Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 417452, 12 pages
doi:10.1155/2010/417452

Research Article

A Note on Nörlund-Type Twisted q-Euler Polynomials and Numbers of Higher Order Associated with Fermionic Invariant q-Integrals

Leechae Jang

Department of Mathematics and Computer Science, KonKuk University, Seol 143-701, Republic of Korea

Correspondence should be addressed to Leechae Jang, leechae.jang@kku.ac.kr

Received 4 April 2010; Accepted 20 June 2010

Academic Editor: Vijay Gupta

We define multiple Nörlund-type twisted q-Euler polynomials and numbers and give interpolation functions of multiple Nörlund-type twisted q-Euler polynomials at negative integers. Furthermore, we investigate some identities related to these polynomials and interpolation functions.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will be, respectively, the ring of p-adic rational integers, the field of p-adic rational numbers, and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1-q|_p < p^{1/(1-p)}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.1}$$

compared with [1–22], for all $x \in \mathbb{Z}_p$.

For a fixed odd positive integer d with $(p, d) = 1$, set

$$X = X_d = \varprojlim_{n} \mathbb{Z}/dp^n\mathbb{Z}, \qquad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \tag{1.2}$$
$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a (\mathrm{mod}\, dp^n)\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^n$. For any $n \in \mathbb{N}$,

$$\mu_q(a + dp^n\mathbb{Z}_p) = \frac{q^a}{[dp^n]_q} \tag{1.3}$$

is known to be a distribution on $\mathbb{Z}_p$, compared with [1–22].

The q-factorial is defined as $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$, and the Gaussian binomial coefficient is also defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!} \tag{1.4}$$

(see [7, 8]).

Note that

$$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{1.5}$$

From (1.4), we note that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q = q^{n-1}\binom{n}{k-1}_q + \binom{n}{k}_q \tag{1.6}$$

(see [7, 8]).

The q-binomial formulae are known as

$$(b:q)_n = (1-b)(1-bq)\cdots\left(1-bq^{n-1}\right) = \sum_{i=0}^{n}\binom{n}{i}_q q^{\binom{i}{2}}(-1)^i b^i,$$
$$\frac{1}{(b:q)_n} = \frac{1}{(1-b)(1-bq)\cdots(1-bq^{n-1})} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}_q b^i. \tag{1.7}$$

The Euler number E_n and polynomials $E_n(x)$ are defined by the generating function in the complex number field as

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\quad (|t|<\pi),$$
$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad (|t|<\pi). \tag{1.8}$$

The nth q-Euler numbers $E_{n,q}$ and the nth q-Euler polynomials $E_{n,q}(x)$ attached to q are defined by the exponential generating functions as

$$F_q(t)=2\sum_{k=0}^{\infty}(-1)^k e^{[k]_q t}=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!},$$
$$F_q(t,x)=2\sum_{k=0}^{\infty}(-1)^k e^{[k+x]_q t}=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}. \tag{1.9}$$

The nth Euler numbers $E_n^{(r)}$ of higher order and the nth Euler polynomials $E_n^{(r)}(x)$ of higher order attached to q are defined by the exponential generating functions as

$$F^{(r)}(t)=\frac{2^r}{(1+e^t)^r}=\sum_{k=0}^{\infty}E_n^{(r)}\frac{t^n}{n!}, \tag{1.10}$$

$$F^{(r)}(t,x)=\frac{2^r}{(1+e^t)^r}e^{xt}=\sum_{k=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}, \tag{1.11}$$

$$F^{(-r)}(t,x)=\frac{(1+e^t)^r}{2^r}e^{xt}=\sum_{k=0}^{\infty}E_n^{(-r)}(x)\frac{t^n}{n!}. \tag{1.11 − 1}$$

