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Research Article

Equivalence of Some Affine Isoperimetric Inequalities

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We establish the equivalence of some affine isoperimetric inequalities which include the L_p-Petty projection inequality, the L_p-Busemann-Petty centroid inequality, the "dual" L_p-Petty projection inequality, and the "dual" L_p-Busemann-Petty inequality. We also establish the equivalence of an affine isoperimetric inequality and its inclusion version for L_p-John ellipsoids.

1. Introduction

In the recent years, the L_p-analogs of the projection bodies and centroid bodies have received considerable attentions [1–7]. Lutwak et al. established the L_p-analog of the Petty projection inequality [4]. It states that *if K is a convex body in $\mathbb{R}^n$, then for $1 \le p < \infty$,*

$$V\left(\Pi_p^* K\right) V(K)^{(n-p)/p} \le \omega_n^{n/p}, \tag{1.1}$$

with an equality if and only if $K$is an ellipsoid. Here, $\Pi_p^* K = (\Pi_p K)^*$ is used to denote the polar body of the L_p-projection body, $\Pi_p K$, of K, and write ω_n for $V(B_n)$, the n-dimensional volume of the unit ball B_n.

They also established the L_p-analog of the Busemann-Petty centroid inequality [4]. It states that *if K is a star body (about the origin) in $\mathbb{R}^n$, then for $1 \le p < \infty$,*

$$V(\Gamma_p K) \ge V(K), \tag{1.2}$$

with an equality if and only if K is a centroid ellipsoid at the origin. Here, $\Gamma_p K$ is the L_p-centroid body of K. It is also shown in [4] that the L_p-Busemann-Petty inequality (1.2) implies L_p-Petty projection inequality (1.1). A quite different proof of the L_p-analog of the Busemann-Petty centroid inequality is obtained by Campi and Gronchi [1].

Recently, Lutwak et al. [8] proved that there is a family of L_p-John ellipsoids, $E_p K$, which can be associated with a fixed convex body K: if K contains the origin in its interior and $p > 0$, among all origin-centered ellipsoids E, the unique ellipsoid $E_p K$ solves the constrained maximization problem:

$$V\left(E_p K\right) = \max_E V(E) \quad \text{subject to } V_p(K, E) \leq V(K). \tag{1.3}$$

Corresponding to Lutwak et al.'s work, Yu et al. [9] proved that there is a family of dual L_p-John ellipsoids, $\widetilde{E}_p K$, which can be associated with a fixed convex body K: if K contains the origin in its interior and $p > 0$, among all origin-centered ellipsoids E, the unique ellipsoid $\widetilde{E}_p K$ solves the constrained maximization problem:

$$V\left(\widetilde{E}_p K\right) = \max_E \frac{1}{V(E)} \quad \text{subject to } \widetilde{V}_{-p}(K, E) \leq V(K). \tag{1.4}$$

Lutwak et al. [8] showed that the following results hold.

Theorem A. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $1 \leq p$, then*

$$\frac{\omega_n}{2^n} V(K) \leq V\left(E_p K\right) \leq V(K), \tag{1.5}$$

with an equality in the right inequality if and only if K is a centered ellipsoid and an equality in the left inequality if K is a parallelotope.

Yu et al. [9] showed a theorem similar to Theorem A, and recently, Lu and Leng [10] gave a strengthened inequality as follows.

Theorem B. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $1 \leq p$, then*

$$V\left(E_p K\right) \leq V\left(\Gamma_{-p} K\right) \leq V(K) \leq V\left(\Gamma_p K\right) \leq V\left(\widetilde{E}_p K\right), \tag{1.6}$$

with an equality if and only if K is a centered ellipsoid. Here, $V(\Gamma_{-p} K) \leq V(K)$ is a dual form of L_p-Busemann-Petty centroid inequality (1.2).

One purpose of this paper is to establish the equivalence of some affine isopermetric inequalities as follows.

Theorem 1.1. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $1 \le p$, then the following inequalities are equivalent:*

$$V(\Gamma_p K) \ge V(K), \tag{1.7}$$

$$V(\Gamma_{-p} K) \le V(K), \tag{1.8}$$

$$V\left(\Pi^*_{-p} K\right)^{-1} V(K)^{(n+p)/p} \le \omega_n^{n/p}, \tag{1.9}$$

$$V\left(\Pi^*_{p} K\right) V(K)^{(n-p)/p} \le \omega_n^{n/p}, \tag{1.10}$$

all above inequalities with an equality if and only if K is a centered ellipsoid.

