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Research Article

Stability Problems of Quintic Mappings in Quasi-β-Normed Spaces

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We investigate the generalized Hyers-Ulam-Rassias stability problem in quasi-β-normed spaces and then the stability by using a subadditive function for the quintic function $f : X \to Y$ such that $2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y) = 20[f(x+y)+f(x-y)]+90f(x)$, for all $x, y \in X$.

1. Introduction

In 1940 Ulam [1] proposed the problem concerning the stability of group homomorphisms as follows. *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given that $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?* In other words, when is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation ? In 1941, Hyers [2] considered the case of approximately additive mappings under the assumption that G_1 and G_2 are Banach spaces.

The famous Hyers stability result that appeared in [2] was generalized by Aoki [3] for the stability of the additive mapping involving a sum of powers of p-norms. In 1978, Th. M. Rassias [4] provided a generalization of Hyers' Theorem for the stability of the linear mapping, which allows the Cauchy difference to be unbounded. This result of Th. M. Rassias lead mathematicians working in stability of functional equations to establish what is known today as Hyers-Ulam-Rassias stability or Cauchy-Rassias stability as well as to introduce new definitions of stability concepts. During the last three decades, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors [5–14]. In particular, J. M. Rassias [15] introduced the quartic functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y). \tag{1.1}$$

It is easy to see that $f(x)=x^4$ is a solution of (1.1) by virtue of the identity

$$(x+2y)^4+(x-2y)^4+6x^4=4(x+y)^4+4(x-y)^4+24y^4. \tag{1.2}$$

For this reason, (1.1) is called a quartic functional equation. Also Chung and Sahoo [16] determined the general solution of (1.1) without assuming any regularity conditions on the unknown function. In fact, they proved that the function $f:\mathbb{R}\to\mathbb{R}$ is a solution of (1.1) if and only if $f(x)=A(x,x,x,x)$, where the function $A:\mathbb{R}^4\to\mathbb{R}$ is symmetric and additive in each variable.

Similar to the quartic functional equation, we may define quintic functional equation as follows.

Definition 1.1. Let X be a linear space and let Y be a real complete linear space. Then a mapping $F:X\to Y$ is called *quintic* if the quintic functional equation

$$2F(2x+y)+2F(2x-y)+F(x+2y)+F(x-2y)=20\left[F(x+y)+F(x-y)\right]+90F(x) \tag{1.3}$$

holds for all $x,y\in X$.

Note that the mapping F is called *quintic* because the following algebraic identity

$$2(2x+y)^5+2(2x-y)^5+(x+2y)^5+(x-2y)^5=20\left[(x+y)^5+(x-y)^5\right]+90x^5 \tag{1.4}$$

holds for all $x,y\in X$.

Let β be a real number with $0<\beta\le 1$ and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We will consider the definition and some preliminary results of a quasi-β-norm on a linear space.

Definition 1.2. Let X be a linear space over a field $\mathbb{K}$. A *quasi-β-norm* $\|\cdot\|$ is a real-valued function on X satisfying the followings.

(1) $\|x\|\ge 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$.

(2) $\|\lambda x\|=|\lambda|^\beta\cdot\|x\|$ for all $\lambda\in\mathbb{K}$ and all $x\in X$.

(3) There is a constant $K\ge 1$ such that $\|x+y\|\le K(\|x\|+\|y\|)$ for all $x,y\in X$.

The pair $(X,\|\cdot\|)$ is called a *quasi-β-normed space* if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-β-normed space.

A quasi-β-norm $\|\cdot\|$ is called a (β,p)-norm $(0<p\le 1)$ if $\|x+y\|^p\le\|x\|^p+\|y\|^p$, for all $x,y\in X$. In this case, a quasi-β-Banach space is called a (β,p)-Banach space; see [17–19].

