

Research Article

Some Starlikeness Criteria for Analytic Functions

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We determine the condition on α , μ , β , and λ for which $|(1-\alpha)(z/f(z))^\mu + \alpha(zf'(z)/f(z))(z/f(z))^{\mu-1}| < \lambda$ implies $f(z) \in S^*(\beta)$, where $S^*(\beta)$ is the class of starlike functions of order β . Some results of Obradović and Owa are extended. We also obtain some new results on starlikeness criteria.

1. Introduction

Let n be a positive integer, and let H_n denote the class of function

$$f(z) = z + \sum_{k=n}^{+\infty} a_{k+1} z^{k+1} \quad (1.1)$$

that are analytic in the unit disk $U = \{z : |z| < 1\}$. For $0 \leq \beta < 1$, let

$$S^*(\beta) = \left\{ f \in H_1 : \operatorname{Re} \frac{zf'(z)}{f(z)} > \beta, z \in U \right\} \quad (1.2)$$

denote the class of starlike function of order β and $S^*(0) = S^*$.

Let $f(z)$ and $F(z)$ be analytic in U ; then we say that the function $f(z)$ is subordinate to $F(z)$ in U , if there exists an analytic function $w(z)$ in U such that $|w(z)| \leq |z|$, and $f(z) \equiv F(w(z))$, denoted that $f < F$ or $f(z) < F(z)$. If $F(z)$ is univalent in U , then the subordination is equivalent to $f(0) = F(0)$ and $f(U) \subset F(U)$ [1].

Let

$$S_\lambda = \{f \in H_1 : |f'(z) - 1| < \lambda, z \in U\}. \quad (1.3)$$

Singh [2] proved that $S_\lambda \subset S^*$ if $0 < \lambda \leq 2/\sqrt{5}$. More recently, Fournier [3, 4] proved that

$$S_\lambda \subset S^* \iff 0 \leq \lambda \leq \frac{2}{\sqrt{5}},$$

$$\rho_\lambda = \begin{cases} \frac{(1-\lambda)(1-\lambda/2)}{1-\lambda^2/4}, & \text{if } 0 \leq \lambda \leq \frac{2}{3}, \\ \frac{(1/2)(1-(5/4)\lambda^2)}{1-\lambda^2/4}, & \text{if } \frac{2}{3} \leq \lambda \leq 1, \end{cases} \quad (1.4)$$

is the order of starlikeness of S_λ . Now, we define

$$U(\lambda, \mu, n) = \left\{ f \in H_n : \left| \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\mu - 1 \right| < \lambda, z \in U \right\}. \quad (1.5)$$

Clearly, $U(\lambda, -1, 1) = S_\lambda$. In 1998, Obradović [5] proved that

$$U(\lambda, \mu, 1) \subset S^* \quad (1.6)$$

if $0 < \mu < 1$ and $0 < \lambda \leq (1-\mu)/\sqrt{(1-\mu)^2 + \mu^2}$. Recently, Obradović and Owa [6] proved that

$$U(\lambda, \mu, n) \subset S^* \quad (1.7)$$

if $0 < \mu < 1$ and $0 < \lambda \leq (n-\mu)/\sqrt{(n-\mu)^2 + \mu^2}$.

In this paper we find a condition on α, μ, β , and λ for which

$$\left| (1-\alpha) \left(\frac{z}{f(z)} \right)^\mu + \alpha \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\mu - 1 \right| < \lambda \quad (1.8)$$

implies $f(z) \in S^*(\beta)$ and extend some results of Obradović and Owa [5, 6]. Also, we obtain some new results on starlikeness criterions.

2. Main Results

For our results we need the following lemma.

Lemma 2.1 (see [6]). *Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ be analytic in \mathcal{U} , $n \geq 1$, and satisfy the condition*

$$p(z) - \frac{1}{\mu} z p'(z) < 1 + \lambda z, \quad 0 < \mu < 1, \quad 0 < \lambda \leq 1. \quad (2.1)$$