Kim [7] defined the nth q-Euler numbers $E_{n,q}^{(r)}$ of higher order, the Euler polynomials $E_{n,q}^{(r)}(x)$ of higher order, and the nth Nörlund-type q-Euler polynomials of higher order which are defined by the exponential generating functions as

$$F_q^{(r)}(t)=2^r\sum_{m=0}^{\infty}(-1)^m\binom{m+r-1}{m}_q e^{[m]_q t}=\sum_{n=0}^{\infty}E_{n,q}^{(r)}\frac{t^n}{n!},$$
$$F_q^{(r)}(t,x)=2^r\sum_{m=0}^{\infty}(-1)^m\binom{m+r-1}{m}_q e^{[m+x]_q t}=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!}, \tag{1.12}$$
$$F_q^{(-r)}(t,x)=\sum_{n=0}^{\infty}E_{n,q}^{(-r)}(x)\frac{t^n}{n!},$$

compared with [6–16, 18–21].

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in \mathrm{UD}(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \tag{1.13}$$

have a limit $l = f'(a)$ as $(x, y) \to (a, a)$, compared with [1–22] (23-24). Note that the bosonic p-adic q-integral of a function $f \in \mathrm{UD}(\mathbb{Z}_p)$ was defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{x=0}^{p^n-1} f(x) q^x \tag{1.14}$$

and that the fermionic p-adic q-integral was defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{x=0}^{p^n-1} f(x) (-q)^x \tag{1.15}$$

(see [1–22] (23-24)). In (1.15), when $q \to 1$, we can obtain

$$L_{-1}(f_1) + L_{-1}(f) = 2f(0), \tag{1.16}$$

where $f_1(x) = f(x + 1)$. If we take $f(x) = e^{tx}$, then we obtain

$$I_{-1}(e^{tx}) = \int_{\mathbb{Z}_p} e^{tx} d\mu_{-1}(x) = \frac{2}{e^t + 1}. \tag{1.17}$$

In this paper, we define multiple Nörlund-type twisted q-Euler polynomials and give interpolation functions of multiple Nörlund-type twisted q-Euler polynomials at negative integers. Furthermore, we investigate some identities related to these polynomials and interpolation functions.

2. Nörlund-Type Twisted q-Euler Numbers and Polynomials of Higher Order

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Let $C_{p^\infty} = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n}$ be the locally constant space, when $C_{p^n} = \{\xi \in X \mid \xi^{p^n} = 1\}$ is the cyclic group of order p^n. Let $\xi \in C_{p^\infty}$. We define the twisted q-Euler polynomials (see [1–5, 18–21]) as follows:

$$\begin{aligned} E^{(r)}_{m,q,\xi}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_r]_q^n \xi^{x_1+\cdots+x_r} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1+q^l \xi^l} \right)^r \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m [m+x]^n_{q,\xi^m}, \end{aligned} \tag{2.1}$$

where $[a]_{q,\xi} = (1-q^a\xi)/(1-q)$. Let $F^{(r)}_{q,\xi}(t,x) = \sum_{n=0}^{\infty} E^{(r)}_{n,q,\xi}(x)(t^n/n!)$. Then we have

$$F^{(r)}_{q,\xi}(t,x) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m e^{[m+x]_{q,\xi^m} t}. \tag{2.2}$$

In this special case $x = 0$, $E^{(r)}_{n,q,\xi}(0) = E^{(r)}_{n,q,\xi}$ are called the twisted q-Euler numbers of order r. In the sense of the twisted in $(1.11-1)$, we consider the Nörlund-type twisted q-Euler polynomials as follows:

$$G^{(r)}_{q,\xi}(t,x) = F^{(-r)}_{q,\xi}(t,x) = \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} e^{[m+x]_{q,\xi^m} t} = \sum_{n=0}^{\infty} E^{(-r)}_{n,q,\xi}(x) \frac{t^n}{n!}. \tag{2.3}$$

By (2.3), we have

$$E^{(-r)}_{n,q,\xi}(x) = \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} [m+x]^n_{q,\xi^m}. \tag{2.4}$$

From (2.1) and (2.4), we can obtain the following theorem.

