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Research Article

A Strong Limit Theorem for Weighted Sums of Sequences of Negatively Dependent Random Variables

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Applying the moment inequality of negatively dependent random variables which was obtained by Asadian et al. (2006), the strong limit theorem for weighted sums of sequences of negatively dependent random variables is discussed. As a result, the strong limit theorem for negatively dependent sequences of random variables is extended. Our results extend and improve the corresponding results of Bai and Cheng (2000) from the i.i.d. case to ND sequences.

1. Introduction and Lemmas

Definition 1.1. Random variables X and Y are said to be negatively dependent (ND) if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y), \tag{1.1}$$

for all $x, y \in \mathrm{R}$. A collection of random variables is said to be pairwise negatively dependent (PND) if every pair of random variables in the collection satisfies (1.1).

It is important to note that (1.1) implies

$$P(X > x, Y > y) \le P(X > x)P(Y > y), \tag{1.2}$$

for all $x, y \in \mathrm{R}$. Moreover, it follows that (1.2) implies (1.1), and hence, (1.1) and (1.2) are equivalent. However, (1.1) and (1.2) are not equivalent for a collection of 3 or more random variables. Consequently, the following definition is needed to define sequences of negatively dependent random variables.

Definition 1.2. The random variables $X_1, \ldots, X_n$ are said to be negatively dependent (ND) if for all real $x_1, \ldots, x_n$,

$$P\left(\bigcap_{j=1}^{n}(X_j \le x_j)\right) \le \prod_{j=1}^{n} P(X_j \le x_j),$$
$$P\left(\bigcap_{j=1}^{n}(X_j > x_j)\right) \le \prod_{j=1}^{n} P(X_j > x_j). \tag{1.3}$$

An infinite sequence of random variables $\{X_n; n \ge 1\}$ is said to be ND if every finite subset $X_1, \ldots, X_n$ is ND.

Definition 1.3. Random variables $X_1, X_2, \ldots, X_n, n \ge 2$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{cov}\left(f_1(X_i; i \in A_1), f_2\left(X_j; j \in A_2\right)\right) \le 0, \tag{1.4}$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable), such that this covariance exists. An infinite sequence of random variables $\{X_n; n \ge 1\}$ is said to be NA if every finite subfamily is NA.

The definition of PND is given by Lehmann [1], the concept of ND is given by Bozorgnia et al. [2], and the definition of NA is introduced by Joag-Dev and Proschan [3]. These concepts of dependent random variables have been very useful in reliability theory and applications.

Obviously, NA implies ND from the definition of NA and ND. But ND does not imply NA, so ND is much weaker than NA. Because of the wide applications of ND random variables, the notions of ND dependence of random variables have received more and more attention recently. A series of useful results have been established (cf: [2, 4–10]). Hence, extending the limit properties of independent or NA random variables to the case of ND variables is highly desirable and of considerably significance in the theory and application.

Strong convergence is one of the most important problems in probability theory. Some recent results can be found in Wu and Jiang [11], Chen and Gan [12], and Bai and Cheng [13]. Bai and Cheng [13] gave the following Theorem.

Theorem 1.4. *Suppose that $1 < \alpha,\ \beta < \infty,\ 1 \le p < 2$, and $1/p = 1/\alpha + 1/\beta$. Let $\{X, X_n; n \ge 1\}$ be a sequence of i.i.d. random variables satisfying $EX = 0$, and let $\{a_{nk}; 1 \le k \le n, n \ge 1\}$ be an array of real constants such that*

$$\limsup_{n \to \infty}\left(\frac{1}{n}\sum_{k=1}^{n}|a_{nk}|^{\alpha}\right)^{1/\alpha} < \infty. \tag{1.5}$$

If $E|X|^{\beta} < \infty$, *then*

$$\lim_{n\to\infty} n^{-1/p} \sum_{k=1}^{n} a_{nk} X_k = 0, \quad \text{a.s.} \tag{1.6}$$

In this paper, we study the strong convergence for negatively dependent random variables. Our results generalize and improve the above Theorem.

In the following, let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n. The symbol c stands for a generic positive constant which may differ from one place to another. And $S_n \widehat{=} \sum_{j=1}^{n} X_j$.

Lemma 1.5 (see [2]). *Let* $X_1, \ldots, X_n$ *be ND random variables and let* $\{f_n; n \geq 1\}$ *be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing), then* $\{f_n(X_n); n \geq 1\}$ *is still a sequence of ND r.v.s.*

Lemma 1.6 (see [14]). *Let* $\{X_n; n \geq 1\}$ *be an ND sequence with* $EX_n = 0$ *and* $E|X_n|^p < \infty$, $p \geq 2$, *then*

$$E|S_n|^p \leq c_p \left\{ \sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\}, \tag{1.7}$$

where $c_p > 0$ *depends only on* p.

