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Research Article

On the Stability of Generalized Quartic Mappings in Quasi-β-Normed Spaces

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We investigate the generalized Hyers-Ulam-Rassias stability problem in quasi-β-normed spaces and then the stability by using a subadditive function for the generalized quartic function $f : X \to Y$ such that $f(ax+by)+f(ax-by)-2a^2(a^2-b^2)f(x) = (ab)^2[f(x+y)+f(x-y)]-2b^2(a^2-b^2)f(y)$, where $a \neq 0$, $b \neq 0$, $a \pm b \neq 0$, for all $x, y \in X$.

1. Introduction

One of the interesting questions concerning the stability problems of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation? Such an idea was suggested in 1940 by Ulam [1] as follows. *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?* In other words, we are looking for situations when the homomorphisms are stable; that is, if a mapping is almost a homomorphism, then there exists a true homomorphism near it. In 1941, Hyers [2] considered the case of approximately additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. The famous Hyers stability result that appeared in [2] was generalized in the stability involving a sum of powers of norms by Aoki [3]. In 1978, Rassias [4] provided a generalization of Hyers Theorem which allows the Cauchy difference to be unbounded. During the last decades, stability problems of various functional equations have been extensively studied and generalized by a number of authors [5–10]. In particular, Rassias [11] introduced the quartic functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y). \tag{1.1}$$

It is easy to see that $f(x)=x^4$ is a solution of (1.1) by virtue of the identity

$$(x+2y)^4+(x-2y)^4+x^4=4(x+y)^4+4(x-y)^4+24y^4. \tag{1.2}$$

For this reason, (1.1) is called a quartic functional equation. Also Chung and Sahoo [12] determined the general solution of (1.1) without assuming any regularity conditions on the unknown function. In fact, they proved that the function $f:\mathbb{R}\to\mathbb{R}$ is a solution of (1.1) if and only if $f(x)=A(x,x,x,x)$, where the function $A:\mathbb{R}^4\to\mathbb{R}$ is symmetric and additive in each variable. Lee and Chung [13] introduced a quartic functional equation as follows:

$$f(ax+y)+f(ax-y)=a^2f(x+y)+a^2f(x-y)+2a^2\left(a^2-1\right)f(x)-2\left(a^2-1\right)f(y), \tag{1.3}$$

for fixed integer a with $a\neq 0,\pm 1$.

Let β be a real number with $0<\beta\leq 1$ and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We will consider the definition and some preliminary results of a quasi-β-norm on a linear space.

Definition 1.1. Let X be a linear space over a field $\mathbb{K}$. A *quasi-* β*-norm* $\|\cdot\|$ is a real-valued function on X satisfying the followings.

(1) $\|x\|\geq 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$.

(2) $\|\lambda x\|=|\lambda|^{\beta}\cdot\|x\|$ for all $\lambda\in\mathbb{K}$ and all $x\in X$.

(3) There is a constant $K\geq 1$ such that $\|x+y\|\leq K(\|x\|+\|y\|)$ for all $x,y\in X$.

The pair $(X,\|\cdot\|)$ is called a *quasi-* β*-normed space* if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-β-normed space.

A quasi-β-norm $\|\cdot\|$ is called a (β,p)-norm $(0<p\leq 1)$ if $\|x+y\|^p\leq\|x\|^p+\|y\|^p$, for all $x,y\in X$. In this case, a quasi-β-Banach space is called a (β,p)-Banach space; see [14–16].

In this paper, we consider the following the generalized quartic functional equation:

$$\begin{aligned} &f(ax+by)+f(ax-by)-2a^2\left(a^2-b^2\right)f(x)\\ &\quad=(ab)^2[f(x+y)+f(x-y)]-2b^2\left(a^2-b^2\right)f(y), \end{aligned} \tag{1.4}$$

for fixed integers a and b such that $a\neq 0,\ b\neq 0,\ a\pm b\neq 0$, for all $x,y\in X$. We investigate the generalized Hyers-Ulam-Rassias stability problem in quasi-β-normed spaces and then the stability by using a subadditive function for the generalized quartic function $f:X\to Y$ satisfying (1.4).

