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Research Article

Optimality Conditions of Globally Efficient Solution for Vector Equilibrium Problems with Generalized Convexity

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We study optimality conditions of globally efficient solution for vector equilibrium problems with generalized convexity. The necessary and sufficient conditions of globally efficient solution for the vector equilibrium problems are obtained. The Kuhn-Tucker condition of globally efficient solution for vector equilibrium problems is derived. Meanwhile, we obtain the optimality conditions for vector optimization problems and vector variational inequality problems with constraints.

1. Introduction

Throughout the paper, let X, Y, and Z be real Hausdorff topological vector spaces, $D \subset X$ a nonempty subset, and 0_Y denotes the zero element of Y. Let $C \subset Y$ and $K \subset Z$ be two pointed convex cones (see [1]) such that $\operatorname{int} C \neq \emptyset$, $\operatorname{int} K \neq \emptyset$, where $\operatorname{int} C$ denotes the interior of C. Let $g : D \to Z$ be a mapping and let $F : D \times D \to Y$ be a mapping such that $F(x,x) = 0$, for all $x \in D$. For each $x \in D$, we denote $F(x,D) = \bigcup_{y \in D} F(x,y)$ and define the constraint set

$$A = \{x \in D : g(x) \in -K\}, \tag{1.1}$$

which is assumed to be nonempty.

Consider the vector equilibrium problems with constraints (for short, VEPC): finding $x \in A$ such that

$$F(x,y) \notin -P, \quad \forall y \in A, \tag{VEPC}$$

where $P \cup \{0_Y\}$ is a convex cone in Y.

Vector equilibrium problems, which contain vector optimization problems, vector variational inequality problems, and vector complementarity problems as special case, have been studied by Ansari et al. [2, 3], Bianchi et al. [4], Fu [5], Gong [6], Gong and Yao [7, 8], Hadjisavvas and Schaible [9], Kimura and Yao [10–13], Oettli [14], and Zeng et al. [15]. But so far, most papers focused mainly on the existence of solutions and the properties of the solutions, there are few papers which deal with the optimality conditions. Giannessi et al. [16] turned the vector variational inequalities with constraints into another vector variational inequalities without constraints. They gave sufficient conditions for efficient solution and weakly efficient solution of the vector variational inequalities in finite dimensional spaces. Morgan and Romaniello [17] gave scalarization and Kuhn-Tucker-like conditions for weak vector generalized quasivariational inequalities in Hilbert space by using the concept of subdifferential of the function. Gong [18] presented the necessary and sufficient conditions for weakly efficient solution, Henig efficient solution, and superefficient solution for the vector equilibrium problems with constraints under the condition of cone-convexity. However, the condition of cone-convexity is too strong. Some generalized convexity has been developed, such as cone-preinvexity (see [19]), cone-convexlikeness (see [20]), cone-subconvexlikeness (see [21]), and generalized cone-convexlikeness (see [22]). Among them, the generalized cone-subconvexlikeness has received more attention. Then, it is important to give the optimality conditions for the solution of (VEPC) under conditions of generalized convexity. Moreover, it appears that no work has been done on the Kuhn-Tucker condition of solution for (VEPC). This paper is the effort in this direction.

In the paper, we study the optimality conditions for the vector equilibrium problems. Firstly, we present the necessary and sufficient conditions for globally efficient solution of (VEPC) under generalized cone-subconvexlikeness. Secondly, we prove that the Kuhn-Tucker condition for (VEPC) is both necessary and sufficient under the condition of cone-preinvexity. Meanwhile, we obtain the optimality conditions for vector optimization problems with constraints and vector variational inequality problems with constraints in Section 4.

2. Preliminaries and Definitions

Let Y^*, Z^* be the dual space of Y, Z, respectively, then the dual cone of C is defined as

$$C^* = \{\varphi \in Y^* : \varphi(c) \geq 0,\ \forall c \in C\}. \tag{2.1}$$

The set of strictly positive functional in C^* is denoted by C^{+i}, that is,

$$C^{+i} = \{\varphi \in C^* : \varphi(c) > 0,\ \forall c \in C \setminus \{0_Y\}\}. \tag{2.2}$$

It is well known that

(i) if $C^{+i} \neq \emptyset$, then C has a base;

(ii) if Y is a Hausdorff locally convex space, then $C^{+i} \neq \emptyset$ if and only if C has a base;

(iii) if Y is a separable normed space and C is a pointed closed convex cone, then C^{+i} is nonempty (see [1]).