The following lemma is known, see, for example, Wu, 2006 [15].

Lemma 1.7. *Let* $\{X_n; n \geq 1\}$ *be an arbitrary sequence of random variables. If there exist an r.v.* X *and a constant* c *such that* $P(|X_n| \geq x) \leq cP(|X| \geq x)$ *for* $n \geq 1$ *and* $x > 0$, *then for any* $u > 0$, $t > 0$, *and* $n \geq 1$,

$$\begin{gathered} E|X_n|^u I_{(|X_n| \leq t)} \leq c\left(E|X|^u I_{(|X| \leq t)} + t^u P(|X| > t)\right), \\ E|X_n|^u I_{(|X_n| > t)} \leq cE|X|^u I_{(|X| > t)}. \end{gathered} \tag{1.8}$$

2. Main Results and Proof

Theorem 2.1. *Suppose that* $\alpha, \beta > 0$, $0 < p < 2$, *and* $1/p = 1/\alpha + 1/\beta$. *Let* $\{X_n; n \geq 1\}$ *be a sequence of ND random variables, there exist an r.v.* X *and a constant* c *satisfying*

$$\begin{gathered} P(|X_n| \geq x) \leq cP(|X| \geq x), \quad \forall n \geq 1,\ x > 0, \\ E|X|^{\beta} < \infty. \end{gathered} \tag{2.1}$$

If $\beta > 1$, *further assume that* $EX_n = 0$. *Let* $\{a_{nk}; 1 \leq k \leq n, n \geq 1\}$ *be an array of real numbers such that*

$$\sum_{k=1}^{n} |a_{nk}|^{\alpha} \ll n, \tag{2.2}$$

then

$$\lim_{n\to\infty} n^{-1/p}\sum_{k=1}^{n} a_{nk}X_k = 0 \quad \text{a.s.} \tag{2.3}$$

Corollary 2.2. *Suppose that $\alpha,\beta > 0$, $0 < p < 2$, and $1/p = 1/\alpha + 1/\beta$. Let $\{X_n; n \geq 1\}$ be a sequence of ND identically distributed random variables with $E|X_1|^\beta < \infty$. If $\beta > 1$, further assume that $EX_1 = 0$. Let $\{a_{nk}; 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers such that* (2.2) *holds, then* (2.3) *holds.*

Taking $a_{nk} \equiv 1$ in Corollary 2.2, then (2.2) is always valid for any $\alpha > 0$. Hence, for any $0 < p < \min(\beta, 2)$, letting $\alpha = p\beta/(\beta - p) > 0$, we can obtain the following corollary.

Corollary 2.3. *Let $\{X_n; n \geq 1\}$ be a sequence of ND identically distributed random variables with $E|X_1|^\beta < \infty$. If $\beta > 1$, further assume that $EX_1 = 0$, then for any $0 < p < \min(\beta, 2)$,*

$$\lim_{n\to\infty} n^{-1/p}\sum_{k=1}^{n} X_k = 0, \quad \text{a.s.} \tag{2.4}$$

Remark 2.4. Theorem 2.1 improves and extends Theorem 1.4 of Bai and Cheng [13] for i.i.d. case to ND random variables, removes the identically distributed condition, and expands the ranges α, β, and p, respectively.

Proof of Theorem 2.1. For any $\gamma > 0$, by (2.2), the Hölder inequality and the c_r inequality, we have

$$\sum_{k=1}^{n}|a_{nk}|^\gamma \leq \begin{cases} \left(\sum_{k=1}^{n}|a_{nk}|^\alpha\right)^{\gamma/\alpha}\left(\sum_{k=1}^{n}1\right)^{1-\gamma/\alpha} \ll n, \\ \left(\sum_{k=1}^{n}|a_{nk}|^\alpha\right)^{\gamma/\alpha} \ll n^{\gamma/\alpha} \end{cases} = n^{\max(1,\gamma/\alpha)}. \tag{2.5}$$

For any $1 \leq k \leq n,\ n \geq 1$, let

$$\begin{aligned} Y_k &\widehat{=} -n^{1/\beta} I_{(X_k < -n^{1/\beta})} + X_k I_{(|X_k| \leq n^{1/\beta})} + n^{1/\beta} I_{(X_k > n^{1/\beta})}, \\ Z_k &\widehat{=} X_k - Y_k = \left(X_k + n^{1/\beta}\right) I_{(X_k < -n^{1/\beta})} + \left(X_k - n^{1/\beta}\right) I_{(X_k > n^{1/\beta})}. \end{aligned} \tag{2.6}$$

