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Research Article

Strong Convergence Theorems for Common Fixed Points of Multistep Iterations with Errors in Banach Spaces

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We establish strong convergence theorem for multi-step iterative scheme with errors for asymptotically nonexpansive mappings in the intermediate sense in Banach spaces. Our results extend and improve the recent ones announced by Plubtieng and Wangkeeree (2006), and many others.

1. Introduction

Let C be a subset of real normal linear space X. A mapping $T : C \to C$ is said to be asymptotically nonexpansive on C if there exists a sequence $\{r_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that for each $x, y \in C$,

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|, \quad \forall n \geq 1. \tag{1.1}$$

If $r_n \equiv 0$, then T is known as a nonexpansive mapping. T is called asymptotically nonexpansive in the intermediate sense [1] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \leq 0. \tag{1.2}$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense.

Let C be a nonempty subset of normed space X, and Let $T_i : C \to C$ be m mappings. For a given $x_1 \in C$ and a fixed $m \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of all positive integers), compute the iterative sequences $x_n^{(1)}, \ldots, x_n^{(m)}$defined by

$$\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)} T_i^k x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)}, \\
x_n^{(2)} &= \alpha_n^{(2)} T_i^k x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}, \\
x_n^{(3)} &= \alpha_n^{(3)} T_i^k x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}, \\
&\vdots \\
x_n^{(m-1)} &= \alpha_n^{(m-1)} T_i^k x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)}, \\
x_{n+1} = x_n^{(m)} &= \alpha_n^{(m)} T_i^k x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)}, \quad \forall n \geq 1,
\end{aligned} \tag{1.3}$$

where $n = (k-1)m + i$, $\{u_n^{(1)}\}, \{u_n^{(2)}\}, \ldots, \{u_n^{(m)}\}$ are bounded sequences in C and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$, are appropriate real sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \ldots, m\}$.

The purpose of this paper is to establish a strong convergence theorem for common fixed points of the multistep iterative scheme with errors for asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space. The results presented in this paper extend and improve the corresponding ones announced by Plubtieng and Wangkeeree [2], and many others.

2. Preliminaries

Definition 2.1 (see [1]). A Banach space X is said to be a uniformly convex if the modulus of convexity of X is

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \|x-y\| = \epsilon\right\} > 0, \quad \forall \epsilon \in (0,2]. \tag{2.1}$$

Lemma 2.2 (see [3]). *Let $\{a_n\}, \{b_n\}$, and $\{\gamma_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1+\gamma_n) a_n + b_n, \quad \forall n \geq 1, \tag{2.2}$$

where $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then

(1) $\lim_{n\to\infty} a_n$ *exists;*

(2) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 (see [4]). *Let X be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that*

$$\begin{gathered}\limsup_{n\to\infty}\|x_n\| \leq a,\\ \limsup_{n\to\infty}\|y_n\| \leq a,\\ \lim_{n\to\infty}\|t_nx_n+(1-t_n)y_n\| = a,\end{gathered} \tag{2.3}$$

for some $a \geq 0$. Then

$$\lim_{n\to\infty}\|x_n-y_n\| = 0. \tag{2.4}$$

3. Main Results

Lemma 3.1. *Let X be a uniformly convex Banach space, $\{x_n\}$, $\{y_n\}$ are two sequences of X, $\alpha,\beta \in (0,1)$ and $\{\alpha_n\}$ be a real sequence. If there exists $n_0 \in \mathbb{N}$ such that*

(i) $0 < \alpha \leq \alpha_n \leq \beta < 1$ *for all* $n \geq n_0$;

(ii) $\limsup_{n\to\infty}\|x_n\| \leq a$;

(iii) $\limsup_{n\to\infty}\|y_n\| \leq a$;

(iv) $\lim_{n\to\infty}\|\alpha_nx_n+(1-\alpha_n)y_n\| = a$,

then $\lim_{n\to\infty}\|x_n-y_n\| = 0$.