Remark 2.1. The positive cone in many common Banach spaces possesses strictly positive functionals. However, this is not always the case (see [23]).

Let $M \subset Y$ be an arbitrary nonempty subset and $\mathrm{cone}_+(M) = \bigcup\{\lambda x : \lambda > 0,\ x \in M\}$. The symbol $\mathrm{cl}(M)$ denotes the closure of M, and $\mathrm{cone}(M)$ denotes the generated cone of M, that is, $\mathrm{cone}(M) = \bigcup\{\lambda x : \lambda \geq 0,\ x \in M\}$. When M is a convex, so is $\mathrm{cone}(M)$.

Remark 2.2. Obviously, we have

(i) $\mathrm{cone}(M) = \mathrm{cone}_+(M) \bigcup\{0_Y\}$;

(ii) $\mathrm{cl}(\mathrm{cone}(M)) = \mathrm{cl}(\mathrm{cone}_+(M))$;

(iii) if $P \subset Y$ satisfying for all $\lambda > 0$, $\lambda P \subset P$, then $\mathrm{cone}_+(M + P) = \mathrm{cone}_+(M) + P$.

Several definitions of generalized convex mapping have been introduced in literature.

(1) Let $S_0 \subset X$ be a nonempty convex subset and let $C \subset Y$ be a convex cone. A mapping $f : S_0 \to Y$ is called C-convex, if for all $x_1, x_2 \in S_0$, for all $\lambda \in (0,1)$, we have

$$\lambda f(x_1) + (1-\lambda) f(x_2) - f(\lambda x_1 + (1-\lambda)x_2) \in C. \tag{2.3}$$

(2) Let $D \subset X$ be a nonempty subset and let $C \subset Y$ be a convex cone.

(i) A mapping $f : D \to Y$ is called C-convexlike (see [20]), if for all $x_1, x_2 \in D$, for all $\lambda \in (0,1)$, there exists $x_3 \in D$ such that

$$\lambda f(x_1) + (1-\lambda) f(x_2) - f(x_3) \in C. \tag{2.4}$$

(ii) f is said to be C-subconvexlike (see [21]), if there exists $\theta \in \mathrm{int}\, C$ such that for all $x_1, x_2 \in D$, for all $\lambda \in (0,1)$, for all $\varepsilon > 0$, there exists $x_3 \in D$ such that

$$\varepsilon\theta + f(x_1) + (1-\lambda) f(x_2) - f(x_3) \in C. \tag{2.5}$$

(iii) f is said to be generalized C-subconvexlike (see [22]), if there exists $\theta \in \mathrm{int}\, C$ such that for all $x_1, x_2 \in D$, for all $\lambda \in (0,1)$, for all $\varepsilon > 0$, there exists $x_3 \in D$, $\rho > 0$ such that

$$\varepsilon\theta + \lambda f(x_1) + (1-\lambda) f(x_2) - \rho f(x_3) \in C. \tag{2.6}$$

A nonempty subset $S \subset X$ is called invex with respect to η, if there exists a mapping $\eta : S \times S \to X$ such that for any $x, y \in S$, and $t \in [0,1]$, $x + t\eta(y,x) \in S$.

(3) Let $S \subset X$ be a invex set with respect to η. A mapping $f : S \to Y$ is said to be C-preinvex with respect to η (see [19]), if for any $x, y \in S$, and $t \in [0,1]$, we have

$$(1-t) f(x) + t f(y) - f(x + t\eta(y,x)) \in C. \tag{2.7}$$

Remark 2.3. (i) From [21], we know that f is C-convexlike on D if and only if $f(D)+C$ is a convex set and f is C-subconvexlike on D if and only if $f(D)+\operatorname{int}C$ is a convex set.

(ii) If $f(D)+C$ is a convex set, so is $\operatorname{int}(\operatorname{cl}(f(D)+C))$. By Lemma 2.5 of [24], $f(D)+\operatorname{int}C$ is convex. This shows that C-convexlikeness implies C-subconvexlikeness. But in general the converse is not true (see [21]).

(iii) It is clear that C-subconvexlikeness implies generalized C-subconvexlikeness. But in general the converse is not true (see [22]).