Theorem 2.1. *For $r \in \mathbb{N}$, $n \geq 0$, and $\varepsilon \in T_p$, let*

$$2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-\xi)^m e^{[m+x]_{q,\xi^m} t} = \sum_{n=0}^{\infty} E^{(r)}_{n,q,\xi}(x) \frac{t^n}{n!}. \tag{2.5}$$

Then

$$\begin{aligned} E^{(r)}_{n,q,\xi}(x) &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(\frac{1}{1+q^l\xi^l}\right)^r \\ &= 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m[m+x]^n_{q,\xi^m}, \\ E^{(-r)}_{n,q,\xi}(x) &= \frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}q^{lx}\left(1+q^l\xi^l\right)^r \\ &= \frac{1}{2^r}\sum_{m=0}^{r}\binom{r}{m}[m+x]^n_{q,\xi^m}. \end{aligned} \tag{2.6}$$

$E^{(-r)}_{n,q,\xi}(0) = E^{(-r)}_{n,q,\xi}$ are called the twisted q-Euler numbers of higher order. For $h \in \mathbb{Z}$, $r \in \mathbb{N}$, let us define the twisted q-Euler polynomials of higher order as follows:

$$E^{(h,r)}_{n,q,\xi}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(h-j)x_j}\xi^{\sum_{j=1}^{r}x_j}[x+x_1+\cdots+x_r]^n_q d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{2.7}$$

Then

$$\begin{aligned} E^{(h,r)}_{n,q,\xi}(x) &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{\left(1/(1+q^l\xi^l)\right)_r} \\ &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{\left(-q^{h-1+l}\xi^l;q^{-1}\right)_r} \\ &= 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q q^{(h-r)m}(-1)^m[x+m]^n_{q,\xi^m}. \end{aligned} \tag{2.8}$$

Let $F^{(h,r)}_{q,\xi}(t,x) = \sum_{m=0}^{\infty}E^{(h,r)}_{n,q,\xi}(x_1)(t^n/n!)$. By (2.7), we easily see that

$$E^{(h,r)}_{n,q,\xi}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(h-j)x_j}\xi^{\sum_{j=1}^{r}x_j}[x+x_1+\cdots+x_r]^n_q d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{2.9}$$

Thus, we obtain the following theorem.

Theorem 2.2. *For $h \in \mathbb{Z}$, $r \in \mathbb{N}$, $n \geq 0$, and $\varepsilon \in T_p$, let*

$$2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q q^{(h-r)m}(-1)^m e^{[m+x]_{q,\xi^m}t} = \sum_{n=0}^{\infty}E^{(h,r)}_{n,q,\xi}(x)\frac{t^n}{n!}. \tag{2.10}$$

Then

$$E_{n,q,\xi}^{(h,r)}(x) = \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{(-qh^{h-r+l};q)_r}$$
$$= 2^r\sum_{m=0}^{\infty} q^{(h-r)m}(-1)^m\binom{m+r-1}{m}_q(-1)^m[m+x]_{q,\xi^m}^n. \tag{2.11}$$

Now, we define the Nörlund-type twisted q-Euler polynomials of higher order as follows:

$$E_{n,q,\xi}^{(h,-r)}(x) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{l(x_1+\cdots+x_r)}q^{\sum_{j=1}^{r}(h-j)x_j}\xi^{l(x_1+\cdots+x_r)}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)}. \tag{2.12}$$

Let $F_{q,\xi}^{(h,-r)}(t,x) = \sum_{m=0}^{\infty} E_{n,q,\xi}^{(h,-r)}(x_1)(t^n/n!)$. By (2.12), we have

$$F_{q,\xi}^{(h,-r)}(t,x) = \frac{1}{2^r}\sum_{m=0}^{r} q^{\binom{m}{2}}q^{(h-r)m}\binom{r}{m}_q e^{[m+x]_{q,\xi^m}t}. \tag{2.13}$$

Thus, we obtain the following theorem.

