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Research Article

On the Exponent of Convergence for the Zeros of the Solutions of $y'' + Ay' + By = 0$

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Let B and C be entire functions of order less than 1 with $C \not\equiv 0$ and B transcendental. We prove that every solution $f \not\equiv 0$ of the equation $y'' + Ay' + By = 0$, $A(z) = C(z)e^{\alpha z}$, $\alpha \in \mathbb{C} \setminus \{0\}$ being has zeros with infinite exponent of convergence.

1. Introduction

It is assumed that the reader of this paper is familiar with the basic concepts of Nevanlinna theory [1, 2] such as $T(r,f)$, $m(r,f)$, $N(r,f)$, and $S(r,f)$. Suppose that f is a meromorphic function, then the order of growth of the function f and the exponent of convergence of the zeros of f are defined, respectively, as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \qquad \lambda(f) = \limsup_{r\to\infty} \frac{\log N(r,1/f)}{\log r}. \tag{1.1}$$

Let E be a measurable subset of $[1,+\infty)$. The lower logarithmic density and the upper logarithmic density of E are defined, respectively, by

$$\underline{\log\ \mathrm{dens}}(E) = \liminf_{r\to\infty} \frac{\int_1^r (\chi(t)dt/t)}{\log r}, \qquad \overline{\log\ \mathrm{dens}}(E) = \limsup_{r\to\infty} \frac{\int_1^r (\chi(t)dt/t)}{\log r}, \tag{1.2}$$

where $\chi(t)$ is the characteristic function of E defined as

$$\chi(t) = \begin{cases} 1, & \text{if } t \in E, \\ 0, & \text{if } t \notin E. \end{cases} \tag{1.3}$$

Now let us recall some of the previous results on the linear differential equation

$$y'' + e^{-z}y' + B(z)y = 0, \tag{1.4}$$

where $B(z)$ is an entire function of finite order, When $B(z)$ is polynomial, many authors [3–6] have studied the properties of the solutions of (1.4). If $B(z)$ is a transcendental entire function with $\rho(B) \neq 1$, Gundersen [7] proved that every nontrivial solution of (1.4) has infinite order of growth. In [8], Wang and Laine considered the nonhomogeneous equation of type

$$y'' + A_1(z)e^{az}y' + A_0(z)e^{bz}y = H(z), \tag{1.5}$$

where $A_0(z), A_1(z), H(z)$ are entire functions of order less than one and a, b are complex numbers. In fact, they have proved the following theorem.

Theorem 1.1. *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, H are entire functions of order less than one, and suppose that $a, b \in \mathbb{C}$ with $ab \neq 0$ and $a \neq b$. Then every nontrivial solution of* (1.5) *is of infinite order.*

Corollary 1.2. *Suppose that $B(z) = h(z)e^{bz}$, where h is a nonvanishing entire function with $\rho(h) < 1$ and $b \in \mathbb{C}$ with $b \neq 0, -1$. Then every nontrivial solution of* (1.4) *is of infinite order.*

2. Results

We observe that all the above results concern the order of growth only. In this paper, we are going to prove the following theorem which concerns the exponent of convergence.

Theorem 2.1. *Let B and C be entire functions of order less than 1 with $C \not\equiv 0$ and B being transcendental. Then every solution $f \not\equiv 0$ of the equation*

$$y'' + Ay' + By = 0, \qquad A(z) = C(z)e^{\alpha z}, \quad \alpha \in \mathbb{C} \setminus \{0\}, \tag{2.1}$$

has zeros with infinite exponent of convergence.

The hypothesis that B is transcendental is not redundant since Frei [4] has shown that

$$y'' + e^{-z}y' + Ky = 0 \tag{2.2}$$

has solutions of finite order, for certain constants K.

We notice that Theorem 2.1 fails for $\rho(B) \geq 1$. For any entire function D the function $f = e^D$ solves (2.1) with

$$-B = \frac{f''}{f} + A\frac{f'}{f} = D'' + D'^2 + AD. \tag{2.3}$$

3. Some Lemmas

Throughout this paper we need the following lemmas. In 1965, Hayman [9] proved the following lemma.

