent, then the above subordination is equivalent to $f(0) = g(0)$ and $f(E) \subseteq g(E)$.

For any two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in E), \tag{1.2}$$

the convolution (Hadamard product) of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in E). \tag{1.3}$$

We denote by $S^*(\alpha)$, $C(\alpha)$, $(0 \leq \alpha < 1)$, the classes of starlike and convex functions of order α, respectively, defined by

$$S^*(\alpha) = \left\{ f \in A\colon \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in E \right\},$$
$$C(\alpha) = \left\{ f \in A\colon zf'(z) \in S^*(\alpha),\ z \in E \right\}. \tag{1.4}$$

For $\alpha = 0$, we have the well-known classes of starlike and convex univalent functions denoted by S^* and C, respectively.

Let $P_k(\alpha)$ be the class of functions $p(z)$ analytic in the unit disc E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{p(z) - \alpha}{1 - \alpha} \right| d\theta \leq k\pi, \tag{1.5}$$

where $z = re^{i\theta}$, $k \geq 2$, and $0 \leq \alpha < 1$. For $\alpha = 0$, we obtain the class P_k introduced in [1]. Also, for $p \in P_k(\alpha)$, we can write $p(z) = (1-\alpha)q_1(z) + \alpha$, $q_1 \in P_k$. We can also write, for $p \in P_k(\alpha)$,

$$p(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\alpha)ze^{-it}}{1 - ze^{-it}} d\mu(t), \qquad z \in E, \tag{1.6}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2\pi, \qquad \int_0^{2\pi} |d\mu(t)| \leq k\pi. \tag{1.7}$$

For (1.6) together with (1.7), see [2]. Since $\mu(t)$ has a bounded variation on $[0, 2\pi]$, we may write $\mu(t) = A(t) - B(t)$, where $A(t)$ and $B(t)$ are two non-negative increasing functions on $[0, 2\pi]$ satisfying (1.7). Thus, if we set $A(t) = ((k/4) + (1/2))\mu_1(t)$ and $B(t) = ((k/4) - (1/2))\mu_2(t)$, then (1.6) becomes

$$\begin{aligned} p(z) = {} & \left(\frac{k}{4} + \frac{1}{2}\right) \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\alpha)ze^{-it}}{1 - ze^{-it}} d\mu_1(t) \\ & - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\alpha)ze^{-it}}{1 - ze^{-it}} d\mu_2(t). \end{aligned} \tag{1.8}$$

Now, using Herglotz-Stieltjes formula for the class $P(\alpha)$ and (1.8), we obtain

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \qquad z \in E, \tag{1.9}$$

where $P(\alpha)$ is the class of functions with real part greater than α and $p_i \in P(\alpha)$, for $i = 1, 2$.

We define the following classes:

$$R_k(\alpha) = \left\{ f:\ f \in A \text{ and } \frac{zf'(z)}{f(z)} \in P_k(\alpha),\ 0 \leq \alpha < 1 \right\},$$
$$V_k(\alpha) = \left\{ f:\ f \in A \text{ and } \frac{(zf'(z))'}{f'(z)} \in P_k(\alpha),\ 0 \leq \alpha < 1 \right\}. \tag{1.10}$$

We note that

$$f \in V_k(\alpha) \Longleftrightarrow zf' \in R_k(\alpha). \tag{1.11}$$

For $\alpha = 0$, we obtain the well-known classes R_k and V_k of analytic functions with bounded radius and bounded boundary rotations, respectively. These classes are studied by Noor [3–5] in more details. Also it can easily be seen that $R_2(\alpha) = S^*(\alpha)$ and $V_2(\alpha) = C(\alpha)$.

Goel [6] proved that $f \in C(\alpha)$ implies that $f \in S^*(\beta)$, where

$$\beta = \beta(\alpha) = \begin{cases} \dfrac{4^{\alpha}(1-2\alpha)}{4 - 2^{2\alpha+1}}, & \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\ln 2}, & \alpha = \dfrac{1}{2}, \end{cases} \tag{1.12}$$

and this result is sharp.