Then

$$p(z) < 1 + \lambda_1 z, \quad (2.2)$$

where

$$\lambda_1 = \frac{\lambda \mu}{n - \mu}. \quad (2.3)$$

Theorem 2.2. *Let $0 \leq \mu < 1$, $n\alpha > \mu$, $0 \leq \beta < 1$, and*

$$M_n(\alpha, \beta, \mu) = \begin{cases} \frac{\alpha(n\alpha - \mu)(1 - \beta)}{\alpha(n + \mu - \mu\beta) - 2\mu}, & \text{if } \alpha \geq \alpha_2, \\ \frac{(n\alpha - \mu)\sqrt{2\alpha(1 - \beta) - 1}}{\sqrt{n^2\alpha^2 + 2[\mu^2(1 - \beta) - n\mu]\alpha}}, & \text{if } \alpha_1 < \alpha < \alpha_2, \\ \frac{(n\alpha - \mu)(1 - \beta)}{n - \mu(1 - \beta)}, & \text{if } 0 < \alpha \leq \alpha_1, \end{cases} \quad (2.4)$$

where

$$\alpha_1 = \frac{n - \mu(1 - \beta)}{n(1 - \beta)}, \quad (2.5)$$

$$\alpha_2 = \frac{n + 3\mu(1 - \beta) + \sqrt{[n + 3\mu(1 - \beta)]^2 - 8n\mu(1 - \beta)}}{2n(1 - \beta)}.$$

If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ and $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \dots$ are analytic in \mathcal{U} , satisfy

$$q(z) < 1 + \frac{\mu\lambda}{n\alpha - \mu} z, \quad (2.6)$$

$$q(z)[1 - \alpha + \alpha p(z)] < 1 + \lambda z, \quad (2.7)$$

where $0 < \lambda \leq M_n(\alpha, \beta, \mu)$, then

$$\operatorname{Re} p(z) > \beta, \quad \text{for } z \in U. \quad (2.8)$$

Proof. If $\mu = 0$, it is easy to see the result is true. Now, assume $\mu > 0$. Let

$$N = \frac{\mu\lambda}{n\alpha - \mu}. \quad (2.9)$$

If there exists $z_0 \in U$, such that $\operatorname{Re} p(z_0) = \beta$, then we will show that

$$|q(z_0)[1 - \alpha + \alpha p(z_0)] - 1| \geq \lambda \quad (2.10)$$

for $0 < \lambda \leq M_n(\alpha, \beta, \mu)$. Note that $|q(z_0) - 1| \leq N$ for $z \in U$; it is sufficient to show that

$$\alpha|p(z_0) - 1| - N|1 - \alpha + \alpha p(z_0)| \geq \lambda \quad (2.11)$$

for $0 < \lambda \leq M_n(\alpha, \beta, \mu)$. Let $p(z_0) = \beta + iy$, $y \in \mathbb{R}$; then, the left-hand side of (2.11) is

$$\begin{aligned} & \alpha\sqrt{(\beta - 1)^2 + y^2} - N\sqrt{(\alpha\beta + 1 - \alpha)^2 + \alpha^2 y^2} \\ &= \alpha\sqrt{\beta^2 + y^2 + 1 - 2\beta} - N\sqrt{\alpha^2\beta^2 + \alpha^2 y^2 + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2}. \end{aligned} \quad (2.12)$$

Suppose that $x = \beta^2 + y^2$ and note that $(n\alpha - \mu)N = \mu\lambda$; then inequality (2.11) is equivalent to

$$N \leq \frac{\alpha\mu\sqrt{x + 1 - 2\beta}}{n\alpha - \mu + \mu\sqrt{\alpha^2 x + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2}} \quad (2.13)$$

for all $x \geq \beta^2$ and $0 < \lambda \leq M_n(\alpha, \beta, \mu)$. Now, if we define

$$\varphi(x) = \frac{\sqrt{x + 1 - 2\beta}}{n\alpha - \mu + \mu\sqrt{\alpha^2 x + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2}}, \quad x \geq \beta^2, \quad (2.14)$$

then we have

$$\varphi'(x) = \frac{(n\alpha - \mu)\varphi(x) + \mu[1 - 2\alpha(1 - \beta)]}{2\varphi(x)\sqrt{x + 1 - 2\beta}[(n\alpha - \mu) + \mu\varphi(x)]^2}, \quad x > \beta^2, \quad (2.15)$$

where

$$\varphi(x) = \sqrt{\alpha^2 x + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2}. \quad (2.16)$$