Theorem 2.3. *For $h \in \mathbb{Z}$, $r \in \mathbb{N}$, $n \geq 0$, and $\varepsilon \in T_p$, let*

$$\frac{1}{2^r}\sum_{m=0}^{r} q^{\binom{m}{2}}q^{(h-r)m}\binom{r}{m}_q e^{[m+x]_{q,\xi^m}t} = \sum_{n=0}^{\infty} E_{n,q,\xi}^{(h,-r)}(x)\frac{t^n}{n!}. \tag{2.14}$$

Then

$$E_{n,q,\xi}^{(h,-r)}(x) = \frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(-q^{h-r+l}\xi^l;q\right)_r$$
$$= \frac{1}{2^r}\sum_{m=0}^{r} q^{\binom{m}{2}}q^{(h-r)m}\binom{r}{m}_q[m+x]_{q,\xi^m}^n. \tag{2.15}$$

For $h = r$, we have

$$E_{n,q,\xi}^{(r,r)}(x) = \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{(-qh^l;q)_r} = 2^r\sum_{m=0}^{\infty}(-1)^m\binom{m+r-1}{m}_q(-1)^m[m+x]_{q,\xi^m}^n, \tag{2.16}$$

$$E_{n,q,\xi}^{(r,-r)}(x) = \frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(-q^l\xi^l;q\right)_r = \frac{1}{2^r}\sum_{m=0}^{r}q^{\binom{m}{2}}q^{(h-r)m}\binom{r}{m}_q[m+x]_{q,\xi^m}^n. \tag{2.17}$$

Thus, it is easy to see that

$$\begin{aligned}\frac{q^{mx}2^r}{(-q^{m-rl};q)_r} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{j=1}^{r}(m-j)x_j+mx}\xi^{\sum_{j=1}^{r}x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([x+x_1+\cdots+x_r]_q(q-1)+1\right)^m q^{\sum_{j=1}^{r}x_j}\xi^{\sum_{j=1}^{r}x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \sum_{l=0}^{m}\binom{m}{l}(q-1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^l q^{\sum_{j=1}^{r}x_j}\xi^{\sum_{j=1}^{r}x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \sum_{l=0}^{m}\binom{m}{l}(q-1)^l E_{l,q,\xi}^{(0,r)}(x).\end{aligned} \tag{2.18}$$

Equation (2.18) implies that

$$\frac{q^{mx}2^r}{(-q^{m-r}\xi;q)_r} = \sum_{l=0}^{m}\binom{m}{l}(q-1)^l E_{l,q,\xi}^{(0,r)}(x). \tag{2.19}$$

From (1.16), we derive

$$\begin{aligned}&q^{h-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+1+x_1+\cdots+x_r]_q^m q^{\sum_{j=1}^{r}(h-j)x_j}\xi^{\sum_{j=1}^{r}x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\quad= -\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+1+x_1+\cdots+x_r]_q^m q^{\sum_{j=1}^{r}(h-j)x_j}\xi^{\sum_{j=1}^{r}x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\qquad+2\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+1+x_1+\cdots+x_r]_q^m q^{\sum_{j=1}^{r-1}(h-1-j)x_{j+1}}\xi^{\sum_{j=1}^{r-1}x_{j+1}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).\end{aligned} \tag{2.20}$$

By (2.20), we have

$$q^{h-1}E_{n,q,\xi}^{(h,r)}(x+1)+E_{n,q,\xi}^{(h,r)}(x)=2E_{n,q,\xi}^{(h-1,r-1)}(x). \tag{2.21}$$