In this paper, we prove the result of Goel [6] for the classes $V_k(\alpha)$ and $R_k(\alpha)$ by using three different methods. The first one is the same as done by Goel [6], while the second and third are the convolution and subordination techniques.

2. Preliminary Results

We need the following results to obtain our results.

Lemma 2.1. *Let $f \in V_k(\alpha)$. Then there exist $s_1, s_2 \in S^*(\alpha)$ such that*

$$f'(z) = \frac{(s_1(z)/z)^{(k/4)+(1/2)}}{(s_2(z)/z)^{(k/4)-(1/2)}}, \quad z \in E. \tag{2.1}$$

Proof. It can easily be shown that $f \in V_k(\alpha)$ if and only if there exists $g \in V_k$ such that

$$f'(z) = \left(g'(z)\right)^{1-\alpha}, \quad z \in E, \text{ see [2]}. \tag{2.2}$$

From Brannan [7] representation form for functions with bounded boundary rotations, we have

$$g'(z)=\frac{\left(\dfrac{g_1(z)}{z}\right)^{\left(\frac{k}{4}\right)+\left(\frac{1}{2}\right)}}{\left(\dfrac{g_2(z)}{z}\right)^{\left(\frac{k}{4}\right)-\left(\frac{1}{2}\right)}},\quad g_i\in S^*,\ i=1,2. \tag{2.3}$$

Now, it is shown in [8] that for $s_i \in S^*(\alpha)$, we can write

$$s_i(z)=z\left[\frac{g_i(z)}{z}\right]^{1-\alpha},\quad g_i\in S^*,\ i=1,2. \tag{2.4}$$

Using (2.3) together with (2.4) in (2.2), we obtain the required result. □

Lemma 2.2 (see [9]). *Let $u=u_1+iu_2$, $v=v_1+iv_2$, and $\Psi(u,v)$ be a complex-valued function satisfying the conditions:*

(i) *$\Psi(u,v)$ is continuous in a domain $D\subset\mathbb{C}^2$,*

(ii) *$(1,0)\in D$ and $\operatorname{Re}\Psi(1,0)>0$,*

(iii) *$\operatorname{Re}\Psi(iu_2,v_1)\le 0$, whenever $(iu_2,v_1)\in D$ and $v_1\le -(1/2)(1+u_2^2)$.*

If $h(z)=1+c_1z+\cdots$ is a function analytic in E such that $(h(z),zh'(z))\in D$ and $\operatorname{Re}\Psi(h(z),zh'(z))>0$ for $z\in E$, then $\operatorname{Re}h(z)>0$ in E.

Lemma 2.3. *Let $\beta>0$, $\beta+\gamma>0$, and $\alpha\in[\alpha_0,1)$, with*

$$\alpha_0=\max\left\{\frac{\beta-\gamma-1}{2\beta},\frac{-\gamma}{\beta}\right\}. \tag{2.5}$$

If

$$\left\{h(z)+\frac{zh'(z)}{\beta h(z)+\gamma}\right\}\prec\frac{1+(1-2\alpha)z}{1-z}, \tag{2.6}$$

then

$$h(z)\prec Q(z)\prec\frac{1+(1-2\alpha)z}{1-z}, \tag{2.7}$$

where

$$\begin{gathered}Q(z)=\frac{1}{\beta G(z)}-\frac{\gamma}{\beta},\\ G(z)=\int_0^1\left[\frac{1-z}{1-tz}\right]^{2\beta(1-\alpha)}t^{\beta+\gamma-1}dt=\frac{{}_2F_1(2\beta(1-\alpha),1,\beta+\gamma+1;z/(z-1))}{(\beta+\gamma)},\end{gathered} \tag{2.8}$$

${}_2F_1$ denotes Gauss hypergeometric function. From (2.7), one can deduce the sharp result that $h \in P(\beta)$, with

$$\beta = \beta(\alpha, \beta, \gamma) = \min \ \operatorname{Re} Q(z) = Q(-1). \tag{2.9}$$

This result is a special case of the one given in [10, page 113].