Lemma 3.1. *Let the function g be meromorphic of finite order ρ in the plane and let $0 < \delta < 1$. Then*

$$T(2r, g) \leq C(\rho, \delta) T(r, g) \tag{3.1}$$

for all r outside a set E of upper logarithmic density δ, where the positive constant $C(\rho, \delta)$ depends only on ρ and δ.

In 1962, Edrei and Fuchs [10] proved the following lemma.

Lemma 3.2. *Let g be a meromorphic function in the complex plane and let $I = I(r) \subseteq [0, 2\pi]$ have measure $\mu = \mu(r)$. Then*

$$\frac{1}{2\pi}\int_I \log^+\left|g\left(re^{i\theta}\right)\right| d\theta \leq 22\mu\left(1 + \log^+\frac{1}{\mu}\right)T(2r, g). \tag{3.2}$$

In 2007, Bergweiler and Langley [11] proved the following lemma.

Lemma 3.3. *Let H be a transcendental entire function of order $\rho < \infty$. For large $r > 0$ define $\theta(r)$ to be the length of the longest arc of the circle $|z| = r$ on which $|H(z)| > 1$, with $\theta(r) = 2\pi$ if the minimum modulus $m_0(r, H) = \min\{|H(z)| : |z| = r\}$ satisfies $m_0(r, H) > 1$. Then at least one of the following is true:*

(i) *there exists a set $F \subseteq [1, \infty)$ of positive upper logarithmic density such that $m_0(r, H) > 1$ for $r \in F$;*

(ii) *for each $\tau \in (0, 1)$ the set $F_r = \{r : \theta(r) > 2\pi(1 - \tau)\}$ has lower logarithmic density at least $(1 - 2\rho(1 - \tau))/\tau$.*

We deduce the following.

Lemma 3.4. *Let $0 < \epsilon < \pi/4$, let N be a positive integer, and let $G \subseteq [1, \infty)$ have logarithmic density 1. Let F be a transcendental entire function such that $|F(z)| \leq |z|^N$ on a path γ tending to infinity and for all z with $|z| \in G$ and $|\arg z| \leq \pi/2 - \epsilon$. Then F has order at least $\pi/(\pi + 2\epsilon)$.*

Proof. Assume that $\rho(F) = \rho < \infty$ and choose a polynomial P of degree at most $N - 1$ such that

$$H(z) = \frac{F(z) - P(z)}{2z^N} \tag{3.3}$$

is transcendental entire. Then we have $|H(z)| \leq 1$ for all $z \in \gamma$ and for all z with $|z| \in G$ and $|\arg z| \leq \pi/2 - \epsilon$. With the notation of Lemma 3.3, we see that $m_0(r, H) \leq 1$ for all large r, and so we must have case (ii), as well as $\theta(r) \leq \pi + 2\epsilon$ for $r \in G$. Define τ by

$$2\pi(1-\tau) = \pi + 2\epsilon. \tag{3.4}$$

Since G has logarithmic density 1 this gives

$$2\rho(1-\tau) \geq 1, \quad \rho \geq \frac{1}{2(1-\tau)} = \frac{\pi}{\pi + 2\epsilon}. \tag{3.5}$$

□

4. Proof of Theorem 2.1

Let A, B and C be as in the hypotheses. We can assume that $\alpha = 1$. Suppose that f is a solution of (2.1) having zeros with finite exponent of convergence. Then we can write

$$f = \Pi e^h, \tag{4.1}$$

where Π and h are entire functions with $\rho(\Pi) < \infty$. We can assume that $h' \not\equiv 0$, since if h is constant we can replace $h(z)$ by $h(z) + z$ and $\Pi(z)$ by $\Pi(z)e^{-z}$. Using (2.1) and (4.1), we get

$$\frac{\Pi''}{\Pi} + 2\frac{\Pi'}{\Pi}h' + h'' + h'^2 + A\left(\frac{\Pi'}{\Pi} + h'\right) + B = 0. \tag{4.2}$$

Lemma 4.1. *One has $\rho(h) \leq 1$.*

Proof. Suppose that $|h'(z)| \geq 1$. Dividing (4.2) by h', we get

$$|h'(z)| \leq \left|\frac{\Pi''(z)}{\Pi(z)}\right| + 2\left|\frac{\Pi'(z)}{\Pi(z)}\right| + \left|\frac{h''(z)}{h'(z)}\right| + |A(z)|\left(\left|\frac{\Pi'(z)}{\Pi(z)}\right| + 1\right) + |B(z)|. \tag{4.3}$$