Since

$$\varphi(x) = \sqrt{\alpha^2 x^2 + 2\alpha(1-\alpha)\beta + (1-\alpha)^2} > |1-\alpha(1-\beta)|, \quad \text{for } x > \beta^2, \quad (2.17)$$

the denominator of $\varphi'(x)$ is positive. Further, let

$$T(x) = (n\alpha - \mu)\varphi(x) + \mu[1 - 2\alpha(1-\beta)], \quad x \geq \beta^2. \quad (2.18)$$

We have

$$T(x) \geq (n\alpha - \mu)|1 - \alpha(1-\beta)| + \mu[1 - 2\alpha(1-\beta)]. \quad (2.19)$$

If

$$\frac{1}{1-\beta} \leq \alpha, \quad (2.20)$$

we get

$$\begin{aligned} T(x) &\geq n\alpha^2(1-\beta) - [n + 3\mu(1-\beta)]\alpha + 2\mu \\ &= n(1-\beta)(\alpha - r_1)(\alpha - r_2), \end{aligned} \quad (2.21)$$

where

$$\begin{aligned} r_1 &= \frac{n + 3\mu(1-\beta) - \sqrt{[n + 3\mu(1-\beta)]^2 - 8n\mu(1-\beta)}}{2n(1-\beta)}, \\ r_2 &= \frac{n + 3\mu(1-\beta) + \sqrt{[n + 3\mu(1-\beta)]^2 - 8n\mu(1-\beta)}}{2n(1-\beta)}. \end{aligned} \quad (2.22)$$

Note that

$$r_1 < \frac{1}{1-\beta} < r_2. \quad (2.23)$$

We obtain

$$T(x) \geq 0 \quad \text{for } \alpha \geq \alpha_2 = r_2. \quad (2.24)$$

If

$$\frac{1}{2(1-\beta)} \leq \alpha < \frac{1}{1-\beta}, \quad (2.25)$$

we have

$$T(x) \geq \alpha[n - \mu(1 - \beta) - n(1 - \beta)\alpha]. \quad (2.26)$$

Hence we obtain

$$T(x) \geq 0 \quad \text{for } \frac{1}{2(1 - \beta)} \leq \alpha \leq \alpha_1, \quad (2.27)$$

where

$$\alpha_1 = \frac{n - \mu(1 - \beta)}{n(1 - \beta)} < \frac{1}{1 - \beta}. \quad (2.28)$$

If

$$0 < \alpha < \frac{1}{2(1 - \beta)}, \quad (2.29)$$

we have $1 - 2\alpha(1 - \beta) > 0$. It follows that $T(x) > 0$.

Therefore we obtain $\varphi'(x) \geq 0$ for $x > \beta^2$ if $0 < \alpha \leq \alpha_1$ or $\alpha \geq \alpha_2$. It follows that

$$\min_{x \geq \beta^2} \varphi(x) = \varphi(\beta^2) = \begin{cases} \frac{(1 - \beta)}{\alpha(n + \mu - \mu\beta) - 2\mu}, & \text{if } \alpha \geq \alpha_2, \\ \frac{(1 - \beta)}{\alpha[n - \mu(1 - \beta)]}, & \text{if } 0 < \alpha \leq \alpha_1. \end{cases} \quad (2.30)$$

If $\alpha_1 < \alpha < \alpha_2$, we have

$$\lim_{x \rightarrow (\beta^2)^+} T(x) = T(\beta^2) = (n\alpha - \mu)|1 - \alpha(1 - \beta)| + \mu[1 - 2\alpha(1 - \beta)] < 0 \quad (2.31)$$

by (2.13) and (2.21) for $1/(1 - \beta) \leq \alpha < \alpha_2$ and by (2.23) for $\alpha_1 < \alpha < 1/(1 - \beta)$. Note that $T(x)$ is an continuous increasing function for $x \geq \beta^2$, and

$$\lim_{x \rightarrow \infty} T(x) > 0. \quad (2.32)$$

Then there exists a unique $x_0 \in (\beta^2, +\infty)$, such that

$$T(x_0) = 0, \quad \text{or} \quad \varphi'(x_0) = 0. \quad (2.33)$$

Thus, x_0 is the global minimum point of $\varphi(x)$ on $[\beta^2, +\infty)$. It follows from (2.33) that

$$(n\alpha - \mu)\sqrt{\alpha^2 x_0 + 2\alpha(1 - \alpha)\beta + (1 - \alpha)^2} = \mu[2\alpha(1 - \beta) - 1], \quad (2.34)$$

or

$$x_0 = \frac{1}{\alpha^2} \left\{ \frac{\mu^2 [2\alpha(1-\beta) - 1]^2}{(n\alpha - \mu)^2} - 2\alpha(1-\alpha)\beta - (1-\alpha)^2 \right\}. \quad (2.35)$$