For the same reason as (1.1) and (1.2), we call (1.4) generalized quartic functional equation.

2. Quartic Functional Equations

Let X, Y be real vector spaces. In this section, we will investigate that the functional equation (1.1) is equivalent to the presented functional equation (1.4).

Lemma 2.1. *A mapping $f : X \to Y$ satisfies the functional equation* (1.1) *if and only if f satisfies*

$$f(x+ay)+f(x-ay)+2\left(a^2-1\right)f(x)=a^2[f(x+y)+f(x-y)]+2a^2\left(a^2-1\right)f(y), \quad (2.1)$$

where $a \neq 0, a \neq \pm 1$, for all $x, y \in X$.

Proof. We will show it by induction on a. Assume that it holds for all less than equal a. Now, letting x be $x+y$ in (2.1),

$$\begin{aligned} &f(x+(a+1)y)+f(x-(a-1)y)+2\left(a^2-1\right)f(x+y) \\ &\quad = a^2[f(x+2y)+f(x)]+2a^2\left(a^2-1\right)f(y), \end{aligned} \quad (2.2)$$

and also replacing x by $x-y$ in (2.1),

$$\begin{aligned} &f(x+(a-1)y)+f(x-(a+1)y)+2\left(a^2-1\right)f(x-y) \\ &\quad = a^2[f(x)+f(x-2y)]+2a^2\left(a^2-1\right)f(y), \end{aligned} \quad (2.3)$$

for all $x, y \in X$. Adding (2.2) and (2.3), we have

$$\begin{aligned} &f(x+(a+1)y)+f(x-(a+1)y)+f(x+(a-1)y)+f(x-(a-1)y) \\ &\quad +2\left(a^2-1\right)[f(x+y)+f(x-y)] \\ &= a^2[f(x+2y)+f(x-2y)]+2a^2f(x)+4a^2\left(a^2-1\right)f(y), \end{aligned} \quad (2.4)$$

for all $x, y \in X$. By induction steps, we have

$$\begin{aligned} &f(x+(a+1)y)+f(x-(a+1)y)-2\left((a-1)^2-1\right)f(x) \\ &\quad +(a-1)^2[f(x+y)+f(x-y)]+2(a-1)^2\left((a-1)^2-1\right)f(y) \\ &\quad +2\left(a^2-1\right)^2[f(x+y)+f(x-y)] \\ &= a^2[-6f(x)+4[f(x+y)+f(x-y)]+24f(y)] \\ &\quad +2a^2f(x)+4a^2\left(a^2-1\right)f(y). \end{aligned} \quad (2.5)$$

Hence we have

$$f(x+(a+1)y)+f(x-(a+1)y)+2\left((a+1)^2-1\right)f(x) = (a+1)^2[f(x+y)+f(x-y)]+2(a+1)^2\left((a+1)^2-1\right)f(y), \tag{2.6}$$

for all $x, y \in X$. Thus they are equivalent. □

Theorem 2.2. *If a mapping $f : X \to Y$ satisfies the functional equation* (1.4)*, then f satisfies the functional equation* (2.1)*.*

Proof. By letting $x = y = 0$ in (2.1), we have $2a^2(a^2-1)f(0) = 0$. Since $a \neq 0$ and $a \neq \pm 1$, $f(0) = 0$. Putting $x = 0$ in (2.1),

$$f(ay)+f(-ay) = a^2[f(y)+f(-y)]+2a^2\left(a^2-1\right)f(y). \tag{2.7}$$

Now, replacing y by $-y$ in (2.7),

$$f(ay)+f(-ay) = a^2[f(y)+f(-y)]+2a^2\left(a^2-1\right)f(-y). \tag{2.8}$$

By (2.7) and (2.8), we have $2a^2(a^2-1)f(y) = 2a^2(a^2-1)f(-y)$, that is, $f(y) = f(-y)$. Hence f is even. This implies that $2f(ay) = 2a^2f(y)+2a^2(a^2-1)f(y)$, that is, $f(ay) = a^4f(y)$, for all $y \in X$. Now, we will show that (2.1) implies (1.4). By letting $x = bx$ in (2.1), we have

$$f(bx+ay)+f(bx-ay)+2\left(a^2-1\right)f(bx) = a^2[f(bx+y)+f(bx-y)]+2a^2\left(a^2-1\right)f(y). \tag{2.9}$$