Hence, provided r lies outside a set of finite measure,

$$T(r, h') = m(r, h') \leq O(\log r) + T(r, A) + T(r, B) + o(T(r, h')), \tag{4.4}$$

and so, using the fact that B and C have order less than 1, we obtain

$$T(r, h') = O(r). \tag{4.5}$$

This holds outside a set E_0 of finite measure and so for all large r, since we may take $s \notin E_0$

with $r \le s \le 2r$ so that

$$T(r, h') \le T(s, h') = O(s) = O(r). \tag{4.6}$$

Lemma 4.1 is proved. □

Let $M_1, M_2, \ldots$ denote large positive constants. Choose σ with

$$\max\{\rho(B), \rho(C)\} < \sigma < 1. \tag{4.7}$$

There exists an R-set U [2, page 84] such that for all large z outside U, we have

$$\left|\frac{\Pi''(z)}{\Pi(z)}\right| + \left|\frac{\Pi'(z)}{\Pi(z)}\right| + \left|\frac{h''(z)}{h'(z)}\right| \le |z|^{M_1}, \tag{4.8}$$

and using the Poisson-Jensen formula,

$$\left|\log\right||C(z)| \le |z|^{\sigma}. \tag{4.9}$$

Moreover, there exists a set $G \subseteq [1, \infty)$ of logarithmic density 1 such that for $r \in G$ the circle $|z| = r$ does not meet the R-set U.

Lemma 4.2. *The functions h' and $h' + A$ are both transcendental.*

Proof. Let ϵ be small and positive and suppose that h' or $h' + A$ is a polynomial. Let z be large with $z \notin U$ and $|\arg z - \pi| \le \pi/2 - \epsilon$. Since $A(z)$ is small it follows from (4.2) and (4.8) that $B(z) = O(|z|^{M_2})$. Choose θ with $|\theta - \pi| < \epsilon$ such that the intersection of U with the ray L given by $\arg z = \theta$ is bounded. Applying Lemma 3.4 to the function $B(-z)$, with γ a subpath of L, gives $\rho(B) \ge \pi/(\pi + 2\epsilon)$, but ϵ may be chosen arbitrarily small, and this contradicts (4.7). □

The next step is to estimate h' in the right half-plane.

Lemma 4.3. *Let N be a large positive integer and let $0 < \epsilon < 1/2$. Then for large z with*

$$\left|\arg z\right| \le \frac{\pi}{2} - \epsilon, \quad z \notin U \tag{4.10}$$

one has, either

$$\left|h'(z)\right| \le |z|^{N} \tag{4.11}$$

or

$$\left|h'(z) + A(z)\right| \le |z|^{N}. \tag{4.12}$$

Proof. Let z be large and satisfy (4.10), and assume that (4.11) does not hold. Then (4.8) implies that

$$\left|\frac{\Pi'(z)}{\Pi(z)} + h'(z)\right| \geq 1. \tag{4.13}$$

Also, (4.7), and (4.9) give

$$\log|B(z)| \leq |z|^{\sigma}, \quad \log|A(z)| \geq \operatorname{Re}(z) - |z|^{\sigma} \geq \frac{|z|}{2}\cos\left(\frac{\pi}{2} - \epsilon\right) = c_1|z|. \tag{4.14}$$

Here $c_1, c_2, \ldots$ denote positive constants which may depend on ϵ but not on z. Using (4.8), (4.12) and (4.14) we get, from (4.2),

$$\log\left|h'(z)\right| \geq c_2|z|. \tag{4.15}$$

Now divide (4.2) by $h'(z)$. We obtain, using (4.15),

$$h'(z) + A(z)\left(1 + \frac{O\left(|z|^{M_1}\right)}{h'(z)}\right) + O\left(|z|^{M_1}\right) = 0 \tag{4.16}$$

which gives $|h'(z)| \sim |A(z)|$ and (4.12). This proves Lemma 4.3. □

Lemma 4.4. *Let N and ϵ be as in Lemma 4.3. Choose $\theta_0 \in (-\pi/4, \pi/4)$ such that the ray $\arg z = \theta_0$ has bounded intersection with the R-set U. Let V be the union of the ray $\arg z = \theta_0$ and the arcs $|z| = r,\ r \in G,\ |\arg z| \leq \pi/2 - \epsilon$, where $G \subseteq [1, \infty)$ is the set chosen following (4.9). Then one of the following holds:*