Note that (1.7) is the L_p-Busemann-Petty centroid inequality (1.2), (1.8) is the dual form of L_p-Busemann-Petty centroid inequality in Theorem B, (1.9) is a "dual" form of L_p-Petty projection inequality, and (1.10) is the L_p-Petty projection inequality (1.1).

Another purpose of this paper is to establish the follow equivalence of Theorem A and its inclusion version Theorem A′.

Theorem 1.2. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $1 \le p$, then Theorem A is equivalent to Theorem A′.*

Theorem A′. *There exist an ellipsoid E and a parallelotope P such that*

$$\begin{gathered} V(E) = V(K) = V(P), \\ E_p E \supseteq E_p K \supseteq E_p P, \end{gathered} \tag{1.11}$$

where the left inclusion with an equality if and only if K is a centered ellipsoid and the right inclusion with an equality if and only if K is a parallelotope.

Some notation and background material contained in Section 2.

2. Notations and Background Materials

We will work in $\mathbb{R}^n$ equipped with a fixed Euclidean structure and write $|\cdot|$ for the corresponding Euclidean norm. We denote the Euclidean unit ball and the unit sphere by B_n and S^{n-1}, respectively. The volume of appropriate dimension will be denoted by $V(\cdot)$. The group of nonsingular affine transformations of $\mathbb{R}^n$ is denoted by $\mathrm{GL}(n)$. The group of special affine transformations is denoted by $\mathrm{SL}(n)$, these are the members of $\mathrm{GL}(n)$ whose determinant is one. We will write ω_n for the volume of the Euclidean unit ball in $\mathbb{R}^n$. Note that

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(1 + n/2)} \tag{2.1}$$

defines ω_n for all nonnegative real n (not just the positive integers). For real $p \ge 1$, define $c_{n,p} = \omega_{n+p}/\omega_2 \omega_n \omega_{p-1}$.

If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then we will use K^* to denote the *polar body* of K, that is,

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \ \forall y \in K\}. \tag{2.2}$$

From the definition of the polar body, we can easily find that for $\lambda > 0$, there is

$$(\lambda K)^* = \frac{1}{\lambda} K^*. \tag{2.3}$$

If K is a convex body in $\mathbb{R}^n$, then its *support function*, $h_K(\cdot) = h(K, \cdot) : \mathbb{R}^n \to R$, is defined for $x \in \mathbb{R}^n$ by $h(K, x) = \max\{x \cdot y : y \in K\}$. A star body in $\mathbb{R}^n$ is a nonempty compact set K satisfying $[o, x] \subset K$ for all $x \in K$ and such that the *radial function* $\rho_K(\cdot) = \rho(K, \cdot)$, defined by $\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}$, is positive and continuous. Two star bodies K and L are said to be dilates if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is a centered (i.e., symmetric about the origin) convex body, then it follows from the definitions of support and radial functions, and the definition of polar body, that

$$h_K^* = \frac{1}{\rho_K}, \qquad \rho_K^* = \frac{1}{h_K}. \tag{2.4}$$

For L_p-mixed and dual mixed volumes, those formulae are directly given as follows.

It was shown in [11] that corresponding to each convex body $K \in \mathbb{R}^n$ that is containing the origin in its interior, there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1}, such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h_Q(u)^p dS_p(K, u), \tag{2.5}$$

for each convex body Q.

If K, L are star bodies in $\mathbb{R}^n$, then for $p \geq 1$, the dual L_p mixed volume, $\widetilde{V}_{-p}(K, L)$, of K and L was defined by [4]

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+p} \rho_L(u)^{-p} dS(u), \tag{2.6}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the integral representation (2.5), it follows immediately that for each convex body K,

$$V_p(K, K) = V(K). \tag{2.7}$$

From (2.6), of the dual L_p-mixed volume, it follows immediately the for each star body K,

$$\widetilde{V}_{-p}(K, K) = V(K). \tag{2.8}$$

We will require two basic inequalities for the L_p-mixed volume V_p and the dual L_p-mixed volume $\widetilde{V}_{-p}$. The L_p-Minkowski inequality states that for convex bodies K, L [3],

$$V_p(K,L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{2.9}$$

with an equality if and only if K and L are dilates [11]. The dual L_p-Minkowski inequality states that for star bodies K, L [4],

$$\widetilde{V}_{-p}(K,L) \geq V(K)^{(n+p)/n} V(L)^{-p/n}, \tag{2.10}$$

with an equality if and only if K and L are dilates.