By simple calculation, we see that

$$\begin{aligned}
&q^{x}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r+x]_q^n q^{\sum_{j=1}^{r}(h-j+1)x_j}\xi^{\sum_{j=1}^{r}x_j}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&\quad=(q-1)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r+x]_q^{n+1} q^{\sum_{j=1}^{r}(h-j)x_j}\xi^{\sum_{j=1}^{r}x_j}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&\qquad+\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r+x]_q^{m} q^{\sum_{j=1}^{r}(h-j)x_j}\xi^{\sum_{j=1}^{r}x_j}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).
\end{aligned} \tag{2.22}$$

By (2.22), we see that

$$q^{x}E_{n,q,\xi}^{(h+1,r)}(x)=(q-1)E_{n+1,q,\xi}^{(h,r)}(x)+E_{n,q,\xi}^{(h,r)}(x). \tag{2.23}$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $h\in\mathbb{Z}$, $r\in\mathbb{N}$, $n\geq 0$, and $\varepsilon\in T_p$,*

$$\begin{aligned}
&q^{h-1}E_{n,q,\xi}^{(h,r)}(x+1)+E_{n,q,\xi}^{(h,r)}(x)=2E_{n,q,\xi}^{(h-1,r-1)}(x)\ ,\\
&q^{x}E_{n,q,\xi}^{(h+1,r)}(x)=(q-1)E_{n+1,q,\xi}^{(h,r)}(x)+E_{n,q,\xi}^{(h,r)}(x).
\end{aligned} \tag{2.24}$$

Moreover,

$$\frac{q^{mx}2^r}{(-q^{m-r}\xi;q)_r}=\sum_{l=0}^{m}\binom{m}{l}(q-1)^l E_{l,q,\xi}^{(0,r)}(x). \tag{2.25}$$

From (2.16), we note that

$$\begin{aligned}
E_{n,q^{-1},\xi}^{(r,r)}(r-x)&=\frac{2^r}{(1-q^{-1})^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{l(r-x)}}{(-q^{-l}\xi;q^{-1})_r}\\
&=(-1)^n q^{n+\binom{r}{2}}\frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{(-q^{l}\xi;q)_r}\\
&=(-1)^n q^{n+\binom{r}{2}}E_{n,q,\xi}^{(r,r)}(x).
\end{aligned} \tag{2.26}$$

In the case $x=r$, we obtain

$$E_{n,q^{-1},\xi}^{(r,r)}(0)=(-1)^n q^{n+\binom{r}{2}}E_{n,q,\xi}^{(r,r)}(r). \tag{2.27}$$

From (2.21) with $h = r$, we derive

$$q^{r-1}E_{n,q,\xi}^{(r,r)}(x+1) + E_{n,q,\xi}^{(r,r)}(x) = 2E_{n,q,\xi}^{(r-1,r-1)}(x). \tag{2.28}$$

3. Further Remarks on Nörlund-Type Twisted q-Euler Polynomials

In the case $h = 0$, let us consider the polynomials $E_{n,q,\xi}^{(0,r)}(x)$ and $E_{n,q,\xi}^{(0,-r)}(x)$ as follows:

$$\begin{aligned} E_{n,q,\xi}^{(0,r)}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-\sum_{j=1}^{r} jx_j}\xi^{\sum_{j=1}^{r} x_j}[x+x_1+\cdots+x_r]_q^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r), \\ E_{n,q,\xi}^{(0,-r)}(x) &= \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{l(x_1+\cdots+x_r)} q^{-\sum_{j=1}^{r} jx_j}\xi^{l(x_1+\cdots+x_r)} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)}. \end{aligned} \tag{3.1}$$

Then we have

$$\begin{aligned} E_{n,q,\xi}^{(0,r)}(x) &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{(-q^{l-r}\xi;q)_r} \\ &= 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q q^{-rm}(-1)^m[x+m]_{q,\xi^m}^n, \\ E_{n,q,\xi}^{(0,-r)}(x) &= \frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(-q^{l-r}\xi^l;q\right)_r \\ &= \frac{1}{2^r}\sum_{m=0}^{r} q^{\binom{m}{2}} q^{(h-r)m}\binom{r}{m}_q [m+x]_{q,\xi^m}^n. \end{aligned} \tag{3.2}$$

