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Research Article

Sharpening and Generalizations of Shafer's Inequality for the Arc Tangent Function

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Recommended by Martin J. Bohner

We sharpen and generalize Shafer's inequality for the arc tangent function. From this, some known results are refined.

1. Introduction and Main Results

In [1], the following elementary problem was posed, showing that for $x > 0$,

$$\arctan x > \frac{3x}{1 + 2\sqrt{1 + x^2}}. \tag{1.1}$$

In [2], the following three proofs for the inequality (1.1) were provided.

Solution by Grinstein

Direct computation gives

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = \frac{\left(\sqrt{1 + x^2} - 1\right)^2}{\left(1 + x^2\right)\left(1 + 2\sqrt{1 + x^2}\right)^2}, \tag{1.2}$$

where

$$F(x) = \arctan x - \frac{3x}{1 + 2\sqrt{1 + x^2}}. \tag{1.3}$$

Now $\mathrm{d}F(x)/\mathrm{d}x$ is positive for all $x \neq 0$; whence $F(x)$ is an increasing function. Since $F(0) = 0$, it follows that $F(x) > 0$ for $x > 0$.

Solution by Marsh

It follows from $(\cos \phi - 1)^2 \geq 0$ that

$$1 \geq \frac{3 + 6 \cos \phi}{(\cos \phi + 2)^2}. \tag{1.4}$$

The desired result is obtained directly upon integration of the latter inequality with respect to ϕ from 0 to $\arctan x$ for $x > 0$.

Solution by Konhauser

The substitution $x = \tan y$ transforms the given inequality into $y > 3 \sin y/(2+\cos y)$, which is a special case of an inequality discussed on [3, pages 105-106] .

It may be worthwhile to note that the inequality (1.1) is not collected in the authorized monographs [4, 5].

In [4, pages 288-289], the following inequalities for the arc tangent function are collected:

$$\arctan x < \frac{2x}{1 + \sqrt{1 + x^2}}, \tag{1.5}$$

$$\begin{gathered} \frac{x}{1 + x^2} < \arctan x < x, \\ x - \frac{x^3}{3} < \arctan x < x, \end{gathered} \tag{1.6}$$

$$\frac{1}{2x} \ln\left(1 + x^2\right) < \arctan x < (1 + x) \ln(1 + x), \tag{1.7}$$

where $x > 0$. The inequality (1.5) is better than (1.7).

The aim of this paper is to sharpen and generalize inequalities (1.1) and (1.5).

Our results may be stated as the following theorems.

Theorem 1.1. *For $x > 0$, let*

$$f_a(x) = \frac{\left(a + \sqrt{1 + x^2}\right) \arctan x}{x}, \tag{1.8}$$

where a is a real number.

(1) *When* $a \le 1/2$, *the function* $f_a(x)$ *is strictly increasing on* $(0, \infty)$.
(2) *When* $a \ge 2/\pi$, *the function* $f_a(x)$ *is strictly decreasing on* $(0, \infty)$.
(3) *When* $1/2 < a < 2/\pi$, *the function* $f_a(x)$ *has a unique minimum on* $(0, \infty)$.

As direct consequences of Theorem 1.1, the following inequalities may be derived.

Theorem 1.2. *For* $-1 < a \le 1/2$,

$$\frac{(1+a)x}{a+\sqrt{1+x^2}} < \arctan x < \frac{(\pi/2)x}{a+\sqrt{1+x^2}}, \quad x > 0. \tag{1.9}$$

For $1/2 < a < 2/\pi$,

$$\frac{4a(1-a^2)x}{a+\sqrt{1+x^2}} < \arctan x < \frac{\max\{\pi/2, 1+a\}x}{a+\sqrt{1+x^2}}, \quad x > 0. \tag{1.10}$$

For $a \ge 2/\pi$, *the inequality* (1.9) *is reversed.*

Moreover, the constants $1+a$ *and* $\pi/2$ *in inequalities* (1.9) *and* (1.10) *are the best possible.*

2. Remarks

Before proving our theorems, we give several remarks on them.

Remark 2.1. The substitution $x = \tan y$ may transform inequalities in (1.9) and (1.10) into some trigonometric inequalities.

Remark 2.2. The inequality (1.1) is the special case $a = 1/2$ of the left-hand side inequality in (1.9).

Remark 2.3. The inequality (1.5) is the special case $a = 1$ of the reversed version of the left hand-side inequality in (1.9).

Remark 2.4. Let

$$h_x(a) = \frac{a(1-a^2)}{a+\sqrt{1+x^2}} \tag{2.1}$$

for $1/2 < a < 2/\pi$ and $x > 0$. Direct computation gives

$$h_x'(a) = \frac{(1-3a^2)\sqrt{1+x^2} - 2a^3}{\left(a+\sqrt{1+x^2}\right)^2}. \tag{2.2}$$

Hence,

(1) when $2/\pi > a \ge 1/\sqrt{3}$, the derivative $h_x'(a)$ is negative for $x > 0$;
(2) when $1/2 < a < 1/\sqrt{3}$, the derivative $h_x'(a)$ has a unique zero which is the unique maximum point of $h_x(a)$ for $x > 0$.