Switching x and y in the previous equation,

$$f(ax+by)+f(ax-by)+2\left(a^2-1\right)f(by) = a^2[f(x+by)+f(x-by)]+2a^2\left(a^2-1\right)f(x). \tag{2.10}$$

By (2.1) with b, the previous equation implies that

$$\begin{aligned} f(ax+by)+f(ax-by)+2b^4\left(a^2-1\right)f(y) &= a^2b^2[f(x+y)+f(x-y)]+2a^2b^2\left(b^2-1\right)f(y) \\ &\quad -2a^2\left(b^2-1\right)f(x)+2a^2\left(a^2-1\right)f(x). \end{aligned} \tag{2.11}$$

Hence we have

$$f(ax+by)+f(ax-by)-2a^2\left(a^2-b^2\right)f(x) \\ = (ab)^2[f(x+y)+f(x-y)]-2b^2\left(a^2-b^2\right)f(y), \tag{2.12}$$

for all $x, y \in X$. □

Corollary 2.3. *If a mapping $f : X \to Y$ satisfies the functional equation* (1.1)*, then f satisfies the functional equation* (1.4)*.*

3. Stabilities

Throughout this section, let X be a quasi-β-normed space and let Y be a quasi-β-Banach space with a quasi-β-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$. We will investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.4). After then we will study the stability by using a subadditive function. For a given mapping $f : X \to Y$ and all fixed integers a and b with $a \neq 0$, $a \neq 0$, $a \pm b \neq 0$, let

$$Df(x,y) := f(ax+by)+f(ax-by)-2a^2\left(a^2-b^2\right)f(x) \\ +2b^2\left(a^2-b^2\right)f(y)-(ab)^2[f(x+y)+f(x-y)], \quad x, y \in X. \tag{3.1}$$

Theorem 3.1. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$\|Df(x,y)\|_Y \le \phi(x,y), \tag{3.2}$$

and the series $\sum_{j=0}^{\infty}(K/a^{4\beta})^j\phi(a^jx, a^jy)$ converges for all $x, y \in X$. Then there exists a unique generalized quartic mapping $Q : X \to Y$ which satisfies (1.4) *and the inequality*

$$\|f(x)-Q(x)\|_Y \le \frac{K}{2^\beta a^{4\beta}}\sum_{j=0}^{\infty}\left(\frac{K}{a^{4\beta}}\right)^j\phi\left(a^jx,0\right), \tag{3.3}$$

for all $x \in X$.

Proof. By letting $y = 0$ in the inequality (3.2), since $f(0) = 0$, we have

$$\|Df(x,0)\|_Y = \left\|2f(ax)-2a^2(a^2-b^2)f(x)-2(ab)^2f(x)\right\|_Y \\ = \left\|2f(ax)-2a^4f(x)\right\|_Y = \left(2a^4\right)^\beta\left\|f(x)-\frac{1}{a^4}f(ax)\right\|_Y \le \phi(x,0), \tag{3.4}$$

that is,

$$\left\| f(x) - \frac{1}{a^4} f(ax) \right\|_Y \le \frac{1}{2^\beta a^{4\beta}} \phi(x,0), \tag{3.5}$$

for all $x \in X$. Now, putting $x = ax$ and multiplying $1/a^{4\beta}$ in the inequality (3.5), we get