Let us consider the following polynomials:

$$E_{n,q,\xi}^{(h,1)}(x) = \int_{\mathbb{Z}_p} q^{x_1(h-1)}\xi^{x_1}[x+x_1]_q^n d\mu_{-1}(x_1) = \frac{2}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{1+q^{l+h-1}\xi^l}. \tag{3.3}$$

By the simple calculation of fermionic p-adic invariant integral, we see that

$$\begin{aligned} q^x\int_{\mathbb{Z}_p} q^{x_1(h-1)}\xi^{x_1}[x+x_1]_q^n d\mu_{-1}(x_1) &= (q-1)\int_{\mathbb{Z}_p} q^{x_1(h-2)}\xi^{x_1}[x+x_1]_q^{n+1} d\mu_{-1}(x_1) \\ &\quad + \int_{\mathbb{Z}_p} q^{x_1(h-2)}\xi^{x_1}[x+x_1]_q^n d\mu_{-1}(x_1). \end{aligned} \tag{3.4}$$

Thus

$$q^x E_{n,q,\xi}^{(h,1)}(x) = (q-1)E_{n+1,q,\xi}^{(h-1,1)}(x) + E_{n,q,\xi}^{(h-1,1)}(x). \tag{3.5}$$

It is easy to see that

$$\int_{\mathbb{Z}_p} q^{(h-1)x_1}\xi^{x_1}[x+x_1]_q^n d\mu_{-1}(x_1) = \sum_{j=1}^{n}\binom{n}{j}[x]_q^{n-j}q^{jx}\int_{\mathbb{Z}_p}[x_1]_q^j q^{(h-1)x_1}\xi^{x_1}d\mu_{-1}(x_1). \tag{3.6}$$

By (2.20), we can obtain that

$$E_{n,q,\xi}^{(h,1)}(x) = \sum_{j=0}^{n}\binom{n}{j}[x]_q^{n-j}q^{jx}E_{j,q,\xi}^{(h,1)} = \left(q^x E_{q,\xi}^{(h,1)} + [x]_q\right)^n, \quad n \geq 0, \tag{3.7}$$

where we use the technique method notation by replacing $(E_{q,\xi}^{(h,1)})^n$ by $E_{n,q,\xi}^{(h,1)}$, symbolically. From (1.14), we can also derive

$$\int_{\mathbb{Z}_p} q^{(h-1)(x_1+1)}\xi^{x_1+1}[x+x_1+1]_q^n d\mu_{-1}(x_1) + \int_{\mathbb{Z}_p}[x+x_1]_q^n q^{(h-1)x_1}\xi^{x_1}d\mu_{-1}(x_1) = 2[x]_q^n. \tag{3.8}$$

Thus, we obtain that

$$q^{h-1}E_{n,q,\xi}^{(h,1)}(x+1) + E_{n,q,\xi}^{(h,1)}(x) = 2[x]_q^n. \tag{3.9}$$

For $x = 0$, we have

$$q^{h-1}\left(E_{n,q,\xi}^{(h,1)} + 1\right)^n + E_{n,q,\xi}^{(h,1)} = 2\delta_{n,0}, \tag{3.10}$$

where $\delta_{n,0}$ is the Kronecker symbol. It is easy to see that

$$E_{0,q,\xi}^{(h,1)} = \int_{\mathbb{Z}_p} q^{x_1(h-1)}\xi^{x_1}d\mu_{-1}(x_1) = \frac{2}{1+q^{h-1}\xi} = \frac{2}{[2]_{q^{h-1},\xi}}. \tag{3.11}$$

