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Research Article

On the Stability of a General Mixed Additive-Cubic Functional Equation in Random Normed Spaces

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We prove the generalized Hyers-Ulam stability of the following additive-cubic equation $f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2f(kx)-2kf(x)$ in the setting of random normed spaces.

1. Introduction

A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation?

If the problem accepts a unique solution, we say the equation is stable (see [1]). The first stability problem concerning group homomorphisms was raised by Ulam [2] in 1940 and affirmatively solved by Hyers [3]. The result of Hyers was generalized by Rassias [4] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x,y)=f(x+y)-[f(x)+f(y)]$ to be controlled by $\epsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [5], who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$ in the spirit of Th. M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see, e.g., [6–12] and references therein). In addition, J. M. Rassias et al. [13–16] generalized the Hyers stability result by introducing two weaker conditions controlled by the Ulam-Gavruta-Rassias (or UGR) product of different powers of norms and the JM Rassias (or JMR) mixed product-sum of powers of norms, respectively.

The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics (see [17] and the references therein). The generalized Hyers-Ulam stability of different functional equations in random normed spaces, fuzzy normed spaces, and non-Archimedean fuzzy normed spaces has been recently studied in [14–28].

Najati and Eskandani [29] established the general solution and investigated the Ulam-Hyers stability of the following functional equation.

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x), \tag{1.1}$$

with $f(0)=0$ in the quasi-Banach spaces. It is easy to see that the mapping $f(x)=ax^3+bx$ is a solution of the functional equation (1.1), which is called a mixed additive-cubic functional equation, and every solution of the mixed additive-cubic functional equation is said to be a mixed additive-cubic mapping.

In [14–16], we considered the following general mixed additive-cubic functional equation:

$$f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2f(kx)-2kf(x). \tag{1.2}$$

It is easy to show that the function $f(x)=ax^3+bx$ satisfies the functional equation (1.2). We observe that in case $k=2$ (1.2) yields mixed additive-cubic equation (1.1). Therefore, (1.2) is a generalized form of the mixed additive-cubic equation.

In the present paper, we first prove a theorem on stability of equation $g(ax)=a^s g(x)$ $(a, s \in \mathbb{N}, a \geq 2)$ in random normed spaces and derive from it results on stability of equation $f(4x)=10f(2x)-16f(x)$. Next, use those results to establish Ulam-Hyers stability for the general mixed additive-cubic functional equation (1.2) in the setting of random normed spaces. In this way some results will be obtained on stability of the linear functional equations also for the random normed spaces, which correspond, for example, to the papers [30–33].

2. Preliminaries

In the sequel we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [17, 28]. Throughout this paper, the space of all probability distribution functions is denoted by

$$\begin{aligned}\Delta^+ = \{F : \mathbb{R}\cup\{-\infty,+\infty\} \longrightarrow [0,1] : F \text{ is left-continuous and nondecreasing on } \mathbb{R} \\ \text{and } F(0)=0, F(+\infty)=1\},\end{aligned} \tag{2.1}$$

and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty)=1\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{2.2}$$

Definition 2.1 (see [17, 28]). A function $\mathcal{T} : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if $\mathcal{T}$ satisfies the following conditions:

(a) $\mathcal{T}$ is commutative and associative;

(b) $\mathcal{T}$ is continuous;

(c) $\mathcal{T}(a,1) = a$ for all $a \in [0,1]$;

(d) $\mathcal{T}(a,b) \leq \mathcal{T}(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $\mathcal{T}_P(a,b) = ab$, $\mathcal{T}_M(a,b) = \min(a,b)$ and $\mathcal{T}_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm).

Now, if $\mathcal{T}$ is a t-norm and $\{x_i\}$ is a given sequence of numbers in $[0,1]$, we define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}_{i=1}^1 x_1 = x_1$ and $\mathcal{T}_{i=1}^n x_i = \mathcal{T}(\mathcal{T}_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. $\mathcal{T}_{i=n}^\infty x_i$ is defined as $\mathcal{T}_{i=1}^\infty x_{n+i}$.

Definition 2.2 (see [17, 28]). A random normed space (briefly, RN-space) is a triple $(X, \mu, \mathcal{T})$, where X is a vector space, $\mathcal{T}$ is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all x in X, $\alpha \neq 0$ and all $t \geq 0$;

(RN3) $\mu_{x+y}(t+s) \geq \mathcal{T}(\mu_x(t), \mu_y(s))$ for all $x, y \in$ X and all $t, s \geq 0$.

Example 2.3. Let $(\mathrm{X}, \|\cdot\|)$ be a normed space. For all $x \in$ X and $t > 0$, consider $\mu_x(t) = t/(t + \|x\|)$. Then $(\mathrm{X}, \mu, \mathcal{T}_M)$ is a random normed space, where $\mathcal{T}_M$ is the minimum t-norm. This space is called the induced random normed space.

Definition 2.4. Let $(\mathrm{X}, \mu, \mathcal{T})$ be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x}(t) > 1 - \varepsilon$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) > 1 - \varepsilon$ whenever $n \geq m \geq N$.