Remark 2.4. For $\eta(x,y)=x-y$, the invex set is a convex set and the C-preinvex mapping is a convex mapping. However, there are mappings which are C-preinvex but not convex (see [25]).

Relationships among various types of convexity are as shown below:

$$\begin{aligned} &C\text{-convexity} \Longrightarrow C\text{-preinvexity} \Longrightarrow C\text{-convexlikeness} \Longrightarrow C\text{-subconvexlikeness} \\ &\qquad \Longrightarrow \text{generalized } C\text{-subconvexlikeness}. \end{aligned} \tag{2.8}$$

Yang [26] proved the following Lemma in Banach space; Chen and Rong [27] generalized the result to topological vector space.

Lemma 2.5. *Assume that* $\operatorname{int}C\neq\emptyset$. *Then* $f: D\to Y$ *is generalized C-subconvexlike if and only if* $\operatorname{cone}_+(f(D))+\operatorname{int}C$ *is convex.*

Lemma 2.6. *Assume that* (i) $M\subset Y$ *is a nonempty subset and* $C\subset Y$ *is a convex cone with* $\operatorname{int}C\neq\emptyset$. (ii) $\operatorname{cone}_+(M)+\operatorname{int}C$ *is convex. Then* $\operatorname{cl}(\operatorname{cone}(M+C))$ *is also convex.*

Proof. By Lemma 2.5 and Remark 2.1(iii), we deduce that $\operatorname{cone}_+(M+\operatorname{int}C)$ is a convex set. It is not difficult to prove that $\operatorname{cone}(M+\operatorname{int}C)$ is a convex set.

Note that $\operatorname{cl}(\operatorname{cone}(M+C))=\operatorname{cl}(\operatorname{cone}(M+\operatorname{int}C))$ and the closure of a convex set is convex, then $\operatorname{cl}(\operatorname{cone}(M+C))$ is a convex set. The proof is finished. □

Lemma 2.7 (see [1]). *If* $\psi\in K^*\setminus\{0_{Z^*}\}$, $z\in-\operatorname{int}K$, *then* $\langle\psi,z\rangle<0$.

Assume that $\operatorname{int}C\neq\emptyset$, *a vector* $x\in A$ *is called a weakly efficient solution of* (VEPC), *if* x *satisfies*

$$F(x,y)\notin-\operatorname{int}C,\quad \forall y\in A. \tag{2.9}$$

Definition 2.8 (see [6]). Let $C\subset Y$ be a convex cone. Also, $\overline{x}\in A$ is said to be a globally efficient solution of (VEPC), if there exists a pointed convex cone $H\subset Y$ with $C\setminus\{0_Y\}\subset\operatorname{int}H$ such that

$$F(\overline{x},A)\cap(-H\setminus\{0_Y\})=\emptyset. \tag{2.10}$$

Remark 2.9. Obviously, $\overline{x}\in A$ is a globally efficient solution of (VEPC), then $\overline{x}$ is also a weakly efficient solution of (VEPC). But in general the converse is not true (see [6]).

3. Optimality Conditions

Theorem 3.1. *Assume that* (i) $\overline{x} \in A$ *and there exists* $x_0 \in D$ *such that* $g(x_0) \in -\operatorname{int} K$; (ii) $h(y) = (F(\overline{x}, y), g(y))$ *is a generalized* $C \times K$*-subconvexlike on* D. *Then* $\overline{x} \in A$ *is a globally efficient solution of* (VEPC) *if and only if there exists* $\varphi \in C^{+i}$ *and* $\psi \in K^*$ *such that*

$$\langle \varphi, F(\overline{x}, \overline{x}) \rangle + \langle \psi, g(\overline{x}) \rangle = \min_{y \in D} \{ \langle \varphi, F(\overline{x}, y) \rangle + \langle \psi, g(y) \rangle \}, \tag{3.1}$$

$$\langle \psi, g(\overline{x}) \rangle = 0. \tag{3.2}$$

Proof. Assume that $\overline{x} \in A$ is a globally efficient solution of (VEPC), then there exists a pointed convex cone $H \subset Y$ with $C \setminus \{0_Y\} \subset \operatorname{int} H$ such that

$$F(\overline{x}, A) \cap -H = \{0_Y\}. \tag{3.3}$$

Since H is a pointed convex cone with $C \setminus \{0_Y\} \subset \operatorname{int} H$, then

$$(F(\overline{x}, A) + C) \cap - \operatorname{int} H = \emptyset. \tag{3.4}$$

Note that $h(y) = (F(\overline{x}, y), g(y))$, for all $y \in D$ and above formula, it is not difficult to prove

$$(h(D) + C \times K) \cap (-\operatorname{int} H) \times (-\operatorname{int} K) = \emptyset. \tag{3.5}$$