(i) *one has (4.11) for all large z in V;*
(ii) *one has (4.12) for all large z in V.*

Proof. This follows simply from continuity. For each large z in V we have (4.11) or (4.12), but we cannot have both because of (4.14). This proves Lemma 4.4. □

Lemma 4.5. *Let $0 < \epsilon < 1/2$. Then for large $z \notin U$ with $|\arg z - \pi| \leq \pi/2 - \epsilon$, one has*

$$\log^{+}\left|h'(z)\right| = O\left(|z|^{\sigma}\right), \qquad \log^{+}\left|h'(z) + A(z)\right| = O\left(|z|^{\sigma}\right). \tag{4.17}$$

Proof. Let z be as in the hypotheses. Since $A(z) = o(1)$ we only need to prove (4.17) for $|h'(z)|$. Assume that $|h'(z)| \geq 1$. Then dividing (4.2) by h' gives

$$\left|h'(z)\right| \leq |B(z)| + O\left(|z|^{M_1}\right) \tag{4.18}$$

by (4.8), and so (4.17) follows using (4.7). This proves Lemma 4.5. □

Lemma 4.6. *If conclusion (i) of Lemma 4.4 holds then $\rho(h') < 1$, while if conclusion (ii) of Lemma 4.4 holds then $\rho(h' + A) < 1$.*

Proof. Suppose that conclusion (i) of Lemma 4.4 holds. Choose $\delta_1 > 0$ such that

$$\sigma(1 + \delta_1) < 1 \tag{4.19}$$

and let $\delta > 0$ be small compared to δ_1. Assume that ϵ in Lemma 4.4 is small compared to δ, in particular so small that

$$88\epsilon\left(1 + \log\frac{1}{4\epsilon}\right)C(\rho(h), \delta) \le \frac{1}{2}, \tag{4.20}$$

where $C(\rho(h), \delta)$ is the positive constant from Lemma 3.1. Let

$$I(r) = \left[\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\right] \cup \left[\frac{3\pi}{2} - \epsilon, \frac{3\pi}{2} + \epsilon\right], \tag{4.21}$$

and let E be the exceptional set of Lemma 3.1, with $g = h'$. Then for large $r \in G \setminus E$ we have, using (4.20) and Lemmas 3.1, 3.2, and 4.5,

$$\begin{aligned} T(r, h') = m(r, h') &\le O(r^\sigma) + O(\log r) + \frac{1}{2\pi}\int_{I(r)} \log^+\left|h'\left(re^{i\theta}\right)\right| d\theta \\ &\le O(r^\sigma) + 88\epsilon\left(1 + \log\frac{1}{4\epsilon}\right)T(2r, h') \\ &\le O(r^\sigma) + 88\epsilon\left(1 + \log\frac{1}{4\epsilon}\right)C(\rho(h), \delta)T(r, h') \\ &\le O(r^\sigma) + \frac{1}{2}T(r, h'). \end{aligned} \tag{4.22}$$

We then have

$$T(r, h') = O(r^\sigma) \tag{4.23}$$

for large $r \in G \setminus E$. Now take any large r. Since G has logarithmic density 1, while E has upper logarithmic density at most δ, and since δ/δ_1 is small, there exists s with

$$r \le s \le r^{1+\delta_1}, \quad s \in G \setminus E, \quad T(r, h') \le T(s, h') = O(s^\sigma) = O\left(r^{\sigma(1+\delta_1)}\right), \tag{4.24}$$

which proves Lemma 4.6 in this case. The alternative case, in which we have conclusion (ii) in Lemma 4.4, is proved the same way, using $h' + A$ in place of h'. □

To finish the proof suppose first that conclusion (ii) of Lemma 4.4 holds. Then Lemma 3.4 implies that h' has order at least $\pi/(\pi+2\epsilon)$. Since ϵ may be chosen arbitrarily small, this contradicts Lemma 4.6. The same contradiction, with h' replaced by $h'+A$, arises if conclusion (i) of Lemma 4.4 holds, and the proof of the theorem is complete.

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