(3) An RN-space $(X, \mu, \mathcal{T})$ is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

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Theorem 3.1. *Let $s, a \in \mathbb{N}$ with $a \geq 2$, X be a linear space, (Y, μ, T_M) be a complete RN-space, and $g : X \to Y$ be a mapping for which there is $\varphi : X \to D^+$ such that*

$$\mu_{g(ax)-a^s g(x)}(t) \geq \varphi_x(t) \tag{3.1}$$

for all $x \in X$ and $t > 0$. If for some $0 < \alpha < a^s$,

$$\varphi_{ax}(t) \geq \varphi_x\left(\frac{t}{\alpha}\right) \tag{3.2}$$

for all $x \in X$ and $t > 0$, then there exists a uniquely determined mapping $G : X \to Y$ such that $G(ax) = a^s G(x)$ and

$$\mu_{g(x)-G(x)}(t) \geq \varphi_x\left(\frac{(a^s - \alpha)t}{2}\right) \tag{3.3}$$

for all $x \in X$ and $t > 0$.

Proof. Replacing x by $a^i x$ in (3.1) and using (3.2), we get

$$\mu_{(g(a^{i+1}x)/a^{s(i+1)})-(g(a^i x)/a^{si})}\left(\frac{\alpha^i t}{a^{s(i+1)}}\right) \geq \varphi_x(t) \tag{3.4}$$

for all $x \in X, i \in \mathbb{N}$, and $t > 0$. It follows that

$$\begin{aligned} &\mu_{(g(a^n x)/a^{sn})-(g(a^m x)/a^{sm})}\left(\sum_{i=m}^{n-1}\frac{\alpha^i t}{a^{s(i+1)}}\right) \\ &\quad = \mu_{\sum_{i=m}^{n-1}((g(a^{i+1}x)/a^{s(i+1)})-(g(a^i x)/a^{si}))}\left(\sum_{i=m}^{n-1}\frac{\alpha^i t}{a^{s(i+1)}}\right) \geq \varphi_x(t) \end{aligned} \tag{3.5}$$

for all $x \in X, t > 0$ and all nonnegative integers n and m with $n > m$. Hence

$$\mu_{(g(a^n x)/a^{sn})-(g(a^m x)/a^{sm})}(t) \geq \varphi_x\left(t\Big/\sum_{i=m}^{n-1}\frac{\alpha^i}{a^{s(i+1)}}\right) \tag{3.6}$$

for all $x \in X, t > 0$, and $m, n \in \mathbb{N}$ with $n > m$. As $0 < \alpha < a^s$ and $\sum_{i=0}^{\infty}(\alpha^i / a^{s(i+1)}) < \infty$, the right hand side of the inequality tends to 1 as m tend to infinity. Then the sequence $\{g(a^n x)/a^{sn}\}$ is a Cauchy sequence in (Y, μ, T_M). Since (Y, μ, T_M) is a complete RN-space, this sequence converges to some point $G(x) \in Y$. Therefore, we may define $G(x) := \lim_{n\to\infty} g(a^n x)/a^{sn}$ for all $x \in X$. Fix $x \in X$, and put $m = 0$ in (3.6). Then we obtain

$$\mu_{(g(a^n x)/a^{sn})-g(x)}(t) \geq \varphi_x\left(t\Big/\sum_{i=0}^{n-1}\frac{\alpha^i}{a^{s(i+1)}}\right), \tag{3.7}$$

and so, by (RN3), we have

$$\begin{aligned}\mu_{G(x)-g(x)}(t) &\geq \tau_M\left(\mu_{G(x)-(g(a^n x)/a^{sn})}\left(\frac{t}{2}\right), \mu_{(g(a^n x)/a^{sn})-g(x)}\left(\frac{t}{2}\right)\right)\\ &\geq \tau_M\left(\mu_{G(x)-(g(a^n x)/a^{sn})}\left(\frac{t}{2}\right), \varphi_x\left(t/\sum_{i=0}^{n-1}\frac{2\alpha^i}{a^{s(i+1)}}\right)\right)\end{aligned} \tag{3.8}$$

for every $t > 0$. Taking the limit as $n \to \infty$ in (3.8), by $G(x) = \lim_{n\to\infty} g(a^n x)/a^{sn}$, we get (3.3).

To prove the uniqueness of the mapping G, assume that there exists another mapping $H : X \to Y$ which satisfies (3.3) and $H(ax) = a^s H(x)$ for all $x \in X$. Fix $x \in X$. Clearly, $G(a^n x) = a^{sn} G(x)$, and $H(a^n x) = a^{sn} H(x)$ for all $n \in \mathbb{N}$. It follows from (3.2) and (3.3) that

$$\begin{aligned}\mu_{G(x)-H(x)}(t) &\geq \tau_M\left(\mu_{(G(a^n x)/a^{sn})-(g(a^n x)/a^{sn})}\left(\frac{t}{2}\right), \mu_{(g(a^n x)/a^{sn})-(H(a^n x)/a^{sn})}\left(\frac{t}{2}\right)\right)\\ &\geq \varphi_x\left(\frac{(a^s-\alpha)a^{sn}t}{4\alpha^n}\right).\end{aligned} \tag{3.9}$$

Since $\lim_{n\to\infty}(a^s-\alpha)a^{sn}t/(4\alpha^n) = \infty$, we get $\lim_{n\to\infty}\varphi_x((a^s-\alpha)a^{sn}t/(4\alpha^n)) = 1$. Therefore, it follows from (3.9) that $\mu_{G(x)-H(x)}(t) = 1$ for all $t > 0$, and so $G = H$. This completes the proof. □