Since $\operatorname{int} H$ and $\operatorname{int} K$ are two open sets and C, K are two pointed convex cones, by (3.5), we have

$$\operatorname{cl}(\operatorname{cone}(h(D) + C \times K)) \cap (-\operatorname{int} H) \times (-\operatorname{int} K) = \emptyset. \tag{3.6}$$

Moreover, since $h(y) = (F(\overline{x}, y), g(y))$ is a generalized $C \times K$-subconvexlike on D, by Lemma 2.5, $\operatorname{cone}_+(h(D)) + \operatorname{int} H \times \operatorname{int} K$ is convex. This follows from Lemma 2.6 that $\operatorname{cl}(\operatorname{cone}(h(D) + C \times K))$ is convex. By the standard separation theorem (see [1, page 76]), there exists $(\varphi, \psi) \in Y^* \times Z^* \setminus \{0_{Y^*}, 0_{Z^*}\}$ such that

$$\langle (\varphi, \psi), \operatorname{cl}(\operatorname{cone}(h(D) + C \times K)) \rangle > \langle \varphi, -\operatorname{int} C \rangle + \langle \psi, -\operatorname{int} K \rangle. \tag{3.7}$$

Since $\operatorname{cl}(\operatorname{cone}(h(D) + C \times K))$ is a cone, it follows from (3.7) that

$$\langle (\varphi, \psi), \operatorname{cl}(\operatorname{cone}(h(D) + C \times K)) \rangle \geq 0. \tag{3.8}$$

Note that $(0_Y, 0_Z) \in C \times K$, thus $h(D) \subset \operatorname{cl}(\operatorname{cone}(h(D) + C \times K))$. By (3.8), we obtain immediately

$$\langle (\varphi, \psi), h(D) \rangle \geq 0. \tag{3.9}$$

It implies that

$$\langle \varphi, F(\overline{x}, y) \rangle + \langle \psi, g(y) \rangle \geq 0, \quad \forall y \in D. \tag{3.10}$$

On the other hand, by $(0_Y, 0_Z) \in \mathrm{cl}(\mathrm{cone}(h(D) + C \times K))$ and (3.7), we get

$$\langle \varphi, -\operatorname{int} H \rangle + \langle \psi, -\operatorname{int} K \rangle < 0. \tag{3.11}$$

Since for all $h \in \operatorname{int} H$, for all $\lambda > 0$, we have $\lambda h \in \operatorname{int} H$, by (3.11), we get

$$\langle \varphi, h \rangle > \frac{1}{\lambda} \langle \psi, -k_0 \rangle, \quad \forall h \in \operatorname{int} H,\ \forall \lambda > 0,\ k_0 \in \operatorname{int} K. \tag{3.12}$$

Letting $\lambda \to \infty$, we have

$$\langle \varphi, h \rangle \geq 0, \quad \forall h \in \operatorname{int} H. \tag{3.13}$$

Firstly, we prove that

$$\varphi \in H^* \setminus \{0_{Y^*}\}, \quad \psi \in K^*. \tag{3.14}$$

Since H is convex and $\operatorname{int} H$ is nonempty, then $H \subset \mathrm{cl}(H) = \mathrm{cl}(\operatorname{int} H)$. Note that $\varphi \in Y^*$ and (3.13), and we have $\varphi \in H^*$. With similar proof of $\varphi \in H^*$, we can prove that $\psi \in K^*$.

We need to show that $\varphi \neq 0_{Y^*}$.

In fact, if $\varphi = 0_{Y^*}$, then $\psi \in K^* \setminus \{0_{Z^*}\}$. By (3.10), we have

$$\langle \psi, g(y) \rangle \geq 0, \quad \forall y \in D. \tag{3.15}$$

On the other hand, since $\psi \in K^*$, $g(x_0) \in -\operatorname{int} K$, by Lemma 2.7, we have $\langle \psi, g(x_0) \rangle < 0$, which is a contradiction with (3.15).

