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Research Article

Slow Growth for Universal Harmonic Functions

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Given any continuous increasing function $\phi : [0,+\infty[\to]0,+\infty[$ such that $\lim_{t\to\infty} \log\phi(t)/\log t = +\infty$, we show that there are harmonic functions H on $\mathbb{R}^N$ satisfying the inequality $|H(x)| \le \phi(\|x\|)$ for every $x \in \mathbb{R}^N$, which are universal with respect to translations. This answers positively a problem of D. H. Armitage (2005). The proof combines techniques of Dynamical Systems and Operator Theory, and it does not need any result from Harmonic Analysis.

1. Introduction

A classic result of Birkhoff from 1929 [1] says that there are entire functions f whose translates approximate any other entire function as accurately as we want on an arbitrary compact set of the complex plane. More precisely, given any entire function g, there exists an increasing sequence $(n_k)_k$ of integers such that $f(z+n_k) \to g(z)$, uniformly on compact sets of $\mathbb{C}$. Then f is called a *universal* entire function (with respect to translation). In terms of Dynamical Systems, the (continuous and linear) operator $T_1 : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C})$, $g(z) \mapsto g(z+1)$ admits elements f whose orbit $\mathrm{orb}(T_1, f) := \{f, T_1 f, T_1^2 f, \dots\}$ is dense in $\mathscr{H}(\mathbb{C})$. Within the framework of Operator Theory, T_1 is called a *hypercyclic* operator, and f is a hypercyclic vector for T_1. The same phenomenon happens for any translation operator T_a with respect to a nonzero $a \in \mathbb{C}$.

A harmonic function H on $\mathbb{R}^N$ is said to be *universal* with respect to translations if the set of translates $\{H(\cdot + a); a \in \mathbb{R}^N\}$ is dense in the space of all harmonic functions on $\mathbb{R}^N$ with the topology of local uniform convergence, that is, the topology of uniform convergence on compact subsets of $\mathbb{R}^N$, also called compact-open topology. Dzagnidze [2] showed that there are universal harmonic functions on $\mathbb{R}^N$ (see also the recent paper [3] where it is shown that there are harmonic functions which are universal in a stronger sense, which depends on how "frequently" certain orbits visit any neighbourhood). One may think that the growth of a universal function cannot be arbitrarily "controlled" from above by certain prescribed growth. This is certainly the case for polynomial growth since polynomial growth (either in the entire or in the harmonic case) implies that the function is a polynomial and, obviously, the translation of a polynomial is another polynomial of the same degree. But it came as a surprise that if we fix any transcendental growth, one can find universal entire functions that grow more slowly [4, 5]. In the harmonic case, Armitage [6] showed that universal harmonic functions can also have slow growth. More precisely, given any $\phi : [0,+\infty[\to]0,+\infty[$, a continuous increasing function such that

$$\lim_{t\to\infty}\frac{\log\phi(t)}{(\log t)^2}=+\infty, \tag{1.1}$$

then there is a universal harmonic function H on $\mathbb{R}^N$ that satisfies $|H(x)|\le\phi(\|x\|)$ for all $x\in\mathbb{R}^N$. Armitage [6] asked whether the condition (1.1) can be relaxed: is it true that, for every $N>2$, one can find universal harmonic functions on $\mathbb{R}^N$ with arbitrarily slow transcendental growth? In other words, can the exponent 2 of $\log t$ be reduced to 1? We answer positively this question. It is easy to notice that $\lim_{t\to\infty}\log\phi(t)/\log t=+\infty$ is equivalent to saying that $\lim_{t\to\infty}\phi(t)/t^k=+\infty$ for all $k\in\mathbb{N}$, and no universal harmonic function H can satisfy that $\limsup_{x\to\infty}|H(x)|/\|x\|^k<+\infty$ for any $k\in\mathbb{N}$, since this would force H to be a polynomial. This is why we can only consider transcendental growths for universal harmonic functions. Let us recall that the case $N=2$ can be obtained as a consequence of the result in [4, 5] (as was noticed in [6]) since, for example, the real part of a universal entire function is a universal harmonic function on $\mathbb{R}^2$.

We will recall some notions. For the basic theory of harmonic functions we refer the reader to the books [7, 8]. For a good source on the theory of hypercyclic operators and related properties, we refer the reader to [9–12].