Accordingly,

(1) when $2/\pi > a \geq 1/\sqrt{3}$, the function $h_x(a)$ attains its maximum

$$h_x\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3\left[1+\sqrt{3}\,\sqrt{1+x^2}\,\right]}, \tag{2.3}$$

(2) when $1/2 < a < 1/\sqrt{3}$, the unique zero of $h_x'(a)$ equals

$$a_0 = \sqrt{1+x^2}\,\left[\sin\left(\frac{2}{3}\arctan\frac{1}{x}+\frac{\pi}{6}\right)-\frac{1}{2}\right], \tag{2.4}$$

and the function $h_x(a)$ attains its maximum

$$h_x(a_0) = \frac{[\sin((2/3)\arctan(1/x)+\pi/6)-1/2]\left\{1-(1+x^2)[\sin((2/3)\arctan(1/x)+\pi/6)-1/2]^2\right\}}{\sin((2/3)\arctan(1/x)+\pi/6)+1/2} \tag{2.5}$$

for $x > 0$.

In a word, the sharp lower bounds of (1.10) are

$$\arctan x > \frac{8x}{3\left[1+\sqrt{3}\,\sqrt{1+x^2}\,\right]}, \tag{2.6}$$

$$\arctan x > \frac{4x[\sin((2/3)\arctan(1/x)+\pi/6)-1/2]\left\{1-(1+x^2)[\sin((2/3)\arctan(1/x)+\pi/6)-1/2]^2\right\}}{\sin((2/3)\arctan(1/x)+\pi/6)+1/2} \tag{2.7}$$

for $x > 0$. Similarly, the sharp upper bound of (1.10) is

$$\arctan x < \frac{\pi x}{\pi-2+2\sqrt{1+x^2}}, \quad x > 0. \tag{2.8}$$

Remark 2.5. Similar to the deduction of inequalities (2.6) and (2.7), the sharp versions of (1.9) and its reversion are

$$\frac{3x}{1+2\sqrt{1+x^2}} < \arctan x < \frac{\pi x}{1+2\sqrt{1+x^2}}, \quad x > 0, \tag{2.9}$$

$$\frac{\pi^2 x}{4+2\pi\sqrt{1+x^2}} < \arctan x < \frac{(\pi+2)x}{2+\pi\sqrt{1+x^2}}, \quad x > 0. \tag{2.10}$$

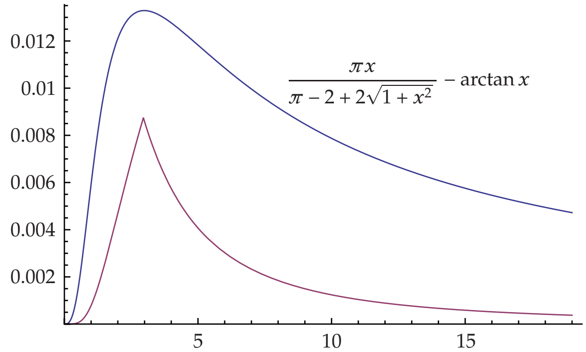

Figure 1: The differences between terms in (2.11).

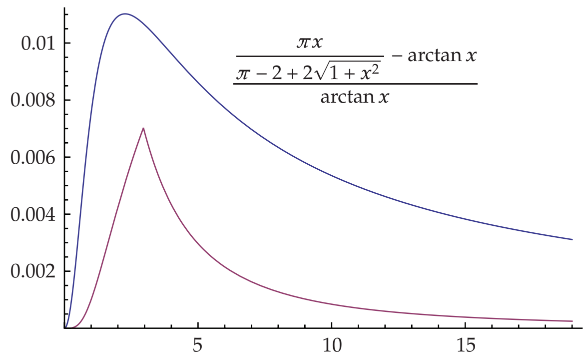

Figure 2: The ratios between terms in (2.11).

Remark 2.6. It is easy to verify that the right-hand side inequalities in (2.9) and (2.10) are included in the inequality (2.8).

By the famous software Mathematica, it is revealed that the inequality (2.7) contains (2.6) and the left-hand side inequality in (2.9), and that the inequality (2.7) and the left-hand side inequality in (2.10) are not included in each other.