Proof. The proof is clear by Lemma 2.3. □

Lemma 3.2. *Let X be a uniformly convex Banach space, let C be a nonempty closed bounded convex subset of X, and let $T_i : C \to C$ be m asymptotically nonexpansive mappings in the intermediate sense such that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Put*

$$G_{ik} = \sup_{x,y\in C}\left(\left\|T_i^kx - T_i^ky\right\| - \|x-y\|\right) \vee 0, \quad \forall k \geq 1, \tag{3.1}$$

so that $\sum_{k=1}^{\infty}G_{ik} < \infty$. Let $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ be real sequences in $[0,1]$ satisfying the following condition:

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for all* $i \in \{1,2,\ldots,m\}$ *and* $n \geq 1$;

(ii) $\sum_{n=1}^{\infty}\gamma_n^{(i)} < \infty$ *for all* $i \in \{1,2,\ldots,m\}$.

If $\{x_n\}$ is the iterative sequence defined by (1.3), then, for each $p \in F = \bigcap_{i=1}^m F(T_i)$, the limit $\lim_{n\to\infty}\|x_n-p\|$ exists.

Proof. For each $q \in F$, we note that

$$
\begin{aligned}
\left\|x_n^{(1)} - q\right\| &= \left\|\alpha_n^{(1)} T_i^k x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - q\right\| \\
&\le \alpha_n^{(1)} \left\|T_i^k x_n - q\right\| + \beta_n^{(1)} \|x_n - q\| + \gamma_n^{(1)} \left\|u_n^{(1)} - q\right\| \\
&\le \alpha_n^{(1)} \|x_n - q\| + \alpha_n^{(1)} G_{ik} + \beta_n^{(1)} \|x_n - q\| + \gamma_n^{(1)} \left\|u_n^{(1)} - q\right\| \\
&= \left(\alpha_n^{(1)} + \beta_n^{(1)}\right) \|x_n - q\| + \alpha_n^{(1)} G_{ik} + \gamma_n^{(1)} \left\|u_n^{(1)} - q\right\| \\
&\le \|x_n - q\| + d_n^{(1)},
\end{aligned} \tag{3.2}
$$

where $d_n^{(1)} = \alpha_n^{(1)} G_{ik} + \gamma_n^{(1)} \|u_n^{(1)} - q\|$. Since

$$
\sum_{n=1}^{\infty} G_{ik} = \sum_{i \in I} \sum_{k=1}^{\infty} G_{ik} < \infty, \tag{3.3}
$$

we see that

$$
\sum_{n=1}^{\infty} d_n^{(1)} < \infty. \tag{3.4}
$$

It follows from (3.2) that

$$
\begin{aligned}
\left\|x_n^{(2)} - q\right\| &\le \alpha_n^{(2)} \left\|x_n^{(1)} - q\right\| + \alpha_n^{(2)} G_{ik} + \beta_n^{(2)} \|x_n - q\| + \gamma_n^{(2)} \left\|u_n^{(2)} - q\right\| \\
&\le \alpha_n^{(2)} \left(\|x_n - q\| + d_n^{(1)}\right) + \alpha_n^{(2)} G_{ik} + \beta_n^{(2)} \|x_n - q\| + \gamma_n^{(2)} \left\|u_n^{(2)} - q\right\| \\
&= \left(\alpha_n^{(2)} + \beta_n^{(2)}\right) \|x_n - q\| + \alpha_n^{(2)} d_n^{(1)} + \alpha_n^{(2)} G_{ik} + \gamma_n^{(2)} \left\|u_n^{(2)} - q\right\| \\
&\le \|x_n - q\| + d_n^{(2)},
\end{aligned} \tag{3.5}
$$

where $d_n^{(2)} = \alpha_n^{(2)} d_n^{(1)} + \alpha_n^{(2)} G_{ik} + \gamma_n^{(2)} \|u_n^{(2)} - q\|$. Since