By a simple calculation, we may obtain

$$\min_{x \geq \beta^2} \varphi(x) = \varphi(x_0) = \frac{\sqrt{2\alpha(1-\beta) - 1}}{\alpha \sqrt{n^2\alpha^2 + 2[\mu^2(1-\beta) - n\mu]\alpha}} \quad (2.36)$$

for $\alpha_1 < \alpha < \alpha_2$. It follows from (2.30) and (2.36) that that inequality (2.13) holds. This shows that inequality (2.10) holds, which contradicts with (2.7). Hence we must have

$$\operatorname{Re} p(z) > \beta, \quad z \in U. \quad (2.37)$$

□

Theorem 2.3. Let $\alpha, \mu, \beta, \lambda$ and $M_n(\alpha, \beta, \mu)$ be defined as in Theorem 2.2. If $f(z) \in H_n$ satisfies

$$\left| (1-\alpha) \left(\frac{z}{f(z)} \right)^\mu + \alpha \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\mu - 1 \right| < \lambda, \quad (2.38)$$

where $0 < \lambda \leq M_n(\alpha, \beta, \mu)$, then $f(z) \in S^*(\beta)$.

Proof. If $\mu = 0$, $M_n(\alpha, \beta, 0) = \alpha(1-\beta)$ and the result is trivial. Now, assume $\mu > 0$. If we put

$$q(z) = \left(\frac{z}{f(z)} \right)^\mu, \quad (2.39)$$

then by some transformations and (2.38) we get

$$q(z) - \frac{\alpha}{\mu} zq'(z) < 1 + \lambda z. \quad (2.40)$$

By Lemma 2.1, we obtain

$$q(z) < 1 + \frac{\mu\lambda}{n\alpha - \mu} z. \quad (2.41)$$

Let

$$p(z) = \frac{zf'(z)}{f(z)}. \quad (2.42)$$

Then we have

$$q(z)[1 - \alpha + \alpha p(z)] < 1 + \lambda z. \quad (2.43)$$

By Theorem 2.2, we get

$$\operatorname{Re} p(z) > \beta, \quad z \in U. \quad (2.44)$$

It follows that $f(z) \in S^*(\beta)$. □

For $\beta = 0$, we get the following corollary.

Corollary 2.4. *Let $0 \leq \mu < 1$, $n\alpha > \mu$, and let*

$$M_n(\alpha, \mu) = \begin{cases} \frac{\alpha(n\alpha - \mu)}{\alpha(n + \mu) - 2\mu}, & \text{if } \alpha \geq \alpha_2, \\ \frac{(n\alpha - \mu)\sqrt{2\alpha - 1}}{\sqrt{n^2\alpha^2 + 2[\mu^2 - n\mu]\alpha}}, & \text{if } \alpha_1 \leq \alpha < \alpha_2, \\ \frac{(n\alpha - \mu)}{n - \mu}, & \text{if } 0 < \alpha < \alpha_1, \end{cases} \quad (2.45)$$

where

$$\alpha_1 = \frac{n - \mu}{n}, \quad (2.46)$$

$$\alpha_2 = \frac{n + 3\mu + \sqrt{(n + 3\mu)^2 - 8n\mu}}{2n}.$$

If $f(z) \in H_n$ satisfies

$$\left| (1 - \alpha) \left(\frac{z}{f(z)} \right)^\mu + \alpha \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\mu - 1 \right| < \lambda, \quad (2.47)$$

where $0 < \lambda \leq M_n(\alpha, \mu)$, then $f(z) \in S^*$.

Corollary 2.5. *Let $0 \leq \mu < 1$, $0 \leq \beta < 1$, and let*

$$M_n(\beta, \mu) = \begin{cases} \frac{(n - \mu)(1 - \beta)}{n - \mu(1 + \beta)}, & \text{if } 1 > \beta \geq \frac{\mu}{n + \mu}, \\ \frac{(n - \mu)\sqrt{1 - 2\beta}}{\sqrt{n^2 + 2[\mu^2(1 - \beta) - n\mu]}}, & \text{if } 0 \leq \beta < \frac{\mu}{n + \mu}. \end{cases} \quad (2.48)$$

If $f(z) \in H_n$ satisfies

$$\left| \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\mu - 1 \right| < \lambda, \quad (2.49)$$

where $0 < \lambda \leq M_n(\beta, \mu)$, then $f(z) \in S^*(\beta)$.