3. Main Results

By using the same method as that of Goel [6], we prove the following result. We include all the details for the sake of completeness.

3.1. First Method

Theorem 3.1. *Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta)$, where $\beta = \beta(\alpha)$ is given by (1.12). This result is sharp.*

Proof. Since $f \in V_k(\alpha)$, we use Lemma 2.1, with relation (1.11) to have

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= \left(\frac{k}{4} + \frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\frac{(zf_1'(z))'}{f_1'(z)} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{(zf_2'(z))'}{f_2'(z)}, \end{aligned} \tag{3.1}$$

where $s_i \in S^*(\alpha)$ and $f_i \in C(\alpha)$, $i = 1, 2$.

Therefore, from (2.4), we have

$$\frac{zf'(z)}{f(z)} = \left(\frac{k}{4} + \frac{1}{2}\right)\frac{z[g_1(z)/z]^{1-\alpha}}{\int_0^z [g_1(\phi)/\phi]^{1-\alpha} d\phi} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{z[g_2(z)/z]^{1-\alpha}}{\int_0^z [g_2(\phi)/\phi]^{1-\alpha} d\phi}, \tag{3.2}$$

that is,

$$\begin{aligned} \frac{zf'(z)}{f(z)} = \left(\frac{k}{4} + \frac{1}{2}\right)&\left[\int_0^z \left[\frac{z}{\phi}\right]^{1-\alpha}\left[\frac{g_1(\phi)}{g_1(z)}\right]^{1-\alpha}\frac{d\phi}{z}\right]^{-1} \\ &- \left(\frac{k}{4} - \frac{1}{2}\right)\left[\int_0^z \left[\frac{z}{\phi}\right]^{1-\alpha}\left[\frac{g_2(\phi)}{g_2(z)}\right]^{1-\alpha}\frac{d\phi}{z}\right]^{-1}, \end{aligned} \tag{3.3}$$

where we integrate along the straight line segment $[0, z]$, $z \in E$.

Writing

$$\frac{zf'(z)}{f(z)} = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} + \frac{1}{2}\right)p_2(z), \tag{3.4}$$

and using (3.3), we have

$$p_i(z) = \left[\int_0^z \left[\frac{z}{\phi}\right]^{1-\alpha} \left[\frac{g_i(\phi)}{g_i(z)}\right]^{1-\alpha} \frac{d\phi}{z}\right]^{-1}, \tag{3.5}$$

where $p_i(0) = 1$ and hence by [11] we have

$$\left|p_i(z) - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}, \quad |z| = r,\ z \in E. \tag{3.6}$$

Therefore,

$$\min_{f_i \in C(\alpha)} \min_{|z|=r} \operatorname{Re}[p_i(z)] = \min_{f_i \in C(\alpha)} \min_{|z|=r} |p_i(z)|. \tag{3.7}$$

Let $z = re^{i\theta}$ and $\phi = Re^{i\theta}$, $0 < R < r < 1$. For fixed z and ϕ, we have from (2.4)

$$\left|\frac{g_i(\phi)}{g_i(z)}\right| \leq \frac{R}{r}\left(\frac{1+r}{1+R}\right)^2. \tag{3.8}$$

Now, using (3.8), we have, for a fixed $z \in E,\ |z| = r$,

$$\left|\int_0^z \left[\frac{z}{\phi}\right]^{1-\alpha} \left[\frac{g_i(\phi)}{g_i(z)}\right]^{1-\alpha} \frac{d\phi}{z}\right| \leq \int_0^r \left(\frac{1+r}{1+R}\right)^{2(1-\alpha)} \frac{dR}{r}. \tag{3.9}$$