$$
\sum_{n=1}^{\infty} G_{ik} < \infty, \qquad \sum_{n=1}^{\infty} d_n^{(1)} < \infty, \tag{3.6}
$$

we see that

$$
\sum_{n=1}^{\infty} d_n^{(2)} < \infty. \tag{3.7}
$$

It follows from (3.5) that

$$\begin{aligned}
\left\|x_n^{(3)}-q\right\| &\leq \alpha_n^{(3)}\left\|x_n^{(2)}-q\right\|+\alpha_n^{(3)}G_{ik}+\beta_n^{(3)}\|x_n-q\|+\gamma_n^{(3)}\left\|u_n^{(3)}-q\right\| \\
&\leq \alpha_n^{(3)}\left(\|x_n-q\|+d_n^{(1)}\right)+\alpha_n^{(3)}G_{ik}+\beta_n^{(3)}\|x_n-q\|+\gamma_n^{(3)}\left\|u_n^{(3)}-q\right\| \\
&= \left(\alpha_n^{(3)}+\beta_n^{(3)}\right)\|x_n-q\|+\alpha_n^{(3)}d_n^{(2)}+\alpha_n^{(3)}G_{ik}+\gamma_n^{(3)}\left\|u_n^{(3)}-q\right\| \\
&\leq \|x_n-q\|+d_n^{(3)},
\end{aligned} \tag{3.8}$$

where $d_n^{(3)}=\alpha_n^{(3)}d_n^{(2)}+\alpha_n^{(3)}G_{ik}+\gamma_n^{(3)}\|u_n^{(3)}-q\|$, and so

$$\sum_{n=1}^{\infty}d_n^{(3)}<\infty. \tag{3.9}$$

By continuing the above method, there are nonnegative real sequences $\{d_n^{(k)}\}$ such that

$$\begin{gathered}
\sum_{n=1}^{\infty}d_n^{(k)}<\infty, \\
\left\|x_n^{(k)}-q\right\|\leq\|x_n-q\|+d_n^{(k)}, \quad \forall k\in\{1,2,\ldots,m\}.
\end{gathered} \tag{3.10}$$

This together with Lemma 2.2 gives that $\lim_{n\to\infty}\|x_n-q\|$ exists. This completes the proof. □

Lemma 3.3. *Let X be a uniformly convex Banach space, let C be a nonempty closed bounded convex subset of X, and let $T_i : C \to C$ be m asymptotically nonexpansive mappings in the intermediate sense such that $F=\bigcap_{i=1}^{m}F(T_i)\neq\emptyset$. Put*

$$G_{ik}=\sup_{x,y\in C}\left(\left\|T_i^kx-T_i^ky\right\|-\|x-y\|\right)\vee 0, \quad \forall k\geq 1, \tag{3.11}$$

so that $\sum_{k=1}^{\infty}G_{ik}<\infty$. Let the sequence $\{x_n\}$ be defined by (1.3) whenever $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ satisfy the same assumptions as in Lemma 3.2 for each $i\in\{1,2,\ldots,m\}$ and the additional assumption that there exists $n_0\in\mathbb{N}$ such that $0<\alpha\leq\alpha_n^{(m-1)}$, $\alpha_n^{(m)}\leq\beta<1$ for all $n\geq n_0$. Then we have the following:

(1) $\lim_{n\to\infty}\|T_i^kx_n^{(m-1)}-x_n\|=0$;

(2) $\lim_{n\to\infty}\|T_i^kx_n^{(m-2)}-x_n\|=0$.