In this paper, we consider the following quintic functional equation:

$$2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y)=20[f(x+y)+f(x-y)]+90f(x), \tag{1.5}$$

for all $x,y\in X$. We investigate the generalized Hyers-Ulam-Rassias stability problem in quasi-β-normed spaces and then the stability by using a subadditive function for the quintic function $f:X\to Y$ satisfying (1.5).

2. Quintic Functional Equations

Lemma 2.1. *Let $F:X\to Y$ be a quintic mapping satisfying* (1.3). *Then*

(1) $F(2^n x)=(2^n)^5F(x)$, *for all $x\in X$ and $n\in\mathbb{N}$,*

(2) $F(0)=0$,

(3) *F is an odd mapping,*

Proof. (1) Letting $y=0$ in (1.3), we have

$$4F(2x)+2F(x)=130F(x), \tag{2.1}$$

that is, $F(2x)=32F(x)$, for all $x\in X$. Now inductively replacing x by $2x$, we have the desired result. (2) Putting $x=y=0$ in (1.3),

$$6F(0)=130F(0). \tag{2.2}$$

Hence $F(0)=0$. (3) Letting $x=0$ in (1.3), we get

$$2F(y)+2F(-y)+F(2y)+F(-2y)=20F(y)+20F(-y)+90F(0), \tag{2.3}$$

for all $y\in X$. By (1) and (2), we have

$$F(y)=-F(-y), \tag{2.4}$$

for all $y\in X$. Thus it is an odd mapping. □

Note that $F(x)=(1/2^{5n})F(2^n x)$, for all $x\in X$ and $n\in\mathbb{N}$.

3. Stabilities

Throughout this section, let X be a quasi-β-normed space and let Y be a quasi-β-Banach space with a quasi-β-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$. We will investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.5). After that we will study the stability by using a subadditive function. For a given mapping $f : X \to Y$, let

$$
\begin{aligned}
Df(x,y) := {} & 2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) \\
& - 20[f(x+y) + f(x-y)] - 90f(x),
\end{aligned}
\tag{3.1}
$$

for all $x, y \in X$.

Theorem 3.1. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0,\infty)$ for which a mapping $f : X \to Y$ satisfies*

$$
\|Df(x,y)\|_Y \leq \phi(x,y) \tag{3.2}
$$

and the series $\sum_{j=0}^{\infty}(K/2^{5\beta})^j \phi(2^j x, 2^j y)$ converges for all $x, y \in X$. Then there exists a unique quintic mapping $Q : X \to Y$ which satisfies (1.3) *and the inequality*

$$
\|f(x) - Q(x)\|_Y \leq \frac{K}{128^\beta}\sum_{j=0}^{\infty}\left(\frac{K}{2^{5\beta}}\right)^j \phi\left(2^j x, 0\right), \tag{3.3}
$$

for all $x \in X$.

Proof. By letting $y = 0$ in the inequality (3.2), we have

$$
\|Df(x,0)\|_Y = \|4f(2x) - 128f(x)\|_Y = 128^\beta \|f(x) - \left(\frac{1}{2}\right)^5 f(2x)\|_Y \leq \phi(x,0), \tag{3.4}
$$

that is,

$$
\left\|f(x) - \frac{1}{2^5}f(2x)\right\|_Y \leq \frac{1}{128^\beta}\phi(x,0), \tag{3.5}
$$

for all $x \in X$. Now, replacing x by $2x$ and multiplying $1/2^{5\beta}$ in the inequality (3.5), we get

$$
\frac{1}{2^{5\beta}}\left\|f(2x) - \frac{1}{2^5}f(2^2x)\right\|_Y \leq \frac{1}{2^{5\beta}}\ \frac{1}{128^\beta}\phi(2x,0), \tag{3.6}
$$

for all $x \in X$. Combining the two equations (3.5) and (3.6), we have

$$
\left\|f(x) - \left(\frac{1}{2^5}\right)^2 f(2^2x)\right\|_Y \leq \frac{K}{128^\beta}\left[\phi(x,0) + \frac{1}{2^{5\beta}}\phi(2x,0)\right], \tag{3.7}
$$