Secondly, we show that $\varphi \in C^{+i}$.

For any $c \in C \setminus \{0_Y\}$, since $C \setminus \{0_Y\} \subset \operatorname{int} H$, then there exists a balanced neighborhood U of zero element such that

$$c + U \subset H. \tag{3.16}$$

Note that $\varphi \neq 0_{Y^*}$, and there exists $-u \in U$ such that $\langle \varphi, u \rangle > 0$.

Since $\varphi \in H^*$, then

$$\langle \varphi, c \rangle \geq \langle \varphi, u \rangle > 0. \tag{3.17}$$

By the arbitrariness of $c \in C \setminus \{0_Y\}$, we have $\varphi \in C^{+i}$.

Lastly, we show that (3.1) and (3.2) hold.

Taking $y = \overline{x}$ in (3.10), we get

$$\langle \psi, g(\overline{x}) \rangle \geq 0. \tag{3.18}$$

Moreover, since $\overline{x} \in A = \{x \in D : g(x) \in -K\}$, $\psi \in K^*$, then

$$\langle \psi, g(\overline{x}) \rangle \leq 0. \tag{3.19}$$

Thus (3.2) holds.

Since $F(\overline{x}, \overline{x}) = 0$ and $\langle \psi, g(\overline{x}) \rangle = 0$, by (3.10), we have

$$\langle \varphi, F(\overline{x}, \overline{x}) \rangle + \langle \psi, g(\overline{x}) \rangle = \min_{y \in D} \{ \langle \varphi, F(\overline{x}, y) \rangle + \langle \psi, g(y) \rangle \}. \tag{3.20}$$

Then (3.1) holds.

Conversely, if $\overline{x} \in A$ is not a globally efficient solution of (VEPC), then for any pointed convex cone $H \subset Y$ with $C \setminus \{0_Y\} \subset \operatorname{int} H$, we have

$$F(\overline{x}, A) \cap (-H \setminus \{0_Y\}) \neq \emptyset. \tag{3.21}$$

By $\varphi \in C^{+i}$, let

$$H_0 = \{ y \in Y : \langle \varphi, y \rangle > 0 \} \cup \{0_Y\}. \tag{3.22}$$

Obviously, H_0 is a pointed convex cone and $C \setminus \{0_Y\} \subset \operatorname{int} H_0$. By (3.21), then there exists $y_0 \in A$ such that

$$F(\overline{x}, y_0) \in F(\overline{x}, A) \cap (-H \setminus \{0_Y\}). \tag{3.23}$$

By the definition of H_0, we get

$$\langle \varphi, F(\overline{x}, y_0) \rangle < 0. \tag{3.24}$$

Moreover, since $y_0 \in A = \{x \in D : g(x) \in -K\}$ and $\psi \in K^*$, then

$$\langle \psi, g(y_0) \rangle \leq 0. \tag{3.25}$$

This together with (3.24) implies that

$$\langle \varphi, F(\overline{x}, y_0) \rangle + \langle \psi, g(y_0) \rangle < 0. \tag{3.26}$$

On the other hand, since $F(\overline{x},\overline{x})=0$, by (3.1) and (3.2), we get

$$\begin{aligned}0&=\langle\varphi,F(\overline{x},\overline{x})\rangle+\langle\psi,g(\overline{x})\rangle\\&=\min_{y\in D}\{\langle\varphi,F(\overline{x},y)\rangle+\langle\psi,g(y)\rangle\}\\&\leq\langle\varphi,F(\overline{x},y_0)\rangle+\langle\psi,g(y_0)\rangle,\end{aligned}\tag{3.27}$$

which contradicts (3.26). The proof is finished. □

Corollary 3.2. *Assume that* (i) $D\subset X$ *is invex with respect to* η*;* (ii) $\overline{x}\in A$ *and there exists* $x_0\in D$ *such that* $g(x_0)\in-\operatorname{int}K$*;* (iii) $F(\overline{x},\cdot)$ *is* C*-preinvex on* D *with respect to* η*, and* $g:D\to Y$ *is* K*-preinvex on* D *with respect to* η*. Then* $\overline{x}\in A$ *is a globally efficient solution of* (VEPC) *if and only if there exist* $\varphi\in C^{+i}$ *and* $\psi\in K^*$ *such that* (3.1) *and* (3.2) *hold.*

Proof. Since $F(\overline{x},\cdot)$ is C-preinvex on D with respect to η, $g:D\to Y$ is K-preinvex on D with respect to η. Then $h(y)=(F(\overline{x},y),g(y))$ is $C\times K$-preinvex on D with respect to η. Thus by Theorem 3.1, the conclusion of Corollary 3.2 holds. □

Remark 3.3. Corollary 3.2 generalizes and improves the recent results of Gong (see [18, Theorem 3.3]). Especially, Corollary 3.2 generalizes and improves in the following several aspects.