The L_p-projection bodies was first introduced by Lutwak et al. in [4], and is defined as the body whose support function, for $u \in S^{n-1}$, is given by

$$h\left(\Pi_p K, u\right)^p = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K,v). \tag{2.11}$$

If K is a star body about the origin in R^n, and $p \geq 1$, the L_p-centroid body $\Gamma_p K$ of K is the origin-symmetric convex body whose support function is given by [4]

$$h\left(\Gamma_p K, u\right)^p = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx. \tag{2.12}$$

The normalized L_p polar projection body of K, $\Gamma_{-p} K$, for $p > 0$, is defined as the body whose radial function, for $u \in S^{n-1}$, is given by [8]

$$\rho\left(\Gamma_{-p} K, u\right)^{-p} = \frac{1}{n c_{n-2,p} V(K)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K,v). \tag{2.13}$$

Here, we introduce a new convex body of K, $\Pi_{-p} K$, for $p > 0$, defined as the body whose radial function, for $u \in S^{n-1}$, that is given by

$$\rho\left(\Pi_{-p} K, u\right)^{-p} = \frac{1}{\omega_n c_{n,p}} \int_K |u \cdot x|^p dx. \tag{2.14}$$

Noting that the normalization is chosen for the standard unit ball B_n in R^n, we have $\Pi_p B_n = \Gamma_p B_n = \Gamma_{-p} B_n = \Pi_{-p} B_n = B_n$. For general reference the reader may wish to consult the books of Gardner [12] and Schneider [13].

3. Proof of the Results

Lemma 3.1. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$\Pi_p^* K = \left(\frac{\omega_n}{V(K)}\right)^{1/p} \Gamma_{-p} K; \tag{3.1}$$

$$\Pi_{-p}^* K = \left(\frac{V(K)}{\omega_n}\right)^{1/p} \Gamma_p K. \tag{3.2}$$

Proof. From the definition (2.11) and (2.13) combined with (2.4), for $u \in S^{n-1}$, we have

$$\rho\left(\Pi_p^* K, u\right)^{-p} = \frac{V(K)}{\omega_n} \rho(\Gamma_{-p} K, u)^{-p}. \tag{3.3}$$

So we get

$$\Pi_p^* K = \left(\frac{\omega_n}{V(K)}\right)^{1/p} \Gamma_{-p} K. \tag{3.4}$$

From the definition (2.12) and (2.14) combined with (2.4), for $u \in S^{n-1}$, we have

$$h\left(\Pi_{-p}^* K, u\right)^p = \frac{V(K)}{\omega_n} h(\Gamma_p K, u)^p. \tag{3.5}$$

So we get

$$\Pi_{-p}^* K = \left(\frac{V(K)}{\omega_n}\right)^{1/p} \Gamma_p K. \tag{3.6}$$

□

Corollary 3.2. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, let $p(K) = V(\Pi_{-p}^* K)^{-1} V(K)^{(n+p)/p}$, then for $\phi \in GL(n)$,*

$$p(\phi K) = p(K). \tag{3.7}$$

Proof. Since for $\phi \in \mathrm{GL}(n)$, $\Gamma_p(\phi K) = \phi\Gamma_p K$ (see [4]), combined with (3.2) and $V(\phi K) = |\det\phi| V(K)$, we know that for $\phi \in \mathrm{GL}(n)$,

$$\begin{aligned}
p(\phi K) &= V\left(\Pi_{-p}^{*}(\phi K)\right)^{-1} V(\phi K)^{(n+p)/p} \\
&= V\left(\left(\frac{V(\phi K)}{\omega_n}\right)^{1/p} \Gamma_p(\phi K)\right)^{-1} V(\phi K)^{(n+p)/p} \\
&= V\left(\left(\frac{|\det\phi| V(K)}{\omega_n}\right)^{1/p} \phi\Gamma_p K\right)^{-1} (|\det\phi| V(K))^{(n+p)/p} \\
&= V\left(\left(\frac{V(K)}{\omega_n}\right)^{1/p} \Gamma_p K\right)^{-1} V(K)^{(n+p)/p} \\
&= V\left(\Pi_{-p}^{*} K\right)^{-1} V(K)^{(n+p)/p} \\
&= p(K).
\end{aligned} \tag{3.8}$$