Corollary 3.2. *Let $s \in \{1,3\}$ be fixed, X be a linear space, (Y, μ, τ_M) be a complete RN-space, and $f : X \to Y$ be a mapping for which there is $\varphi : X \to D^+$ such that*

$$\mu_{f(4x)-10f(2x)+16f(x)}(t) \geq \varphi_x(t) \tag{3.10}$$

for all $x \in X$ and $t > 0$. If for some $0 < \alpha < 2^s$, $\varphi_{2x}(t) \geq \varphi_x(t/\alpha)$ for all $x \in X$ and $t > 0$, then there exists a uniquely determined mapping $F_s : X \to Y$ such that $F_s(2x) = 2^s F_s(x)$ and

$$\mu_{f(2x)-2^{3/s}f(x)-F_s(x)}(t) \geq \varphi_x\left(\frac{(2^s-\alpha)t}{2}\right) \tag{3.11}$$

for all $x \in X$ and $t > 0$.

Theorem 3.3. *Let X be a linear space, (Z, μ', τ_M) be an RN-space, (Y, μ, τ_M) be a complete RN-space, and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\varphi : X \times X \to Z$ such that*

$$\mu_{f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,y)}(t) \tag{3.12}$$

for all $x, y \in X$ and $t > 0$. If for some $0 < \alpha < 2$,

$$\mu'_{\varphi(2x,2y)}(t) \geq \mu'_{\alpha\varphi(x,y)}(t) \tag{3.13}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \psi_x\left(\frac{(2-\alpha)(k^3-k)t}{2}\right) \tag{3.14}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned}
\psi_x(t) := (\mathcal{T}_M)_{i=1}^{32}\Bigg(&\mu'_{\varphi(x/2,(2k+1)x/2)}\left(\frac{t}{384k}\right), \mu'_{\varphi(x/2,(2k-1)x/2)}\left(\frac{t}{384k}\right),\\
&\mu'_{\varphi(x/2,\,3kx/2)}\left(\frac{t}{384}\right), \mu'_{\varphi(0,(3k-1)x/2)}\left(\frac{(k-1)t}{384k}\right), \mu'_{\varphi(x,x)}\left(\frac{t}{96k^2}\right),\\
&\mu'_{\varphi(x/2,kx/2)}\left(\frac{t}{96}\right), \mu'_{\varphi(0,(k+1)x/2)}\left(\frac{(k-1)t}{96k}\right), \mu'_{\varphi(0,(k-1)x)}\left(\frac{(k-1)t}{96k^2}\right),\\
&\mu'_{\varphi(0,kx)}\left(\frac{t}{96(k+1)}\right), \mu'_{\varphi(x,(k+1)x)}\left(\frac{t}{128}\right), \mu'_{\varphi(x,(k-1)x)}\left(\frac{t}{128}\right),\\
&\mu'_{\varphi(0,x)}\left(\frac{(k-1)t}{128}\right), \mu'_{\varphi(0,kx)}\left(\frac{(k-1)t}{128k}\right), \mu'_{\varphi(2x,x)}\left(\frac{t}{32}\right), \mu'_{\varphi(2x,kx)}\left(\frac{t}{16}\right),\\
&\mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{56}\right), \mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{56}\right), \mu'_{3x,x}\left(\frac{t}{56}\right), \mu'_{\varphi(x,x)}\left(\frac{t}{56}\right),\\
&\mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{56}\right), \mu'_{\varphi(0,(k-1)x)}\left(\frac{(k-1)t}{56}\right), \mu'_{\varphi(0,2kx)}\left(\frac{(k-1)t}{56k}\right),\\
&\mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{384k}\right), \mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{384k}\right), \mu'_{\varphi(x,3kx)}\left(\frac{t}{384}\right),\\
&\mu'_{\varphi(0,(3k-1)x)}\left(\frac{(k-1)t}{384k}\right), \mu'_{\varphi(2x,2x)}\left(\frac{t}{96k^2}\right), \mu'_{\varphi(x,kx)}\left(\frac{t}{96}\right),\\
&\mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{96k}\right), \mu'_{\varphi(0,2(k-1)x)}\left(\frac{(k-1)t}{96k^2}\right),\\
&\mu'_{\varphi(0,2kx)}\left(\frac{t}{96(k+1)}\right), \mu'_{\varphi(2x,2kx)}\left(\frac{t}{16}\right)\Bigg).
\end{aligned} \tag{3.15}$$

Proof. Letting $x = 0$ in (3.12), we get

$$\mu_{f(y)+f(-y)}(t) \geq \mu'_{\varphi(0,y)}((k-1)t) \tag{3.16}$$

for all $y \in X$ and $t > 0$. Putting $y = x$ in (3.12), we have

$$\mu_{f((k+1)x)+f((k-1)x)-kf(2x)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,x)}(t) \tag{3.17}$$

for all $x \in X$ and $t > 0$. Replacing x by $2x$ in (3.17), we obtain

$$\mu_{f(2(k+1)x)+f(2(k-1)x)-kf(4x)-2f(2kx)+2kf(2x)}(t) \geq \mu'_{\varphi(2x,2x)}(t) \tag{3.18}$$

for all $x \in X$ and $t > 0$. Letting $y = kx$ in (3.12), we get

$$\mu_{f(2kx)-kf((k+1)x)-kf(-(k-1)x)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,kx)}(t) \tag{3.19}$$