for all $x \in X$. Inductively, since $K \geq 1$, we have

$$\left\| f(x) - \left(\frac{1}{2^5}\right)^n f(2^n x) \right\|_Y \leq \frac{K}{128^\beta} \sum_{j=0}^{n-1} \left(\frac{K}{2^{5\beta}}\right)^j \phi\left(2^j x, 0\right), \tag{3.8}$$

for all $x \in X, n \in \mathbb{N}$. For all n and d with $n < d$ and inductively switching x and $2^n x$ and multiplying $(1/2^{5\beta})^n$ in the inequality (3.5), we have

$$\left\| \left(\frac{1}{2^5}\right)^n f(2^n x) - \left(\frac{1}{2^5}\right)^d f(2^d x) \right\|_Y \leq \frac{K}{128^\beta} \sum_{j=n}^{d-1} \left(\frac{K}{2^{5\beta}}\right)^j \phi\left(2^j x, 0\right), \tag{3.9}$$

for all $x \in X$. Since the right-hand side of the previous inequality tends to 0 as $d \to \infty$, hence $\{(1/2^5)^n f(2^n x)\}$ is a Cauchy sequence in the quasi-β-Banach space Y. Thus we may define

$$Q(x) = \lim_{n \to \infty} \left(\frac{1}{2^5}\right)^n f(2^n x), \tag{3.10}$$

for all $x \in X$. Since $K \geq 1$, replacing x and y by $2^n x$ and $2^n y$, respectively, and dividing by $2^{5\beta n}$ in the inequality (3.2), we have

$$\begin{aligned}
&\left(\frac{1}{2^{5\beta}}\right)^n \|Df(2^n x, 2^n y)\|_Y \\
&\quad = \left(\frac{1}{2^{5\beta}}\right)^n \| f(2^n(2x+y)) + f(2^n(2x-y)) \\
&\qquad\qquad + f(2^n(x+2y)) + f(2^n(x-2y)) \\
&\qquad\qquad - 20\left[f(2^n(x+y)) + f(2^n(x-y))\right] - 90 f(2^n x)\|_Y \\
&\quad \leq \left(\frac{K}{2^{5\beta}}\right)^n \| f(2^n(2x+y)) + f(2^n(2x-y)) \\
&\qquad\qquad + f(2^n(x+2y)) + f(2^n(x-2y)) \\
&\qquad\qquad - 20\left[f(2^n(x+y)) + f(2^n(x-y))\right] - 90 f(2^n x)\|_Y \\
&\quad \leq \left(\frac{K}{2^{5\beta}}\right)^n \phi(2^n x, 2^n y),
\end{aligned} \tag{3.11}$$

for all $x, y \in X$. By taking $n \to \infty$, the definition of Q implies that Q satisfies (1.3) for all $x, y \in X$, that is, Q is the quintic mapping. Also, the inequality (3.8) implies the inequality (3.3). Now, it remains to show the uniqueness. Assume that there exists $T : X \to Y$ satisfying

(1.3) and (3.3). Lemma 2.1 implies that $T(2^n x) = 2^{5n}T(x)$ and $Q(2^n x) = 2^{5n}Q(x)$, for all $x \in X$. Then

$$\begin{aligned}\|T(x) - Q(x)\|_Y &= \left(\frac{1}{2^{5\beta}}\right)^n \|T(2^n x) - Q(2^n x)\|_Y \\ &\le \left(\frac{1}{2^{5\beta}}\right)^n K(\|T(2^n x) - f(2^n x)\|_Y + \|f(2^n x) - Q(2^n x)\|_Y) \\ &\le \frac{2K^2}{128^\beta}\sum_{j=0}^{\infty}\left(\frac{K}{2^{5\beta}}\right)^{s+j}\phi\left(2^{s+j}x, 0\right),\end{aligned} \tag{3.12}$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of Q. □

