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Research Article

A Note on the Integral Inequalities with Two Dependent Limits

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The theorem on the Gronwall's type integral inequalities with two dependent limits is established. In application, the boundedness of the solutions of nonlinear differential equations is presented.

1. Introduction

Integral inequalities play a significant role in the study of qualitative properties of solutions of integral, differential and integro-differential equations (see, e.g., [1–4] and the references given therein). One of the most useful inequalities in the development of the theory of differential equations is given in the following lemma (see [5]).

Lemma 1.1. *Let $u(t)$ and $f(t)$ be real-valued nonnegative continuous functions for all $t \geq 0$. If*

$$u^2(t) \leq c^2 + 2\int_0^t f(s)u(s)ds \tag{1.1}$$

for all $t \geq 0$, where $c \geq 0$ is a real constant, then

$$u(t) \leq c + \int_0^t f(s)ds \tag{1.2}$$

for all $t \geq 0$.

Note that the generalization of this integral inequality and its discrete analogies are given in papers [5–8]. In paper [9] the following useful inequality with two dependent limits was established.

Lemma 1.2. *Let $u(t)$ be a real-valued nonnegative continuous function defined on $[-T, T]$ and let c and a be nonnegative constants. Then the inequality*

$$u(t) \leq c + \operatorname{sgn}(t) \int_{-t}^{t} au(s)ds, \qquad -T \leq t \leq T \tag{1.3}$$

implies that

$$u(t) \leq ce^{2a|t|}, \qquad -T \leq t \leq T. \tag{1.4}$$

The theory of integral inequalities with several dependent limits and its applications to differential equations has been investigated in [10–14].

The present study involves some Gronwall's type integral inequalities with two dependent limits. Section 2 includes some new integral inequalities with two dependent limits and relevant proofs. Subsequently, Section 3 includes an application on the boundedness of the solutions of nonlinear differential equations.

2. A Main Statement

Our main statement is given by the following theorem.

Theorem 2.1. *Let $u(t)$, $a(t)$, $b(t)$, $g(t)$, $h(t)$, and $m(t)$ be real-valued nonnegative continuous functions defined on $\mathbb{R} = (-\infty, \infty)$.*

(i) Let c be a nonnegative constant. If

$$u^2(t) \leq c^2 + 2\operatorname{sgn}(t) \int_{-t}^{t} m(s)u(s)ds \tag{2.1}$$

for $t \in \mathbb{R}$, then

$$u(t) \leq c + \operatorname{sgn}(t) \int_{-t}^{t} m(s)ds \tag{2.2}$$

for all $t \in \mathbb{R}$.

(ii) Let $p > 1$ be a real constant. If

$$u^p(t) \leq a(t) + b(t)\operatorname{sgn}(t) \int_{-t}^{t} \left[g(s)u^p(s) + h(s)u(s)\right]ds \tag{2.3}$$

for $t \in \mathbb{R}$, *then*

$$u(t) \leq \left\{ a(t) + b(t) \exp\left[\operatorname{sgn}(t) \int_{-t}^{t} b(r)\left(g(r) + \frac{1}{p}h(r)\right)dr\right] \times \operatorname{sgn}(t) \int_{-t}^{t} \left[a(s)\left(g(s) + \frac{1}{p}h(s)\right) + \frac{p-1}{p}h(s)\right] \times \exp\left[-\operatorname{sgn}(s) \int_{-s}^{s} b(r)\left(g(r) + \frac{1}{p}h(r)\right)dr\right]ds \right\}^{1/p} \tag{2.4}$$

for all $t \in \mathbb{R}$.

(iii) Let $c(t)$ *be a real-valued positive continuous and nondecreasing function defined on* $\mathbb{R}$ *and* $p > 1$ *be a real constant. If*

$$u^p(t) \leq c^p(t) + b(t)\operatorname{sgn}(t) \int_{-t}^{t} \left[g(s)u^p(s) + h(s)u(s)\right]ds \tag{2.5}$$

for $t \in \mathbb{R}$, *then*

$$u(t) \leq c(t) \left\{ 1 + b(t) \exp\left[\operatorname{sgn}(t) \int_{-t}^{t} b(r)\left(g(r) + h(r)\frac{c^{1-p}(r)}{p}\right)dr\right] \times \operatorname{sgn}(t) \int_{-t}^{t} \left[g(s) + h(s)c^{1-p}(s)\right] \times \exp\left[-\operatorname{sgn}(s) \int_{-s}^{s} b(r)\left(g(r) + h(r)\frac{c^{1-p}(r)}{p}\right)dr\right]ds \right\}^{1/p} \tag{2.6}$$

for all $t \in \mathbb{R}$.