From Corollary 3.2, we know that (1.9) is an affine isoperimetric inequality. □

Lemma 3.3. *If K, L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interior, then the following equalities are equivalent:*

$$V_p(L, \Gamma_p K) = \frac{\omega_n}{V(K)} \widetilde{V}_{-p}\left(K, \Pi_p^{*} L\right), \tag{3.9}$$

$$\frac{V_p(L, \Gamma_p K)}{V(L)} = \frac{\widetilde{V}_{-p}(K, \Gamma_{-p} L)}{V(K)}, \tag{3.10}$$

$$V_p\left(L, \Pi_{-p}^{*} K\right) = \frac{V(L)}{\omega_n} \widetilde{V}_{-p}(K, \Gamma_{-p} L), \tag{3.11}$$

$$V_p\left(L, \Pi_{-p}^{*} K\right) = \widetilde{V}_{-p}\left(K, \Pi_p^{*} L\right). \tag{3.12}$$

Proof. First, from Lemma 3.1, we know that

$$\Pi_p^{*} L = \left(\frac{\omega_n}{V(L)}\right)^{1/p} \Gamma_{-p} L. \tag{3.13}$$

From (2.5) and (2.6), we have for $\lambda > 0$,

$$V_p(K, \lambda L) = \lambda^p V_p(K, L), \tag{3.14}$$

$$\widetilde{V}_{-p}(K, \lambda L) = \lambda^{-p} \widetilde{V}_{-p}(K, L). \tag{3.15}$$

Substitute (3.13) in (3.9) and combine (3.15) to just get (3.10); substitute (3.2) in (3.10) and combine (3.14) to just get (3.11); substitute (3.13) in (3.11) and combine (3.15) to just get (3.12); substitute (3.2) in (3.12) and combine (3.14) to just get (3.9). □

Note. Equation (3.9) is proved in [4] and (3.10) is proved in [10].

Proof of Theorem 1.1. (1.7)⇒(1.8): substituting $K = \Gamma_{-p}L$ in (3.10), followed by (2.8), (2.9), and (1.7), we have for each convex body L that contains the origin in its interior,

$$\begin{aligned}
1 &= \frac{\widetilde{V}_{-p}(\Gamma_{-p}L, \Gamma_{-p}L)}{V(\Gamma_{-p}L)} \\
&= \frac{V_p(L, \Gamma_p\Gamma_{-p}L)}{V(L)} \\
&\geq \frac{V(L)^{(n-p)/n}V(\Gamma_p\Gamma_{-p}L)^{p/n}}{V(L)} \\
&\geq V(L)^{-(p/n)}V(\Gamma_{-p}L)^{p/n}.
\end{aligned} \tag{3.16}$$

(1.8)⇒(1.9): substituting $L = \Pi^*_{-p}K$ in (3.11), followed by (2.7), (2.9), and (1.8), we have

$$\begin{aligned}
\omega_n &= \frac{\omega_n}{V\left(\Pi^*_{-p}K\right)}V_p\left(\Pi^*_{-p}K, \Pi^*_{-p}K\right) \\
&= \widetilde{V}_p\left(K, \Gamma_{-p}\Pi^*_{-p}K\right) \\
&\geq V(K)^{(n+p)/n}V\left(\Gamma_{-p}\Pi^*_{-p}K\right)^{-p/n} \\
&\geq V(K)^{(n+p)/n}V\left(\Pi^*_{-p}K\right)^{-p/n}.
\end{aligned} \tag{3.17}$$

(1.9)⇒(1.10): substituting $K = \Pi^*_pL$ in (3.12), followed by (2.9), we get

$$V\left(\Pi^*_pL\right) = V_p\left(L, \Pi^*_{-p}\Pi^*_pL\right) \geq V(L)^{(n-p)/n}V\left(\Pi^*_{-p}\Pi^*_pL\right)^{p/n}, \tag{3.18}$$

that is,

$$V(L)^{(n-p)/p} \leq V\left(\Pi^*_{-p}\Pi^*_pL\right)^{-1}V\left(\Pi^*_pL\right)^{n/p}. \tag{3.19}$$

So, we have

$$V\left(\Pi^*_pL\right)V(L)^{(n-p)/p} \leq V\left(\Pi^*_{-p}\Pi^*_pL\right)^{-1}V\left(\Pi^*_pL\right)^{(n+p)/p} \leq \omega_n^{n/p}. \tag{3.20}$$