Proof. (1) Taking each $q\in F$, it follows from Lemma 3.2 that $\lim_{n\to\infty}\|x_n-q\|$ exists. Let

$$\lim_{n\to\infty}\|x_n-q\|=a, \tag{3.12}$$

for some $a \geq 0$. We note that

$$\left\| x_n^{(m-1)} - q \right\| \leq \|x_n - q\| + d_n^{(m-1)}, \quad \forall n \geq 1, \tag{3.13}$$

where $\{\mathrm{d}_n^{(m-1)}\}$ is a nonnegative real sequence such that

$$\sum_{n=1}^{\infty} d_n^{(m-1)} < \infty. \tag{3.14}$$

It follows that

$$\begin{aligned} \limsup_{n\to\infty} \left\| x_n^{(m-1)} - q \right\| &\leq \limsup_{n\to\infty} \|x_n - q\| \\ &= \lim_{n\to\infty} \|x_n - q\| \\ &= a, \end{aligned} \tag{3.15}$$

which implies that

$$\begin{aligned} \limsup_{n\to\infty} \left\| T_i^k x_n^{(m-1)} - q \right\| &\leq \limsup_{n\to\infty} \left(\left\| x_n^{(m-1)} - q \right\| + G_{ik} \right) \\ &= \lim_{n\to\infty} \left\| x_n^{(m-1)} - q \right\| \\ &\leq a. \end{aligned} \tag{3.16}$$

Next, we observe that

$$\left\| T_i^k x_n^{(m-1)} - q + \gamma_n^{(m)} \left(u_n^{(m)} - x_n \right) \right\| \leq \left\| T_i^k x_n^{(m-1)} - q \right\| + \gamma_n^{(m)} \left\| \left(u_n^{(m)} - x_n \right) \right\|. \tag{3.17}$$

Thus we have

$$\limsup_{n\to\infty} \left\| T_i^k x_n^{(m-1)} - q + \gamma_n^{(m)} \left(u_n^{(m)} - x_n \right) \right\| \leq a. \tag{3.18}$$

Also,

$$\left\| x_n - q + \gamma_n^{(m)} \left(u_n^{(m)} - x_n \right) \right\| \leq \|x_n - q\| + \gamma_n^{(m)} \left\| u_n^{(m)} - x_n \right\| \tag{3.19}$$

gives that

$$\limsup_{n\to\infty} \left\| x_n - q + \gamma_n^{(m)} \left(u_n^{(m)} - x_n \right) \right\| \leq a. \tag{3.20}$$

Note that

$$\begin{aligned}
a &= \lim_{n\to\infty}\left\|x_n^{(m)} - q\right\| \\
&= \lim_{n\to\infty}\left\|\alpha_n^{(m)} T_i^k x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - q\right\| \\
&= \lim_{n\to\infty}\left\|\alpha_n^{(m)} T_i^k x_n^{(m-1)} + \left(1-\alpha_n^{(m)}\right) x_n - \gamma_n^{(m)} x_n\right. \\
&\quad \left. +\gamma_n^{(m)} u_n^{(m)} - \left(1-\alpha_n^{(m)}\right) q - \alpha_n^{(m)} q\right\| \\
&= \lim_{n\to\infty}\left\|\alpha_n^{(m)} T_i^k x_n^{(m-1)} - \alpha_n^{(m)} q + \alpha_n^{(m)}\gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)}\gamma_n^{(m)} x_n\right. \\
&\quad \left. +\left(1-\alpha_n^{(m)}\right) q - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)}\gamma_n^{(m)} u_n^{(m)} + \alpha_n^{(m)}\gamma_n^{(m)} x_n\right\| \\
&= \lim_{n\to\infty}\left\|\alpha_n^{(m)}\left(T_i^k x_n^{(m-1)} - q + \gamma_n^{(m)}\left(u_n^{(m)} - x_n\right)\right)\right. \\
&\quad \left. +\left(1-\alpha_n^{(m)}\right)\left(x_n - q + \gamma_n^{(m)}\left(u_n^{(m)} - x_n\right)\right)\right\|.
\end{aligned} \tag{3.21}$$

This together with (3.18), (3.20), and Lemma 3.1, gives

$$\lim_{n\to\infty}\left\|T_i^k x_n^{(m-1)} - x_n\right\| = 0. \tag{3.22}$$

This completes the proof of (1).