(iv) Let $k(t,s)$ *and its partial derivative* $\partial k(t,s)/\partial t$ *be real-valued nonnegative continuous functions on* $-\infty < s \leq t < \infty$ *and let* $k(t,s)$ *be even function in* t. *If*

$$u^p(t) \leq a(t) + b(t)\operatorname{sgn}(t) \int_{-t}^{t} k(t,s)\left[g(s)u^p(s) + h(s)u(s)\right]ds \tag{2.7}$$

for $t \in \mathbb{R}$, *then*

$$u(t) \leq \left\{ a(t) + b(t) \exp\left[\operatorname{sgn}(t) \int_{-t}^{t} k(r,r)b(r)\left(g(r) + \frac{1}{p}h(r)\right)dr\right] \times \left[\operatorname{sgn}(t) \int_{0}^{t} \operatorname{sgn}(s) \int_{-s}^{s} \frac{\partial}{\partial s}k(s,r)\left(a(r)\left(g(r) + \frac{1}{p}h(r)\right) + \frac{p-1}{p}h(r)\right)dr \times \exp\left[-\operatorname{sgn}(s) \int_{-s}^{s} k(r,r)b(r)\left(g(r) + \frac{1}{p}h(r)\right)dr\right]\right.$$

$$\times \exp\left(\operatorname{sgn}(t)\int_s^t \operatorname{sgn}(r)\int_{-r}^r \frac{\partial}{\partial r}k(r,y)b(y)\left(g(y)+\frac{1}{p}h(y)\right)dy\,dr\right)ds$$
$$+\operatorname{sgn}(t)\int_{-t}^t k(s,s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^s k(r,r)b(r)\left(g(r)+\frac{1}{p}h(r)\right)dr\right)$$
$$\times\left(a(s)\left(g(s)+\frac{1}{p}h(s)\right)+\frac{p-1}{p}h(s)\right)B_k(t,s)ds\Bigg]\Bigg\}^{1/p}. \tag{2.8}$$

for all $t\in\mathbb{R}$. *Here*

$$B_k(t,s)=\begin{cases}B_{k+}(t,s), & t\ge 0,\ s\in\mathbb{R},\\ B_{k-}(t,s), & t\le 0,\ s\in\mathbb{R},\end{cases} \tag{2.9}$$

where

$$B_{k-}(t,s)=\begin{cases}\exp\left(\int_s^t\int_{-r}^r \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right), & t\le s\le 0,\\ \exp\left(\int_{-s}^t\int_{-r}^r \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right), & 0\le s\le -t,\end{cases} \tag{2.10}$$

$$B_{k+}(t,s)=\begin{cases}\exp\left(\int_s^t\int_{-r}^r \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right), & 0\le s\le t,\\ \exp\left(\int_{-s}^t\int_{-r}^r \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right), & -t\le s\le 0.\end{cases} \tag{2.11}$$

Proof. (i) Define a function $v(t)$ by

$$v(t)=c^2+2\operatorname{sgn}(t)\int_{-t}^t m(s)u(s)ds. \tag{2.12}$$

Note that $v(t)$ is a nonnegative function and $v(0)=c^2$. Then (2.1) can be rewritten as

$$u^2(t)\le v(t),\qquad u(t)\le\sqrt{v(t)}. \tag{2.13}$$

It is easy to see that $v(t)$ is an even function.

First, let $t \geq 0$; then (2.12) can be rewritten as

$$v(t) = c^2 + 2\int_{-t}^{t} m(s)u(s)ds. \tag{2.14}$$

Differentiating (2.14) and using (2.13), we get

$$v'(t) \leq 2m(t)\sqrt{v(t)} + 2m(-t)\sqrt{v(t)}. \tag{2.15}$$

Dividing both sides of (2.15) by $2\sqrt{v(t)}$, we get

$$\frac{v'(t)}{2\sqrt{v(t)}} \leq m(t) + m(-t). \tag{2.16}$$

Integrating the last inequality from 0 to t, we get

$$\sqrt{v(t)} \leq c + \int_{0}^{t} m(s)ds + \int_{0}^{t} m(-s)ds = c + \operatorname{sgn}(t)\int_{-t}^{t} m(s)ds. \tag{2.17}$$