for all $x \in X$ and $t > 0$. Letting $y = (k+1)x$ in (3.12), we have

$$\mu_{f((2k+1)x)+f(-x)-kf((k+2)x)-kf(-kx)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,(k+1)x)}(t) \tag{3.20}$$

for all $x \in X$ and $t > 0$. Letting $y = (k-1)x$ in (3.12), we have

$$\mu_{f((2k-1)x)-(k+2)f(kx)-kf(-(k-2)x)+(2k+1)f(x)}(t) \geq \mu'_{\varphi(x,(k-1)x)}(t) \tag{3.21}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and x in (3.12), respectively, we get

$$\mu_{f((2k+1)x)+f((2k-1)x)-2f(2kx)-kf(3x)+2kf(2x)-kf(x)}(t) \geq \mu'_{\varphi(2x,x)}(t) \tag{3.22}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $3x$ and x in (3.12), respectively, we get

$$\mu_{f((3k+1)x)+f((3k-1)x)-2f(3kx)-kf(4x)-kf(2x)+2kf(3x)}(t) \geq \mu'_{\varphi(3x,x)}(t) \tag{3.23}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and kx in (3.12), respectively, we have

$$\mu_{f(3kx)+f(kx)-kf((k+2)x)-kf(-(k-2)x)-2f(2kx)+2kf(2x)}(t) \geq \mu'_{\varphi(2x,kx)}(t) \tag{3.24}$$

for all $x \in X$ and $t > 0$. Setting $y = (2k+1)x$ in (3.12), we have

$$\mu_{f((3k+1)x)+f(-(k+1)x)-kf(2(k+1)x)-kf(-2kx)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,(2k+1)x)}(t) \tag{3.25}$$

for all $x \in X$ and $t > 0$. Letting $y = (2k-1)x$ in (3.12), we have

$$\mu_{f((3k-1)x)+f(-(k-1)x)-kf(-2(k-1)x)-kf(2kx)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,(2k-1)x)}(t) \tag{3.26}$$

for all $x \in X$ and $t > 0$. Letting $y = 3kx$ in (3.12), we have

$$\mu_{f(4kx)+f(-2kx)-kf((3k+1)x)-kf(-(3k-1)x)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,3kx)}(t) \tag{3.27}$$

for all $x \in X$ and $t > 0$. By (3.16), (3.17), (3.23), (3.25), and (3.26), we get

$$\begin{aligned}
&\mu_{kf(2(k+1)x)+kf(-2(k-1)x)+6f(kx)-2f(3kx)-kf(4x)+2kf(3x)-6kf(x)}(t)\\
&\quad\geq (\tau_M)_{i=1}^{7}\left(\mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{7}\right),\mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{7}\right),\mu'_{\varphi(3x,x)}\left(\frac{t}{7}\right),\mu'_{\varphi(x,x)}\left(\frac{t}{7}\right),\right.\\
&\qquad\left.\mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{7}\right),\mu'_{\varphi(0,(k-1)x)}\left(\frac{(k-1)t}{7}\right),\mu'_{\varphi(0,2kx)}\left(\frac{(k-1)t}{7k}\right)\right)
\end{aligned}\tag{3.28}$$

for all $x \in X$ and $t > 0$. By (3.16), (3.20), and (3.21), we have

$$\begin{aligned}
&\mu_{f((2k+1)x)+f((2k-1)x)-kf((k+2)x)-kf(-(k-2)x)-4f(kx)+4kf(x)}(t)\\
&\quad\geq (\tau_M)_{i=1}^{4}\left(\mu'_{\varphi(x,(k+1)x)}\left(\frac{t}{4}\right),\mu'_{\varphi(x,(k-1)x)}\left(\frac{t}{4}\right),\mu'_{\varphi(0,x)}\left(\frac{(k-1)t}{4}\right),\mu'_{\varphi(0,kx)}\left(\frac{(k-1)t}{4k}\right)\right)
\end{aligned}\tag{3.29}$$

for all $x \in X$ and $t > 0$. It follows from (3.22) and (3.29) that

$$\begin{aligned}
&\mu_{kf((k+2)x)+kf(-(k-2)x)-2f(2kx)+4f(kx)-kf(3x)+2kf(2x)-5kf(x)}(t)\\
&\quad\geq (\tau_M)_{i=1}^{5}\left(\mu'_{\varphi(x,(k+1)x)}\left(\frac{t}{8}\right),\mu'_{\varphi(x,(k-1)x)}\left(\frac{t}{8}\right),\right.\\
&\qquad\left.\mu'_{\varphi(0,x)}\left(\frac{(k-1)t}{8}\right),\mu'_{\varphi(0,kx)}\left(\frac{(k-1)t}{8k}\right),\mu'_{\varphi(2x,x)}\left(\frac{t}{2}\right)\right)
\end{aligned}\tag{3.30}$$