Theorem 3.2. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0, \infty)$ for which a mapping $f : X \to Y$ satisfies*

$$\left\|Df(x, y)\right\|_Y \le \phi(x, y) \tag{3.13}$$

and the series $\sum_{j=1}^{\infty}(2^{5\beta}K)^j \phi(2^{-j}x, 2^{-j}y)$ converges for all $x, y \in X$. Then there exists a unique quintic mapping $Q : X \to Y$ which satisfies (1.3) *and the inequality*

$$\left\|f(x) - Q(x)\right\|_Y \le \frac{1}{128^\beta}\sum_{j=1}^{\infty}\left(2^{5\beta}K\right)^j \phi\left(2^{-j}x, 0\right), \tag{3.14}$$

for all $x \in X$.

Proof. If x is replaced by $(1/2)x$ in the inequality (3.5), then the proof follows from the proof of Theorem 3.1. □

Now we will recall a subadditive function and then investigate the stability under the condition that the space Y is a (β, p)-Banach space. The basic definitions of subadditive functions follow from [19].

A function $\phi : A \to B$ having a domain A and a codomain $(B, \le)$ that are both closed under addition is called

(1) *a subadditive function* if $\phi(x + y) \le \phi(x) + \phi(y)$,

(2) *a contractively subadditive function* if there exists a constant L with $0 < L < 1$ such that $\phi(x + y) \le L(\phi(x) + \phi(y))$,

(3) *an expansively superadditive function* if there exists a constant L with $0 < L < 1$ such that $\phi(x + y) \ge (1/L)(\phi(x) + \phi(y))$,

for all $x, y \in A$.

Theorem 3.3. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0, \infty)$ for which a mapping $f : X \to Y$ satisfies*

$$\|Df(x, y)\|_Y \le \phi(x, y) \tag{3.15}$$

for all $x, y \in X$ *and the map* ϕ *is contractively subadditive with a constant* L *such that* $2^{1-5\beta}L < 1$. *Then there exists a unique quintic mapping* $Q : X \to Y$ *which satisfies* (1.3) *and the inequality*

$$\|f(x) - Q(x)\|_Y \leq \frac{\phi(x,0)}{4^{\beta}\sqrt[p]{2^{5\beta p} - (2L)^p}}, \tag{3.16}$$

for all $x \in X$.

Proof. By the inequalities (3.5) and (3.9) of the proof of Theorem 3.1, we have

$$\begin{aligned}\left\|\frac{1}{2^{5n}}f(2^n x) - \frac{1}{2^{5d}}f(2^d x)\right\|_Y^p &\leq \sum_{j=n}^{d-1}\left(\frac{1}{2^{5\beta}}\right)^{jp}\left\|f(2^j x) - \frac{1}{2^5}f(2^{j+1}x)\right\|_Y^p \\ &\leq \frac{1}{128^{\beta p}}\sum_{j=n}^{d-1}\left(\frac{1}{2^{5\beta}}\right)^{jp}\phi\left(2^j x, 0\right)^p \\ &\leq \frac{1}{128^{\beta p}}\sum_{j=n}^{d-1}\left(\frac{1}{2^{5\beta}}\right)^{jp}(2L)^{jp}\phi(x,0)^p \\ &= \frac{\phi(x,0)^p}{128^{\beta p}}\sum_{j=n}^{d-1}\left(2^{1-5\beta}L\right)^{jp},\end{aligned} \tag{3.17}$$

that is,

$$\left\|\left(\frac{1}{2^5}\right)^n f(2^n x) - \left(\frac{1}{2^5}\right)^d f(2^d x)\right\|_Y^p \leq \frac{\phi(x,0)^p}{128^{\beta p}}\sum_{j=n}^{d-1}\left(2^{1-5\beta}L\right)^{jp}, \tag{3.18}$$