Then

$$\begin{aligned} n^{-1/p}\sum_{k=1}^{n} a_{nk}X_k &= n^{-1/p}\sum_{k=1}^{n} a_{nk}Z_k + n^{-1/p}\sum_{k=1}^{n} a_{nk}EY_k + n^{-1/p}\sum_{k=1}^{n} a_{nk}(Y_k - EY_k) \\ &\widehat{=} I_{n1} + I_{n2} + I_{n3}. \end{aligned} \tag{2.7}$$

By (2.1),

$$\sum_{k=1}^{\infty} P(Z_k \neq 0) = \sum_{k=1}^{\infty} P\left(|X_k| > k^{1/\beta}\right) \ll \sum_{k=1}^{\infty} P\left(|X| > k^{1/\beta}\right) \ll E|X|^{\beta} < \infty. \tag{2.8}$$

Hence, by the Borel-Cantelli lemma, we can get $P(Z_k \neq 0, \text{i.o.}) = 0$. It follows that from (2.2)

$$\begin{aligned}
|I_{n1}| &= n^{-1/p}\left|\sum_{k=1}^{n} a_{nk} Z_k\right| \leq n^{-1/p}\left(\max_{1\leq k\leq n}|a_{nk}|^{\alpha}\right)^{1/\alpha}\sum_{k=1}^{n}|Z_k| \\
&\leq n^{-1/p}\left(\sum_{k=1}^{n}|a_{nk}|^{\alpha}\right)^{1/\alpha}\sum_{k=1}^{n}|Z_k| \\
&\ll n^{-1/\beta}\sum_{k=1}^{n}|Z_k| \longrightarrow 0, \quad \text{a.s.}
\end{aligned} \tag{2.9}$$

If $0 < \beta \leq 1$, by (2.1), (2.5), the Markov inequality, and Lemma 1.7, we have

$$\begin{aligned}
|I_{n2}| &= n^{-1/p}\left|\sum_{k=1}^{n} a_{nk} EY_k\right| \leq n^{-1/p}\sum_{k=1}^{n}|a_{nk}|E|X_k|I_{(|X_k|\leq n^{1/\beta})} + n^{-1/\alpha}\sum_{k=1}^{n}|a_{nk}|P\left(|X_k| > n^{1/\beta}\right) \\
&\ll n^{-1/p}\sum_{k=1}^{n}|a_{nk}|\left(E|X|^{\beta} n^{(1-\beta)/\beta} I_{(|X|\leq n^{1/\beta})} + n^{1/\beta} P\left(|X| > n^{1/\beta}\right)\right) + n^{-1/\alpha}\sum_{k=1}^{n}|a_{nk}|\frac{E|X|^{\beta}}{n} \\
&\ll n^{-1/\alpha-1+\max(1/\alpha,1)} \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{2.10}$$

If $\beta > 1$, once again, using (2.1), (2.5), $EX_k = 0$, the Markov inequality, and Lemma 1.7, we get

$$\begin{aligned}
|I_{n2}| &= n^{-1/p}\left|\sum_{k=1}^{n} a_{nk} EY_k\right| \leq n^{-1/p}\sum_{k=1}^{n}\left(\left|a_{nk}EX_k I_{(|X_k|\leq n^{1/\beta})}\right| + n^{1/\beta}|a_{nk}|P\left(|X_k| > n^{1/\beta}\right)\right) \\
&= n^{-1/p}\sum_{k=1}^{n}\left(\left|a_{nk}EX_k I_{(|X_k|> n^{1/\beta})}\right| + n^{1/\beta}|a_{nk}|P\left(|X_k| > n^{1/\beta}\right)\right) \\
&\ll n^{-1/p}\sum_{k=1}^{n}\left(|a_{nk}|E|X| I_{(|X|> n^{1/\beta})} + n^{1/\beta}|a_{nk}|P\left(|X| > n^{1/\beta}\right)\right) \\
&\leq n^{-1/p}\sum_{k=1}^{n}|a_{nk}|E|X|\left(\frac{|X|}{n^{1/\beta}}\right)^{\beta-1} I_{(|X|> n^{1/\beta})} + n^{-1/\alpha}\sum_{k=1}^{n}|a_{nk}|\frac{E|X|^{\beta}}{n} \\
&\ll n^{-1/\alpha-1+\max(1/\alpha,1)} \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{2.11}$$

Combining with (2.10), we get

$$I_{n2} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.12}$$

Obviously, $Y_k, k \le n$ are monotonic on X_k. By Lemma 1.5, $\{Y_k; k \ge 1\}$ is also a sequence of ND random variables. Choose q such that $q > 1/\min\{1/2, 1/\alpha, 1/\beta, 1/p - 1/2\}$, by the Markov inequality and Lemma 1.6, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(n^{-1/p}\left|\sum_{k=1}^{n} a_{nk}(Y_k - EY_k)\right| > \varepsilon\right) \\
&\quad \ll \sum_{n=1}^{\infty} n^{-q/p} E\left|\sum_{k=1}^{n} a_{nk}(Y_k - EY_k)\right|^q \\
&\quad \ll \sum_{n=1}^{\infty} n^{-q/p} \sum_{k=1}^{n} E|a_{nk}(Y_k - EY_k)|^q + \sum_{n=1}^{\infty} n^{-q/p}\left(\sum_{k=1}^{n} a_{nk}^2 E(Y_k - EY_k)^2\right)^{q/2} \\
&\quad \widehat{=} J_1 + J_2.
\end{aligned} \tag{2.13}$$