(1) The condition that the subset D is convex is extended to invex.

(2) $F(x,y)$ is C-convex in y is extended to C-preinvex in y.

(3) $g(y)$ is K-convex is extended to K-preinvex.

Next, we introduce Gateaux derivative of mapping.

Let $\overline{x}\in X$ and let $f:X\to Y$ be a mapping. f is called Gateaux differentiable at $\overline{x}$ if for any $x\in X$, there exists limit

$$f'_{\overline{x}}(x)=\lim_{t\to 0}\frac{f(\overline{x}+tx)-f(\overline{x})}{t}.\tag{3.28}$$

Mapping $f'_{\overline{x}}:x\to f'_{\overline{x}}(x)$ is called Gateaux derivative of f at $\overline{x}$.

The following theorem shows that the Kuhn-Tucker condition for (VEPC) is both necessary and sufficient.

Theorem 3.4. *Assume that* (i) $C\subset Y$*,* $K\subset Z$ *are closed,* $D\subset X$ *is invex with respect to* η*;* (ii) $\overline{x}\in A$ *and there exists* $x_0\in D$ *such that* $g(x_0)\in-\operatorname{int}K$*;* (iii) $F(\overline{x},\cdot)$ *is* C*-preinvex on* D *with respect to* η *and Gateaux differentiable at* $\overline{x}$*, and* $g:D\to Y$ *is Gateaux differentiable at* $\overline{x}$ *and* K*-preinvex on* D *with respect to* η*; . Then* $\overline{x}\in A$ *is a globally efficient solution of* (VEPC) *if and only if there exists* $\varphi\in C^{+i}$ *and* $\psi\in K^*$ *such that*

$$\langle\varphi,F'_{\overline{x}}(\overline{x},\eta(y,\overline{x}))\rangle+\langle\psi,g'_{\overline{x}}(\eta(y,\overline{x}))\rangle\geq 0,\quad\forall y\in D,\tag{3.29}$$

$$\langle\psi,g(\overline{x})\rangle=0.\tag{3.30}$$

Proof. Assume that $\overline{x} \in A$ is a globally efficient solution of (VEPC), by Corollary 3.2, there exists $\varphi \in C^{+i}$ and $\psi \in K^*$ such that

$$\langle \psi, g(\overline{x}) \rangle = 0, \tag{3.31}$$

$$\langle \varphi, F(\overline{x}, y) - F(\overline{x}, \overline{x}) \rangle + \langle \psi, g(y) - g(\overline{x}) \rangle \geq 0, \quad \forall y \in D. \tag{3.32}$$

Since D is invex with respect to η, then for any $y \in D$,

$$\overline{x} + t\eta(y, \overline{x}) \in D, \quad \forall t \in (0,1). \tag{3.33}$$

By (3.32), for any $t \in (0,1)$, we have

$$\left\langle \varphi, \frac{F(\overline{x}, \overline{x} + t\eta(y, \overline{x})) - F(\overline{x}, \overline{x})}{t} \right\rangle + \left\langle \psi, \frac{g(\overline{x} + t\eta(y, \overline{x})) - g(\overline{x})}{t} \right\rangle \geq 0, \quad \forall y \in D. \tag{3.34}$$

Since $F(\overline{x}, \cdot)$ is Gateaux differentiable at $\overline{x}$, and $g : D \to Y$ is Gateaux differentiable at $\overline{x}$, letting $t \to 0$ in (3.34), we have

$$\langle \varphi, F'_{\overline{x}}(\overline{x}, \eta(y, \overline{x})) \rangle + \langle \psi, g'_{\overline{x}}(\eta(y, \overline{x})) \rangle \geq 0, \quad \forall y \in D. \tag{3.35}$$