A continuous function $T : X\to X$ on a metric space X is *topologically transitive* (resp., *mixing*) if, for any pair $U, V\subset X$ of nonempty open sets, there is $n\in\mathbb{N}$ (resp., $n_0\in\mathbb{N}$) such that $T^n(U)\cap V\neq\emptyset$ (resp., for every $n\ge n_0$). We will work with (continuous and linear) operators $T : X\to X$ on separable, metric, and complete topological vector spaces X. In our framework it is well known that topological transitivity is equivalent to hypercyclicity.

2. Universal Harmonic Functions of Slow Growth and Strong Approximation of Polynomials

Given a transcendental growth determined by $\phi : [0,+\infty[\to]0,+\infty[$, to find universal harmonic functions H whose growth is bounded by ϕ (i.e., $|H(x)|\le\phi(\|x\|)$ for all $x\in\mathbb{R}^N$), it suffices to find a Banach space X consisting of harmonic functions, whose topology is finer than the topology of uniform convergence on compact sets of $\mathbb{R}^N$, containing the harmonic polynomials, and such that, on the one hand, the translation operator $T_a : X\to X$, $G\mapsto G(\cdot+a)$, is (well-defined, continuous and) hypercyclic for every $a\in\mathbb{R}^N\setminus\{0\}$, and on the other hand every harmonic function in the unit ball of X has a growth bounded by ϕ. Indeed, in this case, there are functions H in the unit ball of X whose orbit under T_a is dense in X, therefore every harmonic polynomial can be approximated in the topology of X by a (multiple of a fixed $a\neq 0$) translation of H, and we obtain that the harmonic functions can be approximated, uniformly on compact sets of $\mathbb{R}^N$, by translations of H.

Theorem 2.1. *Let $\phi : [0, +\infty[\to]0, +\infty[$ be a continuous increasing function such that*

$$\lim_{t\to\infty} \frac{\log \phi(t)}{\log t} = +\infty. \tag{2.1}$$

Then there is a Banach space X of harmonic functions, whose topology is finer than the topology of uniform convergence on compact sets of $\mathbb{R}^N$, containing the harmonic polynomials, such that the translation operator $T_a : X \to X$ is a (well-defined and bounded) mixing operator, for any $a \in \mathbb{R}^N \setminus \{0\}$, and every $H \in X$ with $\|H\| < 1$ satisfies $|H(x)| \le \phi(\|x\|)$ for all $x \in \mathbb{R}^N$.

In particular, there are universal harmonic functions of arbitrarily slow transcendental growth.

Proof. Let us first consider the case $N \ge 3$. We fix $a \in \mathbb{R}^N \setminus \{0\}$, and by D_a we denote the corresponding directional derivative operator ($(D_aG)(x) := \lim_{s\to 0}(G(x+sa) - G(x))/s$, $x \in \mathbb{R}^N$). Since D_a commutes with the Laplacian, one can easily construct a basis $\{p_{n,m}\}_{n,m\in\mathbb{N}}$ of the harmonic polynomials such that each $p_{n,m}$ is $k(n,m)$-homogeneous with $k(n,m) \le n+m$, $D_a p_{1,m} = 0$, and $D_a p_{n+1,m} = p_{n,m}$ for every $n, m \in \mathbb{N}$. To illustrate a particular case, for instance, let $N = 3$ and $a = e_1 = (1, 0, \dots, 0)$. A construction of a basis of harmonic polynomials with the above conditions is

$$\begin{array}{lllll}
p_{1,1} = 1, & p_{2,1} = x_1, & p_{3,1} = \dfrac{x_1^2 - x_2^2}{2}, & p_{4,1} = \dfrac{x_1^3 - 3x_1x_2^2}{6}, & \cdots \\
p_{1,2} = x_2, & p_{2,2} = x_1x_2, & p_{3,2} = \dfrac{3x_1^2x_2 - x_2^3}{6}, & \cdots & \cdots \\
p_{1,3} = x_3, & p_{2,3} = x_1x_3, & p_{3,3} = \dfrac{3x_1^2x_3 - x_3^3}{6}, & \cdots & \cdots \\
p_{1,4} = x_2x_3, & p_{2,4} = x_1x_2x_3, & \cdots & \cdots & \cdots \\
p_{1,5} = x_2^2 - x_3^2, & p_{2,5} = x_1\left(x_2^2 - x_3^2\right), & \cdots & \cdots & \cdots \\
p_{1,6} = x_2^2x_3 - x_3^2x_2, & \cdots & \cdots & \cdots & \cdots \\
p_{1,7} = x_2^3 - 3x_3^2x_2, & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array} \tag{2.2}$$

Note that the polynomials constructed above are all homogeneous, and we wrote the part of the basis until degree 3.