$$\frac{1}{a^{4\beta}} \left\| f(ax) - \frac{1}{a^4} f(a^2 x) \right\|_Y \le \frac{1}{2^\beta} \left(\frac{1}{a^{4\beta}} \right)^2 \phi(ax,0), \tag{3.6}$$

for all $x \in X$. Combining (3.5) and (3.6), we have

$$\left\| f(x) - \left(\frac{1}{a^4} \right)^2 f(a^2 x) \right\|_Y \le \frac{K}{2^\beta a^{4\beta}} \left[\phi(x,0) + \frac{1}{a^{4\beta}} \phi(ax,0) \right], \tag{3.7}$$

for all $x \in X$. Inductively, since $K \ge 1$, we have

$$\left\| f(x) - (\frac{1}{a^4})^s f(a^s x) \right\|_Y \le \frac{K}{2^\beta a^{4\beta}} \sum_{j=0}^{s-1} \left(\frac{K}{a^{4\beta}} \right)^j \phi\left(a^j x, 0 \right), \tag{3.8}$$

for all $x \in X$, $s \in \mathbb{N}$. For all s and d with $s < d$ and switching x and $a^s x$ and multiplying $(1/a^{4\beta})^s$ in the inequality (3.5), inductively,

$$\left\| \left(\frac{1}{a^4} \right)^s f(a^s x) - \left(\frac{1}{a^4} \right)^d f(a^d x) \right\|_Y \le \frac{K}{2^\beta a^{4\beta}} \sum_{j=s}^{d-1} \left(\frac{K}{a^{4\beta}} \right)^j \phi\left(a^j x, 0 \right), \tag{3.9}$$

for all $x \in X$. Since the right-hand side of the previous inequality tends to 0 as $d \to \infty$, hence $\{(1/a^4)^s f(a^s x)\}$ is a Cauchy sequence in the quasi-β-Banach space Y. Thus we may define

$$Q(x) = \lim_{s \to \infty} \left(\frac{1}{a^4} \right)^s f(a^s x), \tag{3.10}$$

for all $x \in X$. Since $K \ge 1$, replacing x and y by $a^s x$ and $a^s y$, respectively, and dividing by $a^{4\beta s}$ in the inequality (3.2), we have

$$\begin{aligned} &\left(\frac{1}{a^{4\beta}} \right)^s \| Df(a^s x, a^s y) \|_Y \\ &= \left(\frac{1}{a^{4\beta}} \right)^s \Big\| \left(a^s (ax + by) \right) + f\left(a^s (ax - by) \right) - 2a^2 \left(a^2 - b^2 \right) f(a^s x) \\ &\qquad + 2b^2 (a^2 - b^2) f(a^s y) - (ab)^2 [f(a^s(x+y)) - f(a^s(x-y))] \Big\|_Y \\ &\le \left(\frac{K}{a^{4\beta}} \right)^s \phi(a^s x, a^s y), \end{aligned} \tag{3.11}$$

for all $x, y \in X$. By taking $s \to \infty$, the definition of Q implies that Q satisfies (1.4) for all $x, y \in X$; that is, Q is the generalized quartic mapping. Also, the inequality (3.8) implies the inequality (3.3). Now, it remains to show the uniqueness. Assume that there exists $T : X \to Y$ satisfying (1.4) and (3.3). It is easy to show that for all $x \in X, T(a^s x) = a^{4s}T(x)$ and $Q(a^s x) = a^{4s}Q(x)$, as in the proof of Theorem 2.2. Then

$$\begin{aligned}\|T(x) - Q(x)\|_Y &= \left(\frac{1}{a^{4\beta}}\right)^s \|T(a^s x) - Q(a^s x)\|_Y \\ &\le \left(\frac{1}{a^{4\beta}}\right)^s K\left(\left\|T(a^s x) - f(a^s x)\right\|_Y + \left\|f(a^s x) - Q(a^s x)\right\|_Y\right) \\ &\le \frac{2K^2}{2^\beta a^{4\beta}} \sum_{j=0}^{\infty} \left(\frac{K}{a^{4\beta}}\right)^{s+j} \phi\left(a^{s+j}x, 0\right),\end{aligned} \tag{3.12}$$

for all $x \in X$. By letting $s \to \infty$, we immediately have the uniqueness of Q. □