In conclusion, the following double inequality is the best accurate one:

$$\begin{aligned} &\max\left\{\frac{\pi^2 x}{4+2\pi\sqrt{1+x^2}}, \frac{4x[\sin((2/3)\arctan(1/x)+\pi/6)-1/2]\mathfrak{K}}{\sin((2/3)\arctan(1/x)+\pi/6)+1/2}\right\} \\ &\quad < \arctan x \\ &\quad < \frac{\pi x}{\pi - 2 + 2\sqrt{1+x^2}}, \quad x > 0. \end{aligned} \tag{2.11}$$

where $\mathfrak{R}$ denotes $\{1-(1+x^2)[\sin((2/3)\arctan(1/x)+\pi/6)-1/2]^2\}$.

Remark 2.7. For possible applications of the double inequality (2.11) in the theory of approximations, the accuracy of bounds in (2.11) for the arc tangent function is described by Figures 1 and 2.

The upper curves in Figures 1 and 2 are, respectively, the graphs of the functions

$$\frac{\pi x}{\pi - 2 + 2\sqrt{1+x^2}} - \arctan x, \qquad \frac{\pi x/\left(\pi - 2 + 2\sqrt{1+x^2}\right) - \arctan x}{\arctan x}, \tag{2.12}$$

and the lower curves in Figures 1 and 2 are, respectively, the graphs of the functions

$$\begin{gathered}\arctan x - \max\left\{\frac{\pi^2 x}{4+2\pi\sqrt{1+x^2}}, \frac{\mathfrak{A}}{\sin((2/3)\arctan(1/x)+\pi/6)+1/2}\right\},\\ \frac{\arctan x - \max\left\{\pi^2 x/\left(4+2\pi\sqrt{1+x^2}\right), \mathfrak{A}/(\sin((2/3)\arctan(1/x)+\pi/6)+1/2)\right\}}{\arctan x}\end{gathered} \tag{2.13}$$

on the interval $(0,19)$, where $\mathfrak{A}$ denotes $4x[\sin((2/3)\arctan(1/x)+\pi/6)-1/2]\{1-(1+x^2)[\sin((2/3)\arctan(1/x)+\pi/6)-1/2]^2\}$.

These two figures are plotted by the famous software Mathematica 7.0.

Remark 2.8. The approach below used in the proofs of Theorems 1.1 and 1.2 has been employed in [6–9].

Remark 2.9. This paper is a revised version of the preprint [10].

3. Proofs of Theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1.1. Direct calculation gives

$$f_a'(x) = \frac{(1+x^2)\left(1+a\sqrt{1+x^2}\right)}{x^2(1+x^2)^{3/2}}\left[\frac{x+x^3+ax\sqrt{1+x^2}}{(1+x^2)\left(1+a\sqrt{1+x^2}\right)} - \arctan x\right]. \tag{3.1}$$

Let

$$g_a(x) = \frac{x+x^3+ax\sqrt{1+x^2}}{(1+x^2)\left(1+a\sqrt{1+x^2}\right)} - \arctan x, \tag{3.2}$$

then

$$g_a'(x) = -\frac{x^2\left(2a^2\sqrt{x^2+1}+a-\sqrt{x^2+1}\right)}{(x^2+1)^{3/2}\left(a\sqrt{x^2+1}+1\right)^2}, \tag{3.3}$$

and the function

$$h_a(x) = \frac{2a^2\sqrt{x^2+1} + a - \sqrt{x^2+1}}{\left(a\sqrt{x^2+1} + 1\right)^2} \tag{3.4}$$

has two zeros

$$a_1(x) = -\frac{1+\sqrt{9+8x^2}}{4\sqrt{1+x^2}}, \qquad a_2(x) = \frac{-1+\sqrt{9+8x^2}}{4\sqrt{1+x^2}}. \tag{3.5}$$

Further differentiation yields

$$\begin{aligned} a_1'(x) &= \frac{x\left(1+\sqrt{9+8x^2}\right)}{4(1+x^2)^{3/2}\sqrt{9+8x^2}} > 0, \\ a_2'(x) &= \frac{x\left(\sqrt{9+8x^2}-1\right)}{4(1+x^2)^{3/2}\sqrt{9+8x^2}} > 0. \end{aligned} \tag{3.6}$$

This means that the functions $a_1(x)$ and $a_2(x)$ are increasing on $(0,\infty)$. From

$$\begin{aligned} \lim_{x\to 0^+} a_1(x) &= -1, \qquad \lim_{x\to\infty} a_1(x) = -\frac{\sqrt{2}}{2}, \\ \lim_{x\to 0^+} a_2(x) &= \frac{1}{2}, \qquad \lim_{x\to\infty} a_2(x) = \frac{\sqrt{2}}{2}, \end{aligned} \tag{3.7}$$

it follows that

(1) when $a \le -1$ or $a \ge \sqrt{2}/2$, the derivative $g_a'(x)$ is negative and the function $g_a(x)$ is strictly decreasing on $(0,\infty)$. From

$$\lim_{x\to 0^+} g_a(x) = 0, \qquad \lim_{x\to\infty} g_a(x) = \frac{1}{a} - \frac{\pi}{2}, \tag{3.8}$$

it is deduced that $g_a(x) < 0$ on $(0,\infty)$. Accordingly,

(a) when $a \le -1$, the derivative $f_a'(x) > 0$ and the function $f_a(x)$ is strictly increasing on $(0,\infty)$;