We consider a sequence of positive weights $v := \{v_{n,m}\}_{n,m\in\mathbb{N}}$ such that $v_{n,m} > \sup\{|p_{n,m}(x)|; \|x\| \le n+m\}$ and $v_{n+1,m} > v_{n,m}$, $n, m \in \mathbb{N}$. Let us define the space of harmonic functions of growth controlled by v as

$$X_v = \left\{ H(x) = \sum_m \sum_n \alpha_{n,m} p_{n,m}(x); \|H\| := \sum_m \sum_n |\alpha_{n,m}| v_{n,m} < \infty \right\}. \tag{2.3}$$

The space X_v is continuously included in the space of all harmonic functions on $\mathbb{R}^N$ under the compact-open topology. Indeed, if $H = \sum_m \sum_n \alpha_{n,m} p_{n,m}$, $G = \sum_m \sum_n \beta_{n,m} p_{n,m} \in X_v$, $R > 1$, and $\|x\| \le R$, then

$$\begin{aligned} |H(x) - G(x)| &\le \sum_{n+m<R} \left|\alpha_{n,m} - \beta_{n,m}\right| \left|p_{n,m}(x)\right| + \sum_{n+m\ge R} \left|\alpha_{n,m} - \beta_{n,m}\right| \\ &\le \sum_{n+m<R} \left|\alpha_{n,m} - \beta_{n,m}\right| R^{k(n,m)} \left| p_{n,m}\left(\frac{1}{R}x\right)\right| + \sum_{n+m\ge R} \left|\alpha_{n,m} - \beta_{n,m}\right| v_{n,m} \\ &\le R^R \|H - G\|. \end{aligned} \tag{2.4}$$

It is clear that if we select the sequence v increasing fast enough, we have that every $H \in X := X_v$ with $\|H\| < 1$ satisfies that $|H(x)| \le \phi(\|x\|)$ for all $x \in \mathbb{R}^N$. To prove it, let $\delta := \phi(0) = \inf_{t\in[0,+\infty[} \phi(t) > 0$. Our hypothesis on ϕ implies that

$$\lim_{x\to\infty} \frac{\phi(\|x\|)}{\left|p_{n,m}(x)\right|} = +\infty, \quad n, m \in \mathbb{N}, \tag{2.5}$$

because each $p_{n,m}$ is a polynomial. Let $R_{n,m} > 0$ so that $\phi(\|x\|) > 2^{n+m}|p_{n,m}(x)|$ if $\|x\| > R_{n,m}$, $n, m \in \mathbb{N}$. We suppose that

$$v_{n,m} > 2 + \frac{2}{\delta} \sup\left\{ \left|p_{n,m}(x)\right| ; \|x\| \le R_{n,m} \right\}, \quad n, m \in \mathbb{N}. \tag{2.6}$$

Given any $x \in \mathbb{R}^N$ and $H = \sum_m \sum_n \alpha_{n,m} p_{n,m} \in X_v$ with $\|H\| < 1$, we have

$$\begin{aligned} |H(x)| &\le \sum_{R_{n,m}<\|x\|} |\alpha_{n,m}| \left|p_{n,m}(x)\right| + \sum_{\|x\|\le R_{n,m}} |\alpha_{n,m}| \frac{\delta}{2} v_{n,m} \\ &\le \sum_{R_{n,m}<\|x\|} |\alpha_{n,m}| \frac{\phi(\|x\|)}{2^{n+m}} + \frac{\delta}{2} < \phi(\|x\|). \end{aligned} \tag{2.7}$$

We have that $D_a : X_v \to X_v$ is a bounded operator which acts as

$$D_a f = D_a\left(\sum_m \sum_n \alpha_{n,m} p_{n,m} \right) = \sum_m \sum_{n\ge 2} \alpha_{n,m} p_{n-1,m}, \tag{2.8}$$

since $v_{n+1,m} > v_{n,m}$, $n, m \in \mathbb{N}$.