(2) For each $n \ge 1$,

$$\begin{aligned}
\left\|x_n - q\right\| &= \left\|x_n - T_i^k x_n^{(m-1)}\right\| + \left\|T_i^k x_n^{(m-1)} - q\right\| \\
&\le \left\|x_n - T_i^k x_n^{(m-1)}\right\| + \left\|x_n^{(m-1)} - q\right\| + G_{ik}.
\end{aligned} \tag{3.23}$$

Since

$$\lim_{n\to\infty}\left\|x_n - T_i^k x_n^{(m-1)}\right\| = 0 = \lim_{n\to\infty} G_{ik}, \tag{3.24}$$

we obtain

$$a = \lim_{n\to\infty}\left\|x_n - q\right\| \le \liminf_{n\to\infty}\left\|x_n^{(m-1)} - q\right\|. \tag{3.25}$$

It follows that

$$\begin{aligned} a &\le \liminf_{n\to\infty}\left\| x_n^{(m-1)} - q \right\| \\ &\le \limsup_{n\to\infty}\left\| x_n^{(m-1)} - q \right\| \\ &\le a, \end{aligned} \tag{3.26}$$

which implies that

$$\lim_{n\to\infty}\left\| x_n^{(m-1)} - q \right\| = a. \tag{3.27}$$

On the other hand, we note that

$$\left\| x_n^{(m-2)} - q \right\| \le \| x_n - q \| + d_n^{(m-2)}, \quad \forall n \ge 1, \tag{3.28}$$

where $\{d_n^{(m-2)}\}$ is a nonnegative real sequence such that

$$\sum_{n=1}^{\infty} d_n^{(m-2)} < \infty. \tag{3.29}$$

Thus we have

$$\begin{aligned} \limsup_{n\to\infty}\left\| x_n^{(m-2)} - q \right\| &\le \limsup_{n\to\infty}\| x_n - q \| \\ &= a, \end{aligned} \tag{3.30}$$

and hence

$$\begin{aligned} \limsup_{n\to\infty}\left\| T_i^k x_n^{(m-2)} - q \right\| &\le \limsup_{n\to\infty}\left(\left\| x_n^{(m-2)} - q \right\| + G_{ik} \right) \\ &\le a. \end{aligned} \tag{3.31}$$

Next, we observe that

$$\left\| T_i^k x_n^{(m-2)} - q + \gamma_n^{(m-1)}\left(u_n^{(m-1)} - x_n \right) \right\| \le \left\| T_i^k x_n^{(m-2)} - q \right\| + \gamma_n^{(m-1)}\left\| u_n^{(m-1)} - x_n \right\|. \tag{3.32}$$

Thus we have

$$\limsup_{n\to\infty}\left\| T_i^k x_n^{(m-2)} - q + \gamma_n^{(m-1)}\left(u_n^{(m-1)} - x_n \right) \right\| \le a. \tag{3.33}$$

Also,

$$\left\|x_n - q + \gamma_n^{(m-1)}\left(u_n^{(\mathrm{m}-1)} - x_n\right)\right\| \leq \|x_n - q\| + \gamma_n^{(m-1)}\left\|u_n^{(m-1)} - x_n\right\| \tag{3.34}$$

gives that

$$\limsup_{n\to\infty}\left\|x_n - q + \gamma_n^{(m-1)}\left(u_n^{(m-1)} - x_n\right)\right\| \leq a. \tag{3.35}$$