Theorem 3.2. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0, \infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$\left\|Df(x, y)\right\|_Y \le \phi(x, y), \tag{3.13}$$

and the series $\sum_{j=1}^{\infty} (a^{4\beta}K)^j \phi(a^{-j}x, a^{-j}y)$ converges for all $x, y \in X$. Then there exists a unique generalized quartic mapping $Q : X \to Y$ which satisfies (2.1) *and the inequality*

$$\left\|f(x) - Q(x)\right\|_Y \le \frac{1}{2^\beta a^{4\beta}} \sum_{j=1}^{\infty} \left(a^{4\beta}K\right)^j \phi\left(a^{-j}x, 0\right), \tag{3.14}$$

for all $x \in X$.

Proof. If x is replaced by $(1/a)x$ in the inequality (3.5), then the proof follows from the proof of Theorem 3.1. □

Now we will recall a subadditive function and then investigate the stability under the condition that the space Y is a (β, p)-Banach space. The basic definitions of subadditive functions follow from [16].

A function $\phi : A \to B$ having a domain A and a codomain $(B, \le)$ that are both closed under addition is called

(1) *a subadditive function* if $\phi(x + y) \le \phi(x) + \phi(y)$,

(2) *a contractively subadditive function* if there exists a constant L with $0 < L < 1$ such that $\phi(x + y) \le L(\phi(x) + \phi(y))$,

(3) *an expansively superadditive function* if there exists a constant L with $0 < L < 1$ such that $\phi(x + y) \ge (1/L)(\phi(x) + \phi(y))$,

for all $x, y \in A$.

Theorem 3.3. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$\|Df(x,y)\|_Y \leq \phi(x,y), \tag{3.15}$$

for all $x, y \in X$ and the map ϕ is contractively subadditive with a constant L such that $a^{1-4\beta}L < 1$. Then there exists a unique generalized quartic mapping $Q : X \to Y$ which satisfies (1.4) and the inequality

$$\|f(x) - Q(x)\|_Y \leq \frac{\phi(x,0)}{2^{\beta}\sqrt[p]{a^{4\beta p} - (aL)^p}}, \tag{3.16}$$

for all $x \in X$.

Proof. By the inequalities (3.5) and (3.9) of the proof of Theorem 3.1, we have

$$\begin{aligned}
\left\|\frac{1}{a^{4s}}f(a^s x) - \frac{1}{a^{4d}}f(a^d x)\right\|_Y^p &\leq \sum_{j=s}^{d-1}\left(\frac{1}{a^{4\beta}}\right)^{jp}\left\|f(a^j x) - \frac{1}{a^4}f(a^{j+1}x)\right\|_Y^p \\
&\leq \frac{1}{2^{\beta p}a^{4\beta p}}\sum_{j=s}^{d-1}\left(\frac{1}{a^{4\beta}}\right)^{jp}\phi\left(a^j x, 0\right)^p \\
&\leq \frac{1}{2^{\beta p}a^{4\beta p}}\sum_{j=s}^{d-1}\left(\frac{1}{a^{4\beta}}\right)^{jp}(aL)^{jp}\phi(x,0)^p \\
&= \frac{\phi(x,0)^p}{2^{\beta p}a^{4\beta p}}\sum_{j=s}^{d-1}\left(a^{1-4\beta}L\right)^{jp},
\end{aligned} \tag{3.17}$$

that is,

$$\left\|\left(\frac{1}{a^4}\right)^s f(a^s x) - \left(\frac{1}{a^4}\right)^d f(a^d x)\right\|_Y^p \leq \frac{\phi(x,0)^p}{2^{\beta p}a^{4\beta p}}\sum_{j=s}^{d-1}\left(a^{1-4\beta}L\right)^{jp}, \tag{3.18}$$