Proof. Note that

$$\begin{aligned} \alpha_1 &= \frac{n - \mu(1 - \beta)}{n(1 - \beta)} \geq 1, \quad \text{for } \beta \geq \frac{\mu}{n + \mu}, \\ \alpha_1 &= \frac{n - \mu(1 - \beta)}{n(1 - \beta)} < 1, \quad \text{for } \beta < \frac{\mu}{n + \mu}, \\ \alpha_2 &= \frac{n + 3\mu(1 - \beta) + \sqrt{[n + 3\mu(1 - \beta)]^2 - 8n\mu(1 - \beta)}}{2n(1 - \beta)} \geq 1. \end{aligned} \quad (2.50)$$

Putting $\alpha = 1$ in Theorem 2.3, we obtain the above corollary. \square

Remark 2.6. Our results extend the results given by Obradović [5], and Obradović and Owa [6].

Theorem 2.7. Let $0 < \mu < 1$, $0 \leq \beta < 1$, $\operatorname{Re}\{c\} > -\mu$, and let

$$\beta_n(\beta, \mu) = \begin{cases} \frac{(n - \mu)(1 - \beta)|n + c - \mu|}{[n - \mu(1 - \beta)]|c - \mu|}, & \text{if } \beta \geq \frac{\mu}{n + \mu}, \\ \frac{(n - \mu)\sqrt{1 - 2\beta}|n + c - \mu|}{\sqrt{n^2 + 2[\mu^2(1 - \beta) - n\mu]}|c - \mu|}, & \text{if } \beta < \frac{\mu}{n + \mu}. \end{cases} \quad (2.51)$$

If $f(z) \in H_n$ satisfies

$$\left| \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\mu - 1 \right| < \lambda, \quad (2.52)$$

where $0 < \lambda \leq \beta_n(\beta, \mu)$, and

$$F(z) = z \left[\frac{c - \mu}{z^{c - \mu}} \int_0^z \left(\frac{t}{f(t)} \right)^\mu t^{c - \mu - 1} dt \right]^{-1/\mu}, \quad (2.53)$$

then $F(z) \in S^*(\beta)$.

Proof. Let

$$Q(z) = F'(z) \left(\frac{z}{F(z)} \right)^{1+\mu}. \quad (2.54)$$

Then from (2.52) and (2.53) we obtain

$$Q(z) + \frac{1}{c-\mu} Q'(z) = f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z. \quad (2.55)$$

Hence, by using Theorem 1 given by Hallenbeck and Ruscheweyh [7], we have that

$$Q(z) < 1 + \lambda_1 z, \quad \lambda_1 = \frac{|c-\mu|\lambda}{|n+c-\mu|} z, \quad (2.56)$$

and the desired result easily follows from Corollary 2.5. \square

For $c = \mu + 1$, we have the following corollary.

Corollary 2.8. *Let $0 < \mu < 1$, $0 \leq \beta < 1$, and let*

$$\beta_n(\beta, \mu) = \begin{cases} \frac{(n-\mu)(1-\beta)(n+1)}{[n-\mu(1-\beta)]}, & \text{if } 1 > \beta \geq \frac{\mu}{n+\mu}, \\ \frac{(n-\mu)\sqrt{1-2\beta}(n+1)}{\sqrt{n^2+2[\mu^2(1-\beta)-n\mu]}}, & \text{if } 0 \leq \beta < \frac{\mu}{n+\mu}. \end{cases} \quad (2.57)$$

If $f(z) \in H_n$ satisfies

$$\left| \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\mu - 1 \right| < \lambda, \quad (2.58)$$

where $0 < \lambda \leq \beta_n(\beta, \mu)$, and

$$F(z) = z \left[\frac{1}{z} \int_0^z \left(\frac{t}{f(t)} \right)^\mu dt \right]^{-1/\mu}, \quad (2.59)$$

then $F(z) \in S^*(\beta)$.

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