Note that

$$\begin{aligned} a &= \lim_{n\to\infty}\left\|x_n^{(m-1)} - q\right\| \\ &= \lim_{n\to\infty}\left\|\alpha_n^{(m-1)}T_i^k x_n + \beta_n^{(m-1)}x_n + \gamma_n^{(m-1)}u_n^{(m-1)} - q\right\| \\ &= \lim_{n\to\infty}\left\|\alpha_n^{(m-1)}\left(T_i^k x_n^{(m-2)} - q + \gamma_n^{(m-1)}\left(u_n^{(m-1)} - x_n\right)\right)\right. \\ &\quad \left. + \left(1 - \alpha_n^{(m-1)}\right)\left(x_n - q + \gamma_n^{(m-1)}\left(u_n^{(m-1)} - x_n\right)\right)\right\|. \end{aligned} \tag{3.36}$$

Therefore, it follows from (3.33), (3.35), and Lemma 3.1 that

$$\lim_{n\to\infty}\left\|T_i^k x_n^{(m-2)} - x_n\right\| = 0. \tag{3.37}$$

This completes the proof. □

Theorem 3.4. *Let X be a uniformly convex Banach space and let C be a nonempty closed bounded convex subset of X. Let $T_i : C \to C$ be m asymptotically nonexpansive mappings in the intermediate sense such that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and there exists one member T in $\{T_i\}_{i=1}^m$which is completely continuous. Put*

$$G_{ik} = \sup_{x,y\in C}\left(\left\|T_i^k x - T_i^k y\right\| - \|x - y\|\right) \vee 0, \quad \forall k \geq 1, \tag{3.38}$$

so that $\sum_{k=1}^\infty G_{ik} < \infty$. Let the sequence $\{x_n\}$ be defined by (1.3) whenever $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ satisfy the same assumptions as in Lemma 3.2 for each $i \in \{1,2,\dots,m\}$ and the additional assumption that there exists $n_0 \in \mathbb{N}$ such that $0 < \alpha \leq \alpha_n^{(m-1)}$, $\alpha_n^{(m)} \leq \beta < 1$ for all $n \geq n_0$. Then $\{x_n^{(k)}\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^m$.

Proof. From Lemma 3.3, it follows that

$$\lim_{n\to\infty}\left\|T_i^k x_n^{(m-1)} - x_n\right\| = 0 = \lim_{n\to\infty}\left\|T_i^k x_n^{(m-2)} - x_n\right\|, \tag{3.39}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|x_n^{(m)}-x_n\right\| \\ &\le \alpha_n^{(m)}\left\|T_i^k x_n^{(m-1)}-x_n\right\|+\gamma_n^{(m-1)}\left\|u_n^{(m-1)}-x_n\right\| \longrightarrow 0, \quad (n\longrightarrow\infty),\end{aligned} \tag{3.40}$$

and so

$$\|x_{n+l}-x_n\| \longrightarrow 0, \quad (n\longrightarrow\infty). \tag{3.41}$$

It follows from (3.22), (3.37) that

$$\begin{aligned}\left\|T_n^k x_n-x_n\right\| &\le \left\|T_i^k x_n-T_i^k x_n^{(m-1)}\right\|+\left\|T_i^k x_n^{(m-1)}-x_n\right\| \\ &\le \left\|x_n-x_n^{(m-1)}\right\|+G_{ik}+\left\|T_i^k x_n^{(m-1)}-x_n\right\| \\ &\le \alpha_n^{(m-1)}\left\|T_i^k x_n^{(m-2)}-x_n\right\|+G_{ik}+\gamma_n^{(m-1)}\left\|u_n^{(m-1)}-x_n\right\| \\ &\quad+\left\|T_i^k x_n^{(m-1)}-x_n\right\| \longrightarrow 0, \quad (n\longrightarrow\infty).\end{aligned} \tag{3.42}$$

Let $\sigma_n=\|T_i^k x_n-x_n\|$ for all $n>n_0$. Then we have

$$\begin{aligned}\|x_n-T_nx_n\| &\le \left\|x_n-T_n^k x_n\right\|+\left\|T_n^k x_n-T_nx_n\right\| \\ &\le \left\|x_n-T_i^k x_n\right\|+L\left\|T_n^{k-1}x_n-x_n\right\| \\ &\le \sigma_n+L\left[\left\|T_n^{k-1}x_n-T_{n-m}^{k-1}x_{n-m}\right\|+\left\|T_{n-m}^{k-1}x_{n-m}-x_{n-m}\right\|+\|x_{n-m}-x_n\|\right].\end{aligned} \tag{3.43}$$