Let

$$T(r) = \int_0^r \left(\frac{1+r}{1+R}\right)^{2(1-\alpha)} \frac{dR}{r}, \tag{3.10}$$

with $R = rt$, $0 < t < 1$, we have

$$T(r) = \int_0^1 \left(\frac{1+r}{1+rt}\right)^{2(1-\alpha)} dt. \tag{3.11}$$

By differentiating we note that

$$T'(r) = 2(1-\alpha)\int_0^1 \frac{(1-t)}{(1+rt)^2}\left(\frac{1+r}{1+rt}\right)^{(1-2\alpha)} dt > 0, \tag{3.12}$$

and therefore $T(r)$ is a monotone increasing function of r and hence

$$\begin{aligned}\max_{0\le r\le 1} T(r) = T(1) &= 2^{2(1-\alpha)}\int_0^1 \frac{dt}{(1+t)^{2(1-\alpha)}}\\ &= \begin{cases} \dfrac{(2-4^{(1-\alpha)})}{(2\alpha-1)}, & \text{if } \alpha \neq \dfrac{1}{2}\\ 2\ln 2, & \text{if } \alpha = \dfrac{1}{2}. \end{cases}\end{aligned} \tag{3.13}$$

By letting

$$\beta(\alpha) = \min\left[\left|\int_0^z \left[\frac{z}{\phi}\right]^{1-\alpha}\left[\frac{g_i(\phi)}{g_i(z)}\right]^{1-\alpha}\frac{d\phi}{z}\right|\right]^{-1}, \quad z \in E, \tag{3.14}$$

for all $g_i(z) \in S^*$, we obtain the required result from (3.7), (3.13), and (3.14).

Sharpness can be shown by the function $f_0 \in V_k(\alpha)$ given by

$$\frac{(zf_0'(z))'}{f_0'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{1-(1-2\alpha)z}{1+z}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{1+(1-2\alpha)z}{1-z}\right). \tag{3.15}$$

It is easy to check that $f_0 \in R_k(\beta)$, where β is the exact value given by (1.12). □

3.2. *Second Method*

Theorem 3.2. *Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta)$, where*

$$\beta = \frac{1}{4}\left[(2\alpha-1) + \sqrt{4\alpha^2 - 4\alpha + 9}\ \right]. \tag{3.16}$$

Proof. Let

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= (1-\beta)p(z) + \beta\\ &= (1-\beta)\left[\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\right] + \beta\end{aligned} \tag{3.17}$$

$p(z)$ is analytic in E with $p(0) = 1$. Then

$$\frac{(zf'(z))'}{f'(z)} = (1-\beta)p(z) + \beta + \frac{(1-\beta)zp'(z)}{(1-\beta)p(z)+\beta}, \tag{3.18}$$

that is,

$$\begin{aligned}\frac{1}{1-\alpha}\left[\frac{(zf'(z))'}{f'(z)} - \alpha\right] &= \frac{1}{1-\alpha}\left[(1-\beta)p(z) + \beta - \alpha + \frac{(1-\beta)zp'(z)}{(1-\beta)p(z)+\beta}\right] \\ &= \frac{(\beta-\alpha)}{1-\alpha} + \frac{(1-\beta)}{1-\alpha}\left[p(z) + \frac{(1/(1-\beta))zp'(z)}{p(z)+(\beta/(1-\beta))}\right].\end{aligned} \tag{3.19}$$

Since $f \in V_k(\alpha)$, it implies that

$$\frac{(\beta-\alpha)}{1-\alpha} + \frac{(1-\beta)}{1-\alpha}\left[p(z) + \frac{(1/(1-\beta))zp'(z)}{p(z)+(\beta/(1-\beta))}\right] \in P_k, \quad z \in E. \tag{3.20}$$

We define

$$\varphi_{a,b}(z) = \frac{1}{1+b}\frac{z}{(1-z)^a} + \frac{b}{1+b}\frac{z}{(1-z)^{1+a}}, \tag{3.21}$$

with $a = 1/(1-\beta)$, $b = \beta/(1-\beta)$. By using (3.17) with convolution techniques, see [5], we have that

$$\frac{\varphi_{a,b}(z)}{z} * p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{\varphi_{a,b}(z)}{z} * p_1(z)\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[\frac{\varphi_{a,b}(z)}{z} * p_2(z)\right] \tag{3.22}$$

implies

$$p(z) + \frac{azp'(z)}{p(z)+b} = \left(\frac{k}{4}+\frac{1}{2}\right)\left[p_1(z) + \frac{azp_1'(z)}{p_1(z)+b}\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[p_2(z) + \frac{azp_2'(z)}{p_2(z)+b}\right]. \tag{3.23}$$