for all $x \in X$ and $t > 0$. By (3.24) and (3.30), we have

$$\begin{aligned}
&\mu_{f(3kx)-4f(2kx)+5f(kx)-kf(3x)+4kf(2x)-5kf(x)}(t)\\
&\quad\geq (\tau_M)_{i=1}^{6}\left(\mu'_{\varphi(x,(k+1)x)}\left(\frac{t}{16}\right),\mu'_{\varphi(x,(k-1)x)}\left(\frac{t}{16}\right),\mu'_{\varphi(0,x)}\left(\frac{(k-1)t}{16}\right),\right.\\
&\qquad\left.\mu'_{\varphi(0,kx)}\left(\frac{(k-1)t}{16k}\right),\mu'_{\varphi(2x,x)}\left(\frac{t}{4}\right),\mu'_{\varphi(2x,kx)}\left(\frac{t}{2}\right)\right)
\end{aligned}\tag{3.31}$$

for all $x \in X$ and $t > 0$. By (3.16) and (3.25)–(3.27), we have

$$\begin{aligned}
&\mu_{kf(-(k+1)x)-kf(-(k-1)x)-k^2f(2(k+1)x)+k^2f(-2(k-1)x)+k^2f(2kx)-(k^2-1)f(-2kx)+f(4kx)-2f(kx)+2kf(x)}(t)\\
&\quad\geq (\tau_M)_{i=1}^{4}\left(\mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{4k}\right),\mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{4k}\right),\mu'_{\varphi(x,3kx)}\left(\frac{t}{4}\right),\mu'_{\varphi(0,(3k-1)x)}\left(\frac{(k-1)t}{4k}\right)\right)
\end{aligned}\tag{3.32}$$

for all $x \in X$ and $t > 0$. It follows from (3.16), (3.18), (3.19), and (3.32) that

$$\begin{aligned}
&\mu_{f(4kx)-2f(2kx)-k^3f(4x)+2k^3f(2x)}(t) \\
&\geq (\tau_M)_{i=1}^{9}\Bigg(\mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{24k}\right), \mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{24k}\right), \mu'_{\varphi(x,3kx)}\left(\frac{t}{24}\right), \mu'_{\varphi(0,(3k-1)x)}\left(\frac{(k-1)t}{24k}\right), \\
&\qquad \mu'_{\varphi(2x,2x)}\left(\frac{t}{6k^2}\right), \mu'_{\varphi(x,kx)}\left(\frac{t}{6}\right), \mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{6k}\right), \\
&\qquad \mu'_{\varphi(0,2(k-1)x)}\left(\frac{(k-1)t}{6k^2}\right), \mu'_{\varphi(0,2kx)}\left(\frac{t}{6(k+1)}\right)\Bigg)
\end{aligned} \tag{3.33}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}
&\mu_{f(2kx)-2f(kx)-k^3f(2x)+2k^3f(x)}(t) \\
&\geq (\tau_M)_{i=1}^{9}\Bigg(\mu'_{\varphi(x/2,(2k+1)x/2)}\left(\frac{t}{24k}\right), \mu'_{\varphi(x/2,(2k-1)x/2)}\left(\frac{t}{24k}\right), \mu'_{\varphi(x/2,3kx/2)}\left(\frac{t}{24}\right), \\
&\qquad \mu'_{\varphi(0,(3k-1)x/2)}\left(\frac{(k-1)t}{24k}\right), \mu'_{\varphi(x,x)}\left(\frac{t}{6k^2}\right), \mu'_{\varphi(x/2,kx/2)}\left(\frac{t}{6}\right), \\
&\qquad \mu'_{\varphi(0,(k+1)x/2)}\left(\frac{(k-1)t}{6k}\right), \mu'_{\varphi(0,(k-1)x)}\left(\frac{(k-1)t}{6k^2}\right), \mu'_{\varphi(0,kx)}\left(\frac{t}{6(k+1)}\right)\Bigg)
\end{aligned} \tag{3.34}$$

for all $x \in X$ and $t > 0$. By (3.19), we have

$$\mu_{f(4kx)-kf(2(k+1)x)-kf(-2(k-1)x)-2f(2kx)+2kf(2x)}(t) \geq \mu'_{\varphi(2x,2kx)}(t) \tag{3.35}$$

for all $x \in X$ and $t > 0$. From (3.33) and (3.35), we have

$$\begin{aligned}
&\mu_{kf(2(k+1)x)+kf(-2(k-1)x)-k^3f(4x)+(2k^3-2k)f(2x)}(t) \\
&\geq (\tau_M)_{i=1}^{10}\Bigg(\mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{48k}\right), \mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{48k}\right), \mu'_{\varphi(x,3kx)}\left(\frac{t}{48}\right), \mu'_{\varphi(0,(3k-1)x)}\left(\frac{(k-1)t}{48k}\right), \\
&\qquad \mu'_{\varphi(2x,2x)}\left(\frac{t}{12k^2}\right), \mu'_{\varphi(x,kx)}\left(\frac{t}{12}\right), \mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{12k}\right), \mu'_{\varphi(0,2(k-1)x)}\left(\frac{(k-1)t}{12k^2}\right), \\
&\qquad \mu'_{\varphi(0,2kx)}\left(\frac{t}{12(k+1)}\right), \mu'_{\varphi(2x,2kx)}\left(\frac{t}{2}\right)\Bigg)
\end{aligned} \tag{3.36}$$