Conversely, if $\overline{x}$ is not a globally efficient solution of (VEPC), a similar proof of (3.24) in Theorem 3.1, there exists $x_1 \in A$ such that

$$\langle \varphi, F(\overline{x}, x_1) \rangle < 0. \tag{3.36}$$

Since $F(\overline{x}, \overline{x}) = 0$, thus we have

$$\langle \varphi, F(\overline{x}, x_1) - F(\overline{x}, \overline{x}) \rangle < 0. \tag{3.37}$$

Moreover, since $F(\overline{x}, \cdot)$ is C-preinvex on D with respect to η, then for any $\lambda \in (0,1)$, $\overline{x}, x_1 \in D$, we have

$$\lambda F(\overline{x}, x_1) + (1 - \lambda) F(\overline{x}, \overline{x}) - F(\overline{x}, \overline{x} + \lambda\eta(x_1, \overline{x})) \in C. \tag{3.38}$$

This together with C being cone yields that

$$F(\overline{x}, x_1) - F(\overline{x}, \overline{x}) - \frac{F(\overline{x}, \overline{x} + \lambda\eta(x_1, \overline{x})) - F(\overline{x}, \overline{x})}{\lambda} \in C. \tag{3.39}$$

Since C is closed, taking $\lambda \to 0$ in the above formula, we have

$$F(\overline{x}, x_1) - F(\overline{x}, \overline{x}) - F'_{\overline{x}}(\overline{x}, \eta(x_1, \overline{x})) \in C. \tag{3.40}$$

Note that $\varphi \in C^*$, then we have

$$\langle \varphi, F(\overline{x}, x_1) - F(\overline{x}, \overline{x}) \rangle \geq \langle \varphi, F'_{\overline{x}}(\overline{x}, \eta(x_1, \overline{x})) \rangle. \tag{3.41}$$

This together with (3.37) yields that

$$\langle \varphi, F'_{\overline{x}}(\overline{x}, \eta(x_1, \overline{x})) \rangle < 0. \tag{3.42}$$

Moreover, since $x_1 \in A$, $\psi \in K^*$ and $\langle \psi, g(\overline{x}) \rangle = 0$, then we have

$$\langle \psi, g(x_1) - g(\overline{x}) \rangle \leq 0. \tag{3.43}$$

With similar proof of (3.41), we get

$$\langle \psi, g'_{\overline{x}}(\eta(x_1, \overline{x})) \rangle \leq \langle \psi, g(x_1) - g(\overline{x}) \rangle \leq 0. \tag{3.44}$$

This together with (3.42) implies that

$$\langle \varphi, F'_{\overline{x}}(\overline{x}, \eta(x_1, \overline{x})) \rangle + \langle \psi, g'_{\overline{x}}(\eta(x_1, \overline{x})) \rangle < 0, \tag{3.45}$$

which contradicts (3.29). The proof is finished. □

4. Application

As interesting applications of the results of Section 3, we obtain the optimality conditions for vector optimization problems and vector variational inequality problems.

Let $L(X, Y)$ be the space of all bounded linear mapping from X to Y. We denote by $\langle h, x \rangle$ the value of $h \in L(X, Y)$ at x.

Equation (VEPC) includes as a special case a vector variational inequality with constraints (for short, (VVIC)) involving

$$F(x, y) = \langle Tx, y - x \rangle, \tag{4.1}$$

where T is a mapping from D to $L(X, Y)$.

Definition 4.1 (see [18]). If $F(x, y) = \langle Tx, y - x \rangle$, $x, y \in A$, and if $\overline{x} \in A$ is a globally efficient solution of (VEPC), then $\overline{x} \in A$ is called a globally efficient solution of (VVIC).

Theorem 4.2. *Assume that* (i) $C \subset Y$, $K \subset Z$ *are closed,* $D \subset X$ *is a nonempty convex subset;* (ii) $\overline{x} \in A$ *and there exists* $x_0 \in D$ *such that* $g(x_0) \in -\operatorname{int} K$; (iii) $g : D \to Y$ *is Gateaux differentiable at* $\overline{x}$ *and* K*-convex on* D*. Then* $\overline{x} \in A$ *is a globally efficient solution of (VVIC) if and only if there exists* $\varphi \in C^{+i}$ *and* $\psi \in K^*$ *such that*

$$\begin{gathered} \langle \varphi, \langle T\overline{x}, y - \overline{x} \rangle \rangle + \langle \psi, g'_{\overline{x}}(y - \overline{x}) \rangle \geq 0, \quad \forall y \in D, \\ \langle \psi, g(\overline{x}) \rangle = 0. \end{gathered} \tag{4.2}$$