Thus, from (3.20) and (3.23), we have

$$\frac{(\beta-\alpha)}{1-\alpha} + \frac{(1-\beta)}{1-\alpha}\left[p_i(z) + \frac{azp_i'(z)}{p_i(z)+b}\right] \in P, \quad i = 1,2. \tag{3.24}$$

We now form the functional $\Psi(u,v)$ by choosing $u = p_i(z)$, $v = zp_i'(z)$ in (3.24) and note that the first two conditions of Lemma 2.2 are clearly satisfied. We check condition (iii) as follows:

$$\begin{aligned}
\operatorname{Re}[\psi(iu_2, v_1)] &= \frac{1}{1-\alpha}\left[(\beta-\alpha) + \operatorname{Re}\left(\frac{v_1}{iu_2 + (\beta/(1-\beta))}\right)\right] \\
&= \frac{1}{1-\alpha}\left[(\beta-\alpha) + \frac{v_1(\beta/(1-\beta))}{u_2^2 + (\beta/(1-\beta))^2}\right] \\
&\leq \frac{1}{1-\alpha}\left[(\beta-\alpha) - \frac{1}{2}\frac{(1+u_2^2)(\beta/(1-\beta))}{u_2^2 + (\beta/(1-\beta))^2}\right] \\
&= \frac{2(\beta-\alpha)\left(u_2^2 + (\beta/(1-\beta))^2\right) - (1+u_2^2)(\beta/(1-\beta))}{2\left(u_2^2 + (\beta/(1-\beta))^2\right)(1-\alpha)} \\
&= \frac{\left[2(\beta-\alpha)\left(\beta^2/(1-\beta)^2\right) - (\beta/(1-\beta))\right] + (2\beta - 2\alpha - (\beta/(1-\beta)))u_2^2}{2\left(u_2^2 + (\beta/(1-\beta))^2\right)(1-\alpha)} \\
&= \frac{A + Bu_2^2}{2C}, \quad 2C > 0,
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
A &= \frac{\beta}{(1-\beta)^2}[2(\beta-\alpha)\beta - (1-\beta)], \\
B &= \frac{1}{1-\beta}(2(\beta-\alpha)(1-\beta) - \beta), \\
C &= (1-\alpha)\left(u_2^2 + \left(\frac{\beta}{1-\beta}\right)^2\right) > 0.
\end{aligned} \tag{3.26}$$

The right-hand side of (3.25) is negative if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we have

$$\beta = \beta(\alpha) = \frac{1}{4}\left[(2\alpha - 1) + \sqrt{4\alpha^2 - 4\alpha + 9}\ \right], \tag{3.27}$$

and from $B \leq 0$, it follows that $0 \leq \beta < 1$.

Since all the conditions of Lemma 2.2 are satisfied, it follows that $p_i \in P$ in E for $i = 1, 2$ and consequently $p \in P_k$ and hence $f \in R_k(\beta)$, where β is given by (3.16). The case $k = 2$ is discussed in [12]. □

3.3. Third Method

Theorem 3.3. *Let $f \in V_k(\alpha)$. Then $f \in R_k(\beta)$, where*

$$\beta = \beta_1(\alpha, 1, 0) = \begin{cases} \dfrac{2\alpha - 1}{2 - 2^{2(1-\alpha)}}, & \text{if } \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\ln 2}, & \text{if } \alpha = \dfrac{1}{2}. \end{cases} \tag{3.28}$$

Proof. Let

$$\frac{zf'(z)}{f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)}, \tag{3.29}$$

and let

$$\frac{zs_i'(z)}{s_i(z)} = p_i(z), \quad i = 1, 2. \tag{3.30}$$

Then p, p_i are analytic in E with $p(0) = 1$, $p_i(0) = 1$, $i = 1, 2$.