(b) when $a \ge \sqrt{2}/2$, the derivative $f_a'(x)$ is negative and the function $f_a(x)$ is strictly decreasing on $(0,\infty)$;

(2) when $1/2 \ge a \ge 0$, the derivative $g_a'(x)$ is positive and the function $g_a(x)$ is increasing on $(0,\infty)$. By (3.8), it follows that the function $g_a(x)$ is positive on $(0,\infty)$. Thus, the derivative $f_a'(x)$ is positive and the function $f_a(x)$ is strictly increasing on $(0,\infty)$;

(3) when $1/2 < a < \sqrt{2}/2$, the derivative $g_a'(x)$ has a unique zero which is a minimum of $g_a(x)$ on $(0,\infty)$. Hence, by the second limit in (3.8), it may be deduced that

(a) when $2/\pi \le a < \sqrt{2}/2$, the function $g_a(x)$ is negative on $(0,\infty)$, so the derivative $f_a'(x)$ is also negative and the function $f_a(x)$ is strictly decreasing on $(0,\infty)$;

(b) when $1/2 < a < 2/\pi$, the function $g_a(x)$ has a unique zero which is also a unique zero of the derivative $f_a'(x)$, and so the function $f_a(x)$ has a unique minimum of the function $f_a(x)$ on $(0,\infty)$.

On the other hand, the derivative $f_a'(x)$ can be rewritten as

$$f_a'(x) = \frac{1+x^2}{x^2(1+x^2)^{3/2}}\left[\frac{x+x^3+ax\sqrt{1+x^2}}{1+x^2} - \left(1+a\sqrt{1+x^2}\right)\arctan x\right], \tag{3.9}$$

and the function

$$G_a(x) = \frac{x+x^3+ax\sqrt{1+x^2}}{1+x^2} - \left(1+a\sqrt{1+x^2}\right)\arctan x \tag{3.10}$$

satisfies

$$G_a'(x) = -\frac{x\left[x\left(a-\sqrt{x^2+1}\right)+a(x^2+1)\arctan x\right]}{(x^2+1)^{3/2}}. \tag{3.11}$$

When $a \le 0$, the derivative $G_a'(x)$ is positive and the function $G_a(x)$ is strictly increasing on $(0,\infty)$. Since $\lim_{x\to 0^+} G_a(x) = 0$, the function $G_a(x)$ is positive, and so the derivative $f_a'(x)$ is positive, on $(0,\infty)$ for $a \le 0$. Consequently, when $a \le 0$, the function $f_a(x)$ is strictly increasing on $(0,\infty)$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Direct calculation yields

$$\lim_{x\to 0^+} f_a(x) = 1+a, \qquad \lim_{x\to\infty} f_a(x) = \frac{\pi}{2}. \tag{3.12}$$

By the increasing monotonicity in Theorem 1.1, it follows that $1+a < f_a(x) < \pi/2$ for $a \le 1/2$, which can be rewritten as (1.9) for $-1 < a \le 1/2$. Similarly, the reversed version of the inequality (1.9) and the right-hand side inequality in (1.10) can be procured.

When $1/2 < a < 2/\pi$, the unique minimum point $x_0 \in (0,\infty)$ of the function $f_a(x)$ satisfies

$$\arctan x_0 = \frac{x_0+x_0^3+ax_0\sqrt{1+x_0^2}}{(1+x_0^2)\left(1+a\sqrt{1+x_0^2}\right)}, \tag{3.13}$$

and so the minimum of $f_a(x)$ on $(0,\infty)$ is

$$\begin{aligned}
f_a(x_0) &= \frac{x_0 + x_0^3 + a x_0\sqrt{1+x_0^2}}{(1+x_0^2)\left(1+a\sqrt{1+x_0^2}\right)} \cdot \frac{a+\sqrt{1+x_0^2}}{x_0} \\
&= \frac{\left(a+\sqrt{1+x_0^2}\right)\left(1+x_0^2+a\sqrt{1+x_0^2}\right)}{(1+x_0^2)\left(1+a\sqrt{1+x_0^2}\right)} \\
&= \frac{(a+u)^2}{u(1+au)} \\
&> 4a\left(1-a^2\right),
\end{aligned} \tag{3.14}$$

where $u = \sqrt{1+x_0^2} \in (1,\infty)$, as a result, the left-hand side inequality in (1.10) follows. The proof of Theorem 1.2 is complete. □

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