for all $x \in X$ and $t > 0$. Also, from (3.28) and (3.36), we get

$$\begin{aligned}
&\mu_{2f(3kx)-6f(kx)+(k-k^3)f(4x)-2kf(3x)+(2k^3-2k)f(2x)+6kf(x)}(t) \\
&\geq (\tau_M)_{i=1}^{17}\Bigg(\mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{14}\right), \mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{14}\right), \\
&\quad \mu'_{\varphi(3x,x)}\left(\frac{t}{14}\right), \mu'_{\varphi(x,x)}\left(\frac{t}{14}\right), \mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{14}\right), \\
&\quad \mu'_{\varphi(0,(k-1)x)}\left(\frac{(k-1)t}{14}\right), \mu'_{\varphi(0,2kx)}\left(\frac{(k-1)t}{14k}\right), \mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{96k}\right), \\
&\quad \mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{96k}\right), \mu'_{\varphi(x,3kx)}\left(\frac{t}{96}\right), \mu'_{\varphi(0,(3k-1)x)}\left(\frac{(k-1)t}{96k}\right), \\
&\quad \mu'_{\varphi(2x,2x)}\left(\frac{t}{24k^2}\right), \mu'_{\varphi(x,kx)}\left(\frac{t}{24}\right), \mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{24k}\right), \\
&\quad \mu'_{\varphi(0,2(k-1)x)}\left(\frac{(k-1)t}{24k^2}\right), \mu'_{\varphi(0,2kx)}\left(\frac{t}{24(k+1)}\right), \mu'_{\varphi(2x,2kx)}\left(\frac{t}{4}\right)\Bigg)
\end{aligned} \tag{3.37}$$

for all $x \in X$ and $t > 0$.

On the other hand, it follows from (3.31) and (3.37) that

$$\begin{aligned}
&\mu_{8f(2kx)-16f(kx)+(k-k^3)f(4x)+(2k^3-10k)f(2x)+16kf(x)}(t) \\
&\geq (\tau_M)_{i=1}^{23}\Bigg(\mu'_{\varphi(x,(k+1)x)}\left(\frac{t}{64}\right), \mu'_{\varphi(x,(k-1)x)}\left(\frac{t}{64}\right), \mu'_{\varphi(0,x)}\left(\frac{(k-1)t}{64}\right), \mu'_{\varphi(0,kx)}\left(\frac{(k-1)t}{64k}\right), \\
&\quad \mu'_{\varphi(2x,x)}\left(\frac{t}{16}\right), \mu'_{\varphi(2x,kx)}\left(\frac{t}{8}\right), \mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{28}\right), \\
&\quad \mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{28}\right), \mu'_{\varphi(3x,x)}\left(\frac{t}{28}\right), \mu'_{\varphi(x,x)}\left(\frac{t}{28}\right), \mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{28}\right), \\
&\quad \mu'_{\varphi(0,(k-1)x)}\left(\frac{(k-1)t}{28}\right), \mu'_{\varphi(0,2kx)}\left(\frac{(k-1)t}{28k}\right), \mu'_{\varphi(x,(2k+1)x)}\left(\frac{t}{192k}\right), \\
&\quad \mu'_{\varphi(x,(2k-1)x)}\left(\frac{t}{192k}\right), \mu'_{\varphi(x,3kx)}\left(\frac{t}{192}\right), \mu'_{\varphi(0,(3k-1)x)}\left(\frac{(k-1)t}{192k}\right), \\
&\quad \mu'_{\varphi(2x,2x)}\left(\frac{t}{48k^2}\right), \mu'_{\varphi(x,kx)}\left(\frac{t}{48}\right), \mu'_{\varphi(0,(k+1)x)}\left(\frac{(k-1)t}{48k}\right) \\
&\quad \mu'_{\varphi(0,2(k-1)x)}\left(\frac{(k-1)t}{48k^2}\right), \mu'_{\varphi(0,2kx)}\left(\frac{t}{48(k+1)}\right), \mu'_{\varphi(2x,2kx)}\left(\frac{t}{8}\right)\Bigg)
\end{aligned} \tag{3.38}$$

for all $x \in X$ and $t > 0$. Therefore by (3.34) and (3.38), we get

$$\mu_{f(4x)-10f(2x)+16f(x)}\left(\frac{t}{k^3-k}\right) \geq \psi_x(t) \tag{3.39}$$

for all $x \in X$ and $t > 0$. By Corollary 3.2, there exists a unique mapping $A : X \to Y$ such that $A(2x) = 2A(x)$ and $\mu_{f(2x)-8f(x)-A(x)}(t) \geq \psi_x((2-\alpha)(k^3-k)t/2)$ for all $x \in X$ and $t > 0$.

It remains to show that A is an additive map. Replacing x, y by $2^n x, 2^n y$ in (3.12) we get

$$\begin{aligned} &\mu_{(1/2^n)[f(k2^n x+2^n y)+f(k2^n x-2^n y)-kf(2^n x+2^n y)-kf(2^n x-2^n y)-2f(k2^n x)+2kf(2^n x)]}(t) \\ &\quad \geq \mu'_{\varphi(x,y)}\left(\frac{2^n t}{\alpha^n}\right) \end{aligned} \tag{3.40}$$