for all $x \in X$, and for all n and d with $n < d$. Hence $\{(1/2^{5n})f(2^n x)\}$ is a Cauchy sequence in the space Y. Thus we may define

$$Q(x) = \lim_{s\to\infty}\frac{1}{2^{5n}}f(2^n x), \tag{3.19}$$

for all $x \in X$. Now, we will show that the map $Q : X \to Y$ is a generalized quintic mapping. Then

$$\begin{aligned}\|DQ(x,y)\|_Y^p &= \lim_{n\to\infty}\frac{\|Df(2^n x, 2^n y)\|_Y^p}{2^{5\beta pn}} \\ &\leq \lim_{n\to\infty}\frac{\phi(2^n x, 2^n y)^p}{2^{5\beta pn}} \\ &\leq \lim_{n\to\infty}\phi(x,y)^p\left(2^{1-5\beta}L\right)^{pn} \\ &= 0,\end{aligned} \tag{3.20}$$

for all $x \in X$. Hence the mapping Q is a quintic mapping. Note that the inequality (3.18) implies the inequality (3.16) by letting $n = 0$ and taking $d \to \infty$. Assume that there exists $T : X \to Y$ satisfying (1.5) and (3.16). We know that $T(2^n x) = 2^{5n} T(x)$, for all $x \in X$. Then

$$\begin{aligned}\left\|T(x) - \left(\frac{1}{2^5}\right)^n f(2^n x)\right\|_Y^p &= \left(\frac{1}{2^{5\beta}}\right)^{pn} \|T(2^n x) - f(2^n x)\|_Y^p \\ &\leq \left(\frac{1}{2^{5\beta}}\right)^{pn} \frac{\phi(2^n x, 0)^p}{4^{\beta p}(2^{5\beta p} - (2L)^p)} \\ &\leq \left(2^{1-5\beta} L\right)^{pn} \frac{\phi(x,0)^p}{4^{\beta p}(2^{5\beta p} - (2L)^p)},\end{aligned} \tag{3.21}$$

that is,

$$\left\|T(x) - \left(\frac{1}{2^5}\right)^n f(2^n x)\right\|_Y \leq \left(2^{1-5\beta} L\right)^n \frac{\phi(x,0)}{4^\beta \sqrt[p]{(2^{5\beta p} - (2L)^p)}}, \tag{3.22}$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of Q. □

Theorem 3.4. *Suppose that there exists a mapping $\phi : X^2 \to \mathbb{R}^+ := [0, \infty)$ for which a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y)\|_Y \leq \phi(x,y) \tag{3.23}$$

for all $x, y \in X$ and the map ϕ is expansively superadditive with a constant L such that $2^{5\beta-1}L < 1$. Then there exists a unique quintic mapping $Q : X \to Y$ which satisfies (1.3) *and the inequality*

$$\|f(x) - Q(x)\|_Y \leq \frac{\phi(x,0)}{4^\beta L \sqrt[p]{2^p - (2^{5\beta}L)^p}}, \tag{3.24}$$

for all $x \in X$.

Proof. By letting $y = 0$ in (3.23), we have

$$\|4f(2x) - 128(x)\|_Y \leq \phi(x,0), \tag{3.25}$$

and then replacing x by $x/2$,

$$\left\|f(x) - 2^5 f\left(\frac{x}{2}\right)\right\|_Y \leq \frac{1}{128^\beta} \phi\left(\frac{x}{2}, 0\right), \tag{3.26}$$

for all $x \in X$. For all n and d with $n < d$, inductively we have

$$\left\| 2^{5n} f\left(\frac{x}{2^n}\right) - 2^{5d} f\left(\frac{x}{2^d}\right) \right\|_Y^p \leq \frac{\phi(x,0)^p}{128^{\beta p}(2L)^p} \sum_{j=n}^{d-1} \left(2^{5\beta-1} L\right)^{jp}, \tag{3.27}$$

for all $x \in X$. The remains follow from the proof of Theorem 3.3. □

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