for all $x \in X$, and for all s and d with $s < d$. Hence $\{(1/a^{4s})f(a^s x)\}$ is a Cauchy sequence in the space Y. Thus we may define

$$Q(x) = \lim_{s\to\infty}\frac{1}{a^{4s}}f(a^s x), \tag{3.19}$$

for all $x \in X$. Now, we will show that the map $Q : X \to Y$ is a generalized quartic mapping. Then

$$\begin{aligned}\|DQ(x,y)\|_Y^p &= \lim_{s\to\infty} \frac{\|Df(a^s x, a^s y)\|_Y^p}{a^{4\beta ps}} \\ &\le \lim_{s\to\infty} \frac{\phi(a^s x, a^s y)^p}{a^{4\beta ps}} \\ &\le \lim_{s\to\infty} \phi(x,y)^p \left(a^{1-4\beta}L\right)^{ps} = 0,\end{aligned} \tag{3.20}$$

for all $x \in X$. Hence the mapping Q is a generalized quartic mapping. Note that the inequality (3.18) implies the inequality (3.16) by letting $s = 0$ and taking $d \to \infty$. Assume that there exists $T : X \to Y$ satisfying (1.4) and (3.16). We know that $T(a^s x) = a^{4s}T(x)$, for all $x \in X$. Then

$$\begin{aligned}\left\|T(x) - \left(\frac{1}{a^4}\right)^s f(a^s x)\right\|_Y^p &= \left(\frac{1}{a^{4\beta}}\right)^{ps} \|T(a^s x) - f(a^s x)\|_Y^p \\ &\le \left(\frac{1}{a^{4\beta}}\right)^{ps} \frac{\phi(a^s x, 0)^p}{2^{\beta p}\left(a^{4\beta p} - (aL)^p\right)} \\ &\le \left(a^{1-4\beta}L\right)^{ps} \frac{\phi(x,0)^p}{2^{\beta p}\left(a^{4\beta p} - (aL)^p\right)},\end{aligned} \tag{3.21}$$

that is,

$$\left\|T(x) - \left(\frac{1}{a^4}\right)^s f(a^s x)\right\|_Y \le \left(a^{1-4\beta}L\right)^s \frac{\phi(x,0)}{2^{\beta} \sqrt[p]{\left(a^{4\beta p} - (aL)^p\right)}}, \tag{3.22}$$

for all $x \in X$. By letting $s \to \infty$, we immediately have the uniqueness of Q. □

Theorem 3.4. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$\|Df(x,y)\|_Y \le \phi(x,y), \tag{3.23}$$

for all $x, y \in X$ and the map ϕ is expansively superadditive with a constant L such that $a^{4\beta-1}L < 1$. Then there exists a unique generalized quartic mapping $Q : X \to Y$ which satisfies (1.4) *and the inequality*

$$\|f(x) - Q(x)\|_Y \le \frac{\phi(x,0)}{2^{\beta}L \sqrt[p]{a^p - (a^{4\beta}L)^p}}, \tag{3.24}$$

for all $x \in X$.

Proof. By letting $y = 0$ in (3.23), we have

$$\left\|2f(ax) - 2a^4 f(x)\right\|_Y \leq \phi(x, 0), \tag{3.25}$$

and then replacing x by x/a,

$$\left\|f(x) - a^4 f\left(\frac{x}{a}\right)\right\|_Y \leq \frac{1}{2^\beta}\phi\left(\frac{x}{a}, 0\right), \tag{3.26}$$

for all $x \in X$. For all s and d with $s < d$, inductively we have

$$\left\|a^{4s} f\left(\frac{x}{a^s}\right) - a^{4d} f\left(\frac{x}{a^d}\right)\right\|_Y^p \leq \frac{\phi(x, 0)^p}{2^{\beta p}(aL)^p}\sum_{j=s}^{d-1}\left(a^{4\beta-1}L\right)^{jp}, \tag{3.27}$$

for all $x \in X$. The remains follow from the proof of Theorem 3.3. □

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