By (3.3), we see that

$$\begin{aligned} E_{n,q,\xi}^{(h,1)}(1-x) &= \int_{\mathbb{Z}_p}[1-x+x_1]_{q^{-1}}^n q^{-x_1(h-1)}\xi^{x_1}d\mu_{-1}(x_1) \\ &= (-1)^n q^{n+h-1}\frac{2}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{1+q^{l+h-1}\xi^l} \\ &= (-1)^n q^{n+h-1}E_{n,q,\xi}^{(h,1)}(x). \end{aligned} \tag{3.12}$$

In particular, for $x = 1$, we obtain that

$$E_{n,q,\xi}^{(h,1)}(0) = (-1)^n q^{n+h-1}E_{n,q,\xi}^{(h,1)}(1) = (-1)^{n-1}q^n E_{n,q,\xi}^{(h,1)}. \tag{3.13}$$

Acknowledgment

This paper was supported by Konkuk University in 2010.

References

[1] M. Can, M. Cenkci, V. Kurt, and Y. Simsek, "Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 135–160, 2009.

[2] M. Cenkci, "The p-adic generalized twisted (h, q)-Euler-l-function and its applications," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 1, pp. 37–47, 2007.

[3] L.-C. Jang and C.-S. Ryoo, "A note on the multiple twisted Carlitz's type q-Bernoulli polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 498173, 7 pages, 2008.

[4] L.-C. Jang, "Multiple twisted q-Euler numbers and polynomials associated with p-adic q-integrals," *Advances in Difference Equations*, vol. 2008, Article ID 738603, 11 pages, 2008.

[5] L.-C. Jang, "A study on the distribution of twisted q-Genocchi polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 181–189, 2009.

[6] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[7] T. Kim, "Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.

[8] T. Kim, "Note on the Euler q-zeta functions," *Journal of Number Theory*, vol. 129, no. 7, pp. 1798–1804, 2009.

[9] T. Kim, "The modified q-Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 2, pp. 161–170, 2008.

[10] T. Kim, "Note on Dedekind type DC sums," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 249–260, 2009.

[11] T. Kim, "On Euler-Barnes multiple zeta functions," *Russian Journal of Mathematical Physics*, vol. 10, no. 3, pp. 261–267, 2003.

[12] T. Kim, "On a q-analogue of the p-adic log gamma functions and related integrals," *Journal of Number Theory*, vol. 76, no. 2, pp. 320–329, 1999.

[13] T. Kim, "On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 779–792, 2007.

[14] T. Kim, "On p-adic interpolating function for q-Euler numbers and its derivatives," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 598–608, 2008.

[15] T. Kim, "On the multiple q-Genocchi and Euler numbers," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 481–486, 2008.

[16] T. Kim, "Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 1, pp. 93–96, 2009.

[17] Y.-H. Kim, W. Kim, and C. S. Ryoo, "On the twisted q-Euler zeta function associated with twisted q-Euler numbers," *Proceedings of the Jangjeon Mathematical Society*, vol. 12, no. 1, pp. 93–100, 2009.

[18] S.-H. Rim, S. J. Lee, E. J. Moon, and J. H. Jin, "On the q-Genocchi numbers and polynomials associated with q-zeta function," *Proceedings of the Jangjeon Mathematical Society*, vol. 12, no. 3, pp. 261–267, 2009.

[19] T. Kim, L.-C. Jang, and H. K. Pak, "A note on q-Euler and Genocchi numbers," *Proceedings of the Japan Academy, Series A*, vol. 77, no. 8, pp. 139–141, 2001.

[20] T. Kim, M.-S. Kim, L. Jang, and S.-H. Rim, "New q-Euler numbers and polynomials associated with p-adic q-integrals," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 243–252, 2007.

[21] S.-H. Rim and T. Kim, "A note on p-adic Euler measure on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 358–361, 2006.

[22] Y. Simsek, O. Yurekli, and V. Kurt, "On interpolation functions of the twisted generalized Frobenius-Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 187–194, 2007.