for all $x, y \in X$ and $t > 0$. Hence

$$\begin{aligned} &\mu_{f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)}(t) \\ &\quad \geq (\tau_M)_{i=1}^{8}\Bigg(\mu_{A(kx+y)-(g(2^n(kx+y)))/2^n}\left(\frac{t}{8}\right), \mu_{A(kx-y)-(g(2^n(kx-y)))/2^n}\left(\frac{t}{8}\right), \\ &\qquad \mu_{A(x+y)-(g(2^n(x+y)))/2^n}\left(\frac{t}{8k}\right), \mu_{A(x-y)-(g(2^n(x-y)))/2^n}\left(\frac{t}{8k}\right), \\ &\qquad \mu_{A(kx)-(g(2^n kx))/2^n}\left(\frac{t}{16}\right), \mu_{A(x)-(g(2^n x))/2^n}\left(\frac{t}{16k}\right), \\ &\qquad \mu'_{\varphi(x,y)}\left(\frac{2^n t}{8\alpha^{n+1}}\right), \mu'_{\varphi(x,y)}\left(\frac{2^n t}{64\alpha^n}\right)\Bigg) \end{aligned} \tag{3.41}$$

for all $x, y \in X$ and $t > 0$. Taking the limit as $n \to \infty$ in (3.41), we conclude that A fulfills (1.2), and so by [16, Lemma 3.1], we see that the mapping $x \to A(2x) - 8A(x)$ is additive, which implies that the mapping A is additive. This completes the proof. □

Similar to Theorem 3.3, one can prove the following result.

Theorem 3.4. *Let X be a linear space, (Z, μ', τ_M) be an RN-space, (Y, μ, τ_M) be a complete RN-space, and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\varphi : X \times X \to Z$ such that*

$$\mu_{f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,y)}(t) \tag{3.42}$$

for all $x, y \in X$ and $t > 0$. If for some $0 < \alpha < 8$, $\mu'_{\varphi(2x,2y)}(t) \geq \mu'_{\alpha\varphi(x,y)}(t)$ for all $x, y \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \psi_x\left(\frac{(8-\alpha)(k^3-k)t}{2}\right) \tag{3.43}$$

for all $x \in X$ and $t > 0$, where $\psi_x(t)$ is defined as in Theorem 3.3.

Remark 3.5. We can also prove Theorems 3.3 and 3.4 for $\alpha > 2$ and $\alpha > 8$, respectively.

Theorem 3.6. *Let X be a linear space, (Z, μ', τ_M) be an RN-space, (Y, μ, τ_M) be a complete RN-space, and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\varphi : X \times X \to Z$ such that*

$$\mu_{f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)}(t) \geq \mu'_{\varphi(x,y)}(t) \tag{3.44}$$

for all $x, y \in X$ and $t > 0$. If for some $0 < \alpha < 2$,

$$\mu'_{\varphi(2x,2y)}(t) \geq \mu'_{\alpha\varphi(x,y)}(t) \tag{3.45}$$

for all $x, y \in X$ and $t > 0$, then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-A(x)-C(x)}(t) \geq \psi_x\left(\frac{3(2-\alpha)(k^3-k)t}{2}\right) \tag{3.46}$$

for all $x \in X$ and $t > 0$, where $\psi_x(t)$ is defined as in Theorem 3.3.

Proof. By Theorems 3.3 and 3.4, there exist an additive mapping $A_1 : X \to Y$ and a cubic mapping $C_1 : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A_1(x)}(t) \geq \psi_x\left(\frac{(2-\alpha)(k^3-k)t}{2}\right), \tag{3.47}$$

$$\mu_{f(2x)-2f(x)-C_1(x)}(t) \geq \psi_x\left(\frac{(8-\alpha)(k^3-k)t}{2}\right) \tag{3.48}$$

for all $x \in X$ and $t > 0$. Therefore from (3.47) and (3.48), we get

$$\mu_{f(x)+(\frac{1}{6})A_1(x)-(\frac{1}{6})C_1(x)}(t) \geq \psi_x\left(\frac{3(2-\alpha)(k^3-k)t}{2}\right) \tag{3.49}$$

for all $x \in X$ and $t > 0$. Letting $A(x) = -(1/6)A_1(x)$ and $C(x) = (1/6)C_1(x)$ for all $x \in X$, it follows from (3.49) that

$$\mu_{f(x)-A(x)-C(x)}(t) \geq \psi_x\left(\frac{3(2-\alpha)(k^3-k)t}{2}\right) \tag{3.50}$$

for all $x \in X$ and $t > 0$. To prove the uniqueness of A and C, let $A', C' : X \to Y$ be another additive and cubic mapping satisfying (3.46). Set $\widetilde{A} = A - A'$ and $\widetilde{C} = C - C'$. So

$$\begin{aligned} \mu_{\widetilde{A}(x)+\widetilde{C}(x)}(t) &\geq \tau_M\left(\mu_{A(x)+C(x)-f(x)}\left(\frac{t}{2}\right), \mu_{f(x)-A'(x)-C'(x)}\left(\frac{t}{2}\right)\right) \\ &\geq \varphi_x\left(\frac{3(2-\alpha)(k^3-k)t}{4}\right) \end{aligned} \tag{3.51}$$

for all $x \in X$ and $t > 0$. By $\widetilde{A}(2x) = 2\widetilde{A}(x)$, $\widetilde{C}(2x) = 8\widetilde{C}(x)$, and (3.51), we get

$$\begin{aligned} \mu_{\widetilde{C}(x)}(t) &\geq \tau_M\left(\mu_{\widetilde{A}(2^n x)+\widetilde{C}(2^n x)}\left(\frac{8^n t}{2}\right), \mu_{\widetilde{A}(2^n x)}\left(\frac{8^n t}{2}\right)\right) \\ &\geq \tau_M\left(\varphi_x\left(\frac{3(2-\alpha)(k^3-k)8^n t}{4\alpha^n}\right), \mu_{\widetilde{A}(x)}\left(\frac{4^n t}{2}\right)\right) \end{aligned} \tag{3.52}$$

for all $x \in X$ and $t > 0$. Since the right hand side of the inequality tends to 1 as n tend to infinity, we find that $\widetilde{C}(x) = 0$. Therefore $\widetilde{C} = 0$, and then $\widetilde{A} = 0$. This completes the proof. □

Remark 3.7. We can formulate similar statements to Theorem 3.6 for $\alpha > 8$.