Logarithmic differentiation yields

$$\begin{aligned} \frac{(zf'(z))'}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} &= \left(\frac{k}{4} + \frac{1}{2}\right)\frac{(zs_1'(z))'}{s_1'(z)} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{(zs_2'(z))'}{s_2'(z)} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\left(p_1(z) + \frac{zp_1'(z)}{p_1(z)}\right) - \left(\frac{k}{4} - \frac{1}{2}\right)\left(p_2(z) + \frac{zp_2'(z)}{p_2(z)}\right). \end{aligned} \tag{3.31}$$

Since $f \in V_k(\alpha)$, it follows that $(zs_i')'/s_i' \in P(\alpha)$, $z \in E$, or $s_i \in C(\alpha)$ for $z \in E$. Consequently,

$$\left(p_i(z) + \frac{zp_i'(z)}{p_i(z)}\right) \in P(\alpha), \tag{3.32}$$

where $zs_i'(z)/s_i(z) = p_i(z)$, $i = 1, 2$. We use Lemma 2.3 with $\gamma = 0$, $\beta = 1 > 0$, $\alpha \in [0, 1)$, and $h = p_i$ in (3.32), to have $p_i \in P(\beta)$, where β is given in (3.28) and this estimate is best possible, extremal function Q is given by

$$Q(z) = \begin{cases} \dfrac{(1-2\alpha)z}{(1-z)\left[1 - (1-z)^{1-2\alpha}\right]}, & \text{if } \alpha \neq \dfrac{1}{2}, \\ \dfrac{z}{(z-1)\log(1-z)}, & \text{if } \alpha = \dfrac{1}{2}, \end{cases} \tag{3.33}$$

see [10]. MacGregor [13] conjectured the exact value given by (3.28). Thus $s_i \in S^*(\beta)$ and consequently $f \in R_k(\beta)$, where the exact value of β is given by (3.28). □

3.4. Application of Theorem 3.3

Theorem 3.4. *Let g and h belong to $V_k(\alpha)$. Then $F(z)$, defined by*

$$F(z) = \int_0^z \left(\frac{g(t)}{t}\right)^{\mu} \left(\frac{h(t)}{t}\right)^{\eta} dt, \tag{3.34}$$

is in the class $V_k(\delta)$, where $0 \leq \mu < \eta \leq 1$, $\delta = \delta(\alpha) = (1 - (\mu + \eta)(1 - \beta))$, and $\beta(\alpha)$ is given by (1.12).

Proof. From (3.34), we can easily write

$$\frac{(zF'(z))'}{F'(z)} = \mu \frac{zg'(z)}{g(z)} + \eta \frac{zh'(z)}{h(z)} + 1 - (\mu + \eta). \tag{3.35}$$

Since g and h belong to $V_k(\alpha)$, then, by Theorem 3.3, $zg'(z)/g(z)$ and $zh'(z)/h(z)$ belong to $P_k(\beta)$, where $\beta = \beta(\alpha)$ is given by (1.12). Using

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= (1 - \beta) q_1(z) + \beta, \quad q_1 \in P_k, \\ \frac{zh'(z)}{h(z)} &= (1 - \beta) q_2(z) + \beta, \quad q_2 \in P_k, \end{aligned} \tag{3.36}$$

in (3.35), we have

$$\frac{1}{1 - \delta}\left[\frac{(zF'(z))'}{F'(z)} - \delta\right] = \frac{\mu}{\mu + \eta} \quad q_1(z) + \frac{\eta}{\mu + \eta} q_2(z). \tag{3.37}$$

Now by using the well-known fact that the class P_k is a convex set together with (3.37), we obtain the required result. □

For $\alpha = 0$, $\mu = 0$, and $\eta = 1$, we have the following interesting corollary.

Corollary 3.5. *Let f belongs to $V_k(0)$. Then $F(z)$, defined by*

$$F(z) = \int_0^z \frac{f(t)}{t} dt \quad \text{(Alexander's integral operator)}, \tag{3.38}$$

is in the class $V_k(1/2)$.

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