The space X_v is naturally isomorphic to $Y_v := \bigoplus^{\ell^1} \ell^1(v_m)$, where $v_m := (v_{n,m})_n$ and $\ell^1(v_m) := \{z = (z_n)_n ; \|z\| := \sum_n |z_n| v_{n,m} < \infty\}$, $m \in \mathbb{N}$. Via the natural isomorphism, the operator D_a is then conjugated to the following operator on Y_v which is the ℓ^1-sum of the backward shift B:

$$L(x_m)_m = (Bx_m)_m, \quad \text{where } Bx_m = B(x_{n,m})_n = (x_{n+1,m})_{n\ge 1}. \tag{2.9}$$

We fix the notation $L = \bigoplus^{\ell^1} B$. Since the translation operator T_a equals e^{D_a}, we thus have that T_a is conjugated to $T = e^L = \bigoplus^{\ell^1} e^B$. Then we are done if we prove that e^L is a mixing operator on Y_v.

Given arbitrary nonempty open sets $U, V \subset Y_v$, we fix $m \in \mathbb{N}$ and pairs $U_k, V_k \subset \ell^1(v_k)$ of nonempty open sets in the corresponding spaces, $k = 1, \ldots, m$, such that

$$\bigoplus_{k=1}^{m} U_k \subset U, \qquad \bigoplus_{k=1}^{m} V_k \subset V. \tag{2.10}$$

The operator e^B is mixing on any weighted ℓ^1-space, as was shown in the proof of Theorem 5.2 of [13]; therefore we find $j_0 \in \mathbb{N}$ such that $e^{jB}(U_k) \cap V_k \neq \emptyset$, $k = 1, \ldots, m$, for every $j \geq j_0$. We finally conclude that

$$T^j(U) \cap V \supset \bigoplus_{k=1}^{m} \left(e^{jB}(U_k) \cap V_k \right) \neq \emptyset, \tag{2.11}$$

and T_a, being conjugated to T, is a mixing operator.

The case $N = 2$ is even simpler since the basis of harmonic polynomials can be taken as $\{p_{n,m}\}_{n\in\mathbb{N}, m=1,2}$ such that $D_a p_{1,m} = 0$ and $D_a p_{n+1,m} = p_{n,m}$ for every $n \in \mathbb{N}$, $m = 1, 2$. The corresponding space Y_v is the direct sum of two weighted ℓ^1-spaces, and a simplification of the above argument does the job. □

Remark 2.2. What we actually showed in the proof of Theorem 2.1 is that there are $H \in X$ with $|H(x)| \leq \phi(\|x\|)$ for all $x \in \mathbb{R}^N$ such that, for any harmonic polynomial P, there is an increasing sequence $(n_k)_k$ of integers such that $H(x + n_k a) \to P(x)$ in the topology of X_v, which is finer than the topology of uniform convergence on compact sets of $\mathbb{R}^N$. Moreover, the function H is universal with respect to translations in the direction of a, a result also stronger than Armitage's in the sense that in [6] all translations were allowed for a universal function.

3. Final Comments

It is possible to obtain the main result of the paper by using tensor product techniques developed in [14]. However the proofs seem to be technically much more complicated since one has to represent the space of harmonic functions on $\mathbb{R}^N$ as a quotient of a direct sum of the complete tensor product of spaces of harmonic functions with fewer variables, while we produced here a, somehow, direct proof. We want to thank Antonio Bonilla for pointing out to us the possibility of a tensor product approach.

When we fix a nonzero $a \in \mathbb{R}^N$, one can consider the parametric family of translation operators $\{T_{ta}\}_{t\geq 0}$ on our space X_v. This is a C_0-semigroup of operators, which we showed to be hypercyclic on X_v. We refer to [12, 13] for the basic theory of hypercyclic C_0-semigroups. In [15] we proved that every hypercyclic vector of a C_0-semigroup is shared by all operators of the semigroup (except, obviously, $T_0 = I$). In particular, if $H \in X_v$ is a universal harmonic function for T_a, then it is also universal (or a hypercyclic vector) for T_{ta} for all $t > 0$. Let us notice that the inheritance of hypercyclicity by discrete subsemigroups depends strongly on the fact that the index set is $\mathbb{R}^+$ or $\mathbb{R}$: there are hypercyclic C_0-semigroups of operators whose index set is a sector of the complex plane and such that no discrete subsemigroup is hypercyclic [16].

The operator T_a can be considered as a (infinite type) differential operator since $T_a = e^{D_a}$. In [17] we develop techniques based on a kind of generalized backward shifts which allow us to extend the main results in this paper and [5] to other differential operators, and then we show the existence of harmonic and entire functions of arbitrarily slow transcendental growth which are universal with respect to some differential operators.

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