By the c_r inequality, (2.1), (2.5), and Lemma 1.7, we have

$$\begin{aligned}
J_1 &\ll \sum_{n=1}^{\infty} n^{-q/p} \sum_{k=1}^{n} |a_{nk}|^q \left(E|X_k|^q I_{(|X_k| \le n^{1/\beta})} + n^{q/\beta} P\left(|X_k| > n^{1/\beta}\right)\right) \\
&\ll \sum_{n=1}^{\infty} n^{-q/p + q/\alpha}\left(E|X|^q I_{(|X| \le n^{1/\beta})} + n^{q/\beta} P\left(|X| > n^{1/\beta}\right)\right) \\
&\ll \sum_{n=1}^{\infty} n^{-q/\beta} \sum_{i=1}^{n} E|X|^q I_{((i-1)^{1/\beta} < |X| \le i^{1/\beta})} + \sum_{n=1}^{\infty} P\left(|X| > n^{1/\beta}\right) \\
&\ll \sum_{i=1}^{\infty} E|X|^q I_{((i-1)^{1/\beta} < |X| \le i^{1/\beta})} \sum_{n=i}^{\infty} n^{-q/\beta} + E|X|^{\beta} \\
&\ll \sum_{i=1}^{\infty} i^{1-q/\beta} E|X|^q I_{((i-1)^{1/\beta} < |X| \le i^{1/\beta})} \\
&\ll \sum_{i=1}^{\infty} E|X|^{\beta} I_{((i-1)^{1/\beta} < |X| \le i^{1/\beta})} \ll E|X|^{\beta} \\
&< \infty.
\end{aligned} \tag{2.14}$$

Next, we prove that $J_2 < \infty$. By (2.5),

$$\sum_{k=1}^{n} a_{nk}^2 \ll \begin{cases} n, & \alpha \geq 2, \\ n^{2/\alpha}, & \alpha < 2. \end{cases} \tag{2.15}$$

And by the Markov inequality,

$$EX^2 I_{(|X| \leq n^{1/\beta})} + n^{2/\beta} P\left(|X| > n^{1/\beta}\right) \leq \begin{cases} E|X|^\beta n^{(1/\beta)(2-\beta)} + n^{2/\beta} n^{-1} E|X|^\beta \ll n^{2/\beta - 1}, & \beta < 2, \\ \ll EX^2 < \infty, & \beta \geq 2. \end{cases} \tag{2.16}$$

By the c_r inequality, the Markov inequality, and Lemma 1.7, combining with (2.15), we get

$$\begin{aligned} \sum_{k=1}^{n} a_{nk}^2 E(Y_k - EY_k)^2 &\ll \sum_{k=1}^{n} a_{nk}^2 \left(EX^2 I\left(|X| \leq n^{1/\beta}\right) + n^{2/\beta} P\left(|X| > n^{1/\beta}\right)\right) \\ &\ll \begin{cases} n^{-1+2/p}, & \alpha < 2,\ \beta < 2, \\ n^{2/\alpha}, & \alpha < 2,\ \beta \geq 2, \\ n^{2/\beta}, & \alpha \geq 2,\ \beta < 2, \\ n, & \alpha \geq 2,\ \beta \geq 2 \end{cases} \\ &\leq n^t, \end{aligned} \tag{2.17}$$

where $t = \max\{-1 + 2/p, 2/\alpha, 2/\beta, 1\}$. Hence, we can obtain the following:

$$J_2 \ll \sum_{n=1}^{\infty} n^{(-1/p + t/2)q} < \infty, \tag{2.18}$$

from $(-(1/p) + (t/2))q = q \cdot \max(-1/2, -1/\beta, -1/\alpha, 1/2 - 1/p) = -q \cdot \min(1/2, 1/\beta, 1/\alpha, 1/p - 1/2 < -1$. By (2.13), (2.14), (2.15), and the Borel-Cantelli lemma,

$$I_{n3} = n^{-1/p} \sum_{k=1}^{n} a_{nk}(Y_k - EY_k) \longrightarrow 0, \quad \text{a.s. } n \longrightarrow \infty. \tag{2.19}$$

Together with (2.7), (2.9), (2.12), and (2.3) holds. □

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