Proof. Let

$$F(x,y) = \langle Tx, y-x\rangle, \quad x,y \in D,$$
$$\eta(y,x) = y-x, \quad x,y \in D, \tag{4.3}$$

then D is invex with respect to $\eta, F(\overline{x},\cdot)$ is Gateaux differentiable at $\overline{x}$ and C-preinvex with respect to η, and $g : D \to Y$ is Gateaux differentiable at $\overline{x}$ and K-preinvex on D with respect to η. Thus the conditions of Theorem 3.4 are satisfied. Note that $\langle \varphi, F'_{\overline{x}}(\overline{x}, \eta(y,\overline{x}))\rangle = \langle \varphi, \langle T\overline{x}, y-\overline{x}\rangle\rangle$, by Theorem 3.4, then the conclusion of Theorem 4.2 holds. □

Another special case of (VEPC) is the vector optimization problem with constraints (for short, VOP):

$$\min f(x)$$
$$\text{subject to } x \in A \tag{VOP}$$

involving

$$F(x,y) = f(y) - f(x), \quad x,y \in D, \tag{4.4}$$

where $f : D \to Y$ is a mapping.

Definition 4.3 (see [18]). If $F(x,y) = f(y) - f(x)$, $x,y \in A$, and if $\overline{x} \in A$ is a globally efficient solution of (VEPC), then $\overline{x} \in A$ is called a globally efficient solution of (VOP).

Theorem 4.4. *Assume that* (i) $C \subset Y$, $K \subset Z$ *are closed,* $D \subset X$ *is invex with respect to* η; (ii) $\overline{x} \in A$ *and there exists* $x_0 \in D$ *such that* $g(x_0) \in -\operatorname{int} K$; (iii) $f : D \to Y$ *is Gateaux differentiable at* $\overline{x}$ *and* C*-preinvex on* D *with respect to* η, *and* $g : D \to Y$ *is Gateaux differentiable at* $\overline{x}$ *and* K*-preinvex on* D *with respect to* η. *Then* $\overline{x} \in A$ *is a globally efficient solution of* (VOP) *if and only if there exists* $\varphi \in C^{+i}$ *and* $\psi \in K^*$ *such that*

$$\langle \varphi, f'_{\overline{x}}(\eta(y,\overline{x})\rangle + \langle \psi, g'_{\overline{x}}(\eta(y,\overline{x}))\rangle \geq 0, \quad \forall y \in D,$$
$$\langle \psi, g(\overline{x})\rangle = 0. \tag{4.5}$$

Proof. Let

$$F(x,y) = f(y) - f(x), \quad x,y \in D, \tag{4.6}$$

then $F(\overline{x},\cdot)$ is Gateaux differentiable at $\overline{x}$ and C-preinvex with respect to η, and $g : D \to Y$ is Gateaux differentiable at $\overline{x}$ and K-preinvex on D with respect to η. Thus the conditions of Theorem 3.4 are satisfied. Note that $\langle \varphi, F'_{\overline{x}}(\overline{x}, \eta(y,\overline{x}))\rangle = \langle \varphi, f'_{\overline{x}}(\eta(y,\overline{x}))\rangle$, by Theorem 3.4, then the conclusion of Theorem 4.4 holds. □

By Theorem 3.1, we have the following result.

Theorem 4.5. *Assume that* (i) $\overline{x} \in A$ *and there exists* $x_0 \in D$ *such that* $g(x_0) \in -\operatorname{int} K$; (ii) $h(y) = (f(y) - f(\overline{x}), g(y))$ *is generalized* $C \times K$*-subconvexlike on* D. *Then* $\overline{x} \in A$ *is a globally efficient solution of* (VOP) *if and only if there exists* $\varphi \in C^{+i}$ *and* $\psi \in K^*$ *such that*

$$\begin{gathered}\langle \varphi, f(\overline{x}) \rangle + \langle \psi, g(\overline{x}) \rangle = \min_{y \in D} \{ \langle \varphi, f(y) \rangle + \langle \psi, g(y) \rangle \}, \\ \langle \psi, g(\overline{x}) \rangle = 0.\end{gathered} \tag{4.7}$$

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