(1.10)⇒(1.7): substituting $L = \Gamma_p K$ in (3.9), followed by (2.7), (2.10), we have

$$\begin{aligned} V(\Gamma_p K) &= V_p(\Gamma_p K, \Gamma_p K) \\ &= \frac{\omega_n}{V(K)} \widetilde{V}_{-p}\left(K, \Pi_p^* \Gamma_p K\right) \\ &\geq \frac{\omega_n}{V(K)} V(K)^{(n+p)/n} V\left(\Pi_p^* \Gamma_p K\right)^{-p/n} \\ &= \omega_n V(K)^{p/n} V\left(\Pi_p^* \Gamma_p K\right)^{-p/n}, \end{aligned} \tag{3.21}$$

that is,

$$V(\Gamma_p K)^{n/p} V\left(\Pi_p^* \Gamma_p K\right) V(K)^{-1} \geq \omega_n^{n/p}. \tag{3.22}$$

Combined with (1.10), we get

$$V(\Gamma_p K)^{n/p} V\left(\Pi_p^* \Gamma_p K\right) V(K)^{-1} \geq \omega_n^{n/p} \geq V(\Gamma_p K)^{(n-p)/p} V\left(\Pi_p^* \Gamma_p K\right), \tag{3.23}$$

that is,

$$V(\Gamma_p K) \geq V(K). \tag{3.24}$$

□

Lemma 3.4 (see [8]). *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $p > 0$, then for $\phi \in GL(n)$,*

$$E_p \phi K = \phi E_p K. \tag{3.25}$$

Proof of Theorem 1.2. Firstly, we prove that Theorem A implies Theorem A′. □

From $V(E_p K) \leq V(K)$, taking $E = (V(K)/V(E_p K))^{1/n} E_p K$, since $V(\lambda K) = \lambda^n V(K)$ for $\lambda > 0$, we know that $V(E) = V(K)$ and followed by Lemma 3.4,

$$E_p E = \left(\frac{V(K)}{V(E_p K)}\right)^{1/n} E_p K \supseteq E_p K, \tag{3.26}$$

where the inclusion with an equality if and only if K is a centered ellipsoid.

Suppose that $E_p K = \widehat{\phi} B_n$, for some $\widehat{\phi} \in \mathrm{GL}(n)$, then

$$V(E_p K) = \left|\det \widehat{\phi}\right| \omega_n. \tag{3.27}$$

Take $P = (\widehat{\phi}/|\det\widehat{\phi}|^{1/n})(V(K)^{1/n}/2)Q$, here Q is the unit cube $[-1,1]^n$. Since Lutwak et al. [8] proved that the L_p-John ellipsoid of the unit cube is B_n, that is, $E_pQ = B_n$, so we have $V(K) = V(P)$ by the fact $V(Q) = 2^n$. Following Lemma 3.4, $E_pQ = B_n$, $E_pK = \widehat{\phi}B_n$, (3.27) and the left inequality of Theorem A, we have

$$\begin{aligned} E_pP &= \left(\frac{V(K)}{2^n\left|\det\widehat{\phi}\right|}\right)^{1/n}\widehat{\phi}E_pQ \\ &= \left(\frac{V(K)}{2^n\left|\det\widehat{\phi}\right|}\right)^{1/n}\widehat{\phi}B_n \\ &= \left(\frac{V(K)\omega_n}{2^nV(E_pK)}\right)^{1/n}E_pK \\ &\subseteq E_pK, \end{aligned} \tag{3.28}$$

where the inclusion with an equality if and only if K is a parallelotope. By (3.26) and (3.28), we know that Theorem A implies Theorem A′.

Secondly, we prove that Theorem A′ implies Theorem A.

On the one hand, since $E_pE \supseteq E_pK$ and $E_pE = E$ by Lemma 3.4, we have

$$V(K) = V(E) = V(E_pE) \geq V(E_pK), \tag{3.29}$$

with an equality holds if and only if K is a centered ellipsoid. On the other hand, suppose that $P = \phi Q$ for some $\phi \in \mathrm{GL}(n)$, then $V(K) = V(P) = |\det\phi|V(Q) = |\det\phi|2^n$, so $|\det\phi| = V(K)/2^n$. Following Theorem A′ and Lemma 3.4, we have

$$E_pK \supseteq E_pP = E_p\phi Q = \phi E_pQ = \phi B_n, \tag{3.30}$$

that is,

$$V(E_pK) \geq V(\phi B_n) = \left|\det\phi\right|V(B_n) = \frac{V(K)}{2^n}\omega_n, \tag{3.31}$$

with an equality if and only if K is a parallelotope. By (3.29) and (3.31), we know that Theorem A′ implies Theorem A.

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