Notice that $n\equiv(n-m)(\operatorname{mod} m)$. Thus $T_n=T_{n-m}$ and the above inequality becomes

$$\|x_n-T_nx_n\| \le \sigma_n+L^2\|x_n-x_{n-m}\|+L\sigma_{n-m}+\|x_{n-m}-x_n\|, \tag{3.44}$$

and so

$$\lim_{n\to\infty}\|x_n-T_nx_n\|=0. \tag{3.45}$$

Since

$$\begin{aligned}\|x_n-T_{n+l}x_n\| &\le \|x_n-x_{n+l}\|+\|x_{n+l}-T_{n+l}x_{n+l}\|+\|T_{n+l}x_{n+l}-T_{n+l}x_n\| \\ &\le (1+L)\|x_n-x_{n+l}\|+\|x_{n+l}-T_{n+l}x_{n+l}\|, \quad \forall l\in\{1,2,\ldots,m\},\end{aligned} \tag{3.46}$$

we have

$$\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0, \quad \forall l \in \{1,2,\ldots,m\}, \tag{3.47}$$

and so

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in \{1,2,\ldots,m\}. \tag{3.48}$$

Since $\{x_n\}$ is bounded and one of T_i is completely continuous, we may assume that T_1 is completely continuous, without loss of generality. Then there exists a subsequence $\{T_1 x_{n_k}\}$ of $\{T_1 x_n\}$ such that $T_1 x_{n_k} \to q \in C$ as $k \to \infty$. Moreover, by (3.48), we have

$$\lim_{n\to\infty} \|x_{n_k} - T_1 x_{n_k}\| = 0, \tag{3.49}$$

which implies that $x_{n_k} \to q$ as $k \to \infty$. By (3.48) again, we have

$$\|q - T_l q\| = \lim_{n\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0, \quad \forall l \in \{1,2,\ldots,m\}. \tag{3.50}$$

It follows that $q \in F$. Since $\lim_{n\to\infty}\|x_n - q\|$ exists, we have

$$\lim_{n\to\infty} \|x_n - q\| = 0, \tag{3.51}$$

that is,

$$\lim_{n\to\infty} x_n^{(m)} = \lim_{n\to\infty} x_n = q. \tag{3.52}$$

Moreover, we observe that

$$\left\|x_n^{(k)} - q\right\| \le \|x_n - q\| + d_n^{(k)}, \tag{3.53}$$

for all $k = 1,2,\ldots,m-1$ and

$$\lim_{n\to\infty} d_n^{(k)} = 0. \tag{3.54}$$

Therefore,

$$\lim_{n\to\infty} x_n^{(k)} = q, \tag{3.55}$$

for all $k = 1,2,\ldots,m-1$. This completes the proof. $\square$

Remark 3.5. Theorem 3.4 improves and extends the corresponding results of Plubtieng and Wangkeeree [2] in the following ways.

(1) The iterative process $\{x_n\}$ defined by (1.3) in [2] is replaced by the new iterative process $\{x_n\}$ defined by (1.3) in this paper.

(2) Theorem 3.4 generalizes Theorem 3.4 of Plubtieng and Wangkeeree [2] from a asymptotically nonexpansive mappings in the intermediate sense to a finite family of asymptotically nonexpansive mappings in the intermediate sense.

Remark 3.6. If $m = 3$ and $T_1 = T_2 = T_3 = T$ in Theorem 3.4, we obtain strong convergence theorem for Noor iteration scheme with error for asymptotically nonexpansive mapping T in the intermediate sense in Banach space, we omit it here.

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