Corollary 3.8. *Let $(X, \|\cdot\|)$ be a normed space, (Z, μ', τ_M) be an RN-space, and (Y, μ, τ_M) be a complete RN-space. Let p be a non-negative real number such that $p \in (0,1) \cup (1,3) \cup (3,\infty)$, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that*

$$\mu_{f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)}(t) \geq \mu'_{(\|x\|^p+\|y\|^p)z_0}(t) \tag{3.53}$$

for all $x, y \in X$ and $t > 0$, then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-A(x)-C(x)}(t) \geq \begin{cases} \mu'_{\|x\|^p z_0}\left(\dfrac{(2-2^p)(k^2-1)t}{256[1+3k]}\right), & p \in (0,1), \\[2ex] \mu'_{\|x\|^p z_0}\left(\dfrac{(2^p-2)(k^2-1)t}{256\left[1+(3k)^3\right]}\right), & p \in (1, \log_2 5), \\[2ex] \mu'_{\|x\|^p z_0}\left(\dfrac{(8-2^p)(k^2-1)t}{256\left[1+(3k)^3\right]}\right), & p \in (\log_2 5, 3), \\[2ex] \mu'_{\|x\|^p z_0}\left(\dfrac{(2^p-8)(k^2-1)t}{256\left[1+(3k)^p\right]}\right), & p \in (3,\infty), \end{cases} \tag{3.54}$$

for all $x \in X$ and $t > 0$.

Corollary 3.9. *Let $(X, \|\cdot\|)$ be a normed space, (Z, μ', τ_M) be an RN-space, and (Y, μ, τ_M) be a complete RN-space. Let r, s be non-negative real numbers such that $\lambda := r+s \in (0,1)\cup(1,3)\cup(3,\infty)$, and let $z_0 \in Z$. If $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\mu_{f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)}(t) \geq \mu'_{[\|x\|^r\|y\|^s+(\|x\|^{r+s}+\|y\|^{r+s})]z_0}(t) \tag{3.55}$$

for all $x, y \in X$ and $t > 0$, then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-A(x)-C(x)}(t) \geq \begin{cases} \mu'_{\|x\|^\lambda z_0}\left(\dfrac{(2-2^\lambda)(k^2-1)t}{256[1+6k]}\right), & \lambda \in (0,1), \\[2ex] \mu'_{\|x\|^\lambda z_0}\left(\dfrac{(2^\lambda-2)(k^2-1)t}{256\left[1+2(3k)^3\right]}\right), & \lambda \in (1, \log_2 5), \\[2ex] \mu'_{\|x\|^\lambda z_0}\left(\dfrac{(8-2^\lambda)(k^2-1)t}{256\left[1+2(3k)^3\right]}\right), & \lambda \in (\log_2 5, 3), \\[2ex] \mu'_{\|x\|^\lambda z_0}\left(\dfrac{(2^\lambda-8)(k^2-1)t}{256\left[1+2(3k)^\lambda\right]}\right), & \lambda \in (3, \infty), \end{cases} \tag{3.56}$$

for all $x \in X$ and $t > 0$.

Now, we give one example to illustrate the main results of Theorem 3.6. This example is a modification of the example of Zhang et al. [34].

Example 3.10. Let $(X, \|\cdot\|)$ be a Banach algebra, x_0 be a unit vector in X, and $\mu_x(t)$ is defined as in Example 2.3. It is easy to see that (X, μ, τ_M) is a complete RN-space.

Define $f : X \to X$ by $f(x) = x^3 + \|x\|^p x_0$ for $x \in X$. For $0 < p < 1$, define

$$\varphi(x, y) = 8k\left(\|x\|^p + \|y\|^p\right)x_0, \quad x, y \in X. \tag{3.57}$$

Since $0 < p < 1$, the inequality $(a+b)^p \leq a^p + b^p$ holds when $a \geq 0$ and $b \geq 0$. A straightforward computation shows that

$$\|f(kx+y) + f(kx-y) - kf(x+y) - kf(x-y) - 2f(kx) + 2kf(x)\| \leq 8k\left(\|x\|^p + \|y\|^p\right) \tag{3.58}$$

for all $x, y \in X$. Therefore, all the conditions of Theorem 3.6 hold, and there exist a unique additive mapping $A : X \to X$ and a unique cubic mapping $C : X \to X$ such that

$$\mu_{f(x)-A(x)-C(x)}(t) \geq \varphi_x\left(\frac{3(2-\alpha)(k^3-k)t}{2}\right) \tag{3.59}$$

for all $x \in X$ and $t > 0$, where $\varphi_x(t)$ is defined as in Theorem 3.3.

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