Second, let $t \leq 0$. Then, (2.12) can be written as

$$v(t) = c^2 - 2\int_{-t}^{t} m(s)u(s)ds. \tag{2.18}$$

Differentiating (2.18) and using (2.13), we get

$$-v'(t) \leq 2m(t)\sqrt{v(t)} + 2m(-t)\sqrt{v(t)}. \tag{2.19}$$

Dividing both sides of (2.19) by $2\sqrt{v(t)}$, we get

$$-\frac{v'(t)}{2\sqrt{v(t)}} \leq m(t) + m(-t). \tag{2.20}$$

Integrating (2.20) from t to 0, we get

$$\sqrt{v(t)} \leq c + \int_{t}^{0} m(s)ds + \int_{t}^{0} m(-s)ds = c + \operatorname{sgn}(t)\int_{-t}^{t} m(s)ds. \tag{2.21}$$

Finally, using (2.17) and (2.21), we obtain

$$\sqrt{v(t)} \le c + \operatorname{sgn}(t) \int_{-t}^{t} m(s)ds. \tag{2.22}$$

The inequality (2.2) follows from (2.13) and (2.22).

(ii) Define a function $v(t)$ by

$$v(t) = \operatorname{sgn}(t) \int_{-t}^{t} \left[g(s)u^p(s) + h(s)u(s)\right]ds. \tag{2.23}$$

It is evident that $v(t)$ is an even and nonnegative function. We have that

$$u^p(t) \le a(t) + b(t)v(t), \qquad u(t) \le [a(t) + b(t)v(t)]^{1/p}. \tag{2.24}$$

Using Young's inequality (see, e.g., [2]), we obtain that

$$u(t) \le \frac{a(t) + b(t)v(t)}{p} + \frac{p-1}{p}. \tag{2.25}$$

Let $t \ge 0$. Then

$$v(t) = \int_{-t}^{t} g(s)u^p(s) + h(s)u(s)ds. \tag{2.26}$$

Differentiating (2.26), we get

$$v'(t) = g(t)u^p(t) + h(t)u(t) + g(-t)u^p(-t) + h(-t)u(-t). \tag{2.27}$$

Using (2.24) and (2.25), we get

$$\begin{aligned} v'(t) \le v(t)&\left[b(t)\left(g(t) + \frac{1}{p}h(t)\right) + b(-t)\left(g(-t) + \frac{1}{p}h(-t)\right)\right] \\ &+ a(t)\left(g(t) + \frac{1}{p}h(t)\right) + a(-t)\left(g(-t) + \frac{1}{p}h(-t)\right) + \frac{p-1}{p}(h(t) + h(-t)). \end{aligned} \tag{2.28}$$

Denoting

$$B(t) = b(t)\left(g(t) + \frac{1}{p}h(t)\right), \qquad A(t) = a(t)\left(g(t) + \frac{1}{p}h(t)\right) + \frac{p-1}{p}h(t), \tag{2.29}$$

we get

$$v'(t) - v(t)[B(t) + B(-t)] \le A(t) + A(-t). \tag{2.30}$$

From that it follows that

$$\begin{aligned}&\exp\left(\int_s^t [B(r)+B(-r)]dr\right)\left(v'(s)-v(s)[B(s)+B(-s)]\right)\\&\quad\le \exp\left(\int_s^t [B(r)+B(-r)]dr\right)(A(s)+A(-s))\end{aligned}\tag{2.31}$$

for any $s \le t$. Integrating the last inequality from 0 to t and using $v(0)=0$, we get

$$v(t) \le \int_0^t (A(s)+A(-s))\exp\left(\int_s^t [B(r)+B(-r)]dr\right)ds.\tag{2.32}$$

It is easy to see that

$$\int_s^t [B(r)+B(-r)]dr = \int_{-t}^t B(r)dr - \int_{-s}^s B(r)dr.\tag{2.33}$$

Then

$$v(t) \le \exp\left(\int_{-t}^t B(r)dr\right)\int_0^t (A(s)+A(-s))\exp\left(-\int_{-s}^s B(r)dr\right)ds.\tag{2.34}$$

Since $0 \le s \le t$, we have that

$$\begin{aligned}v(t) &\le \exp\left(\operatorname{sgn}(t)\int_{-t}^t B(r)dr\right)\\&\quad\times\left[\int_0^t A(s)\exp\left(-\int_{-s}^s B(r)dr\right)ds + \int_{-t}^0 A(s)\exp\left(-\int_s^{-s} B(r)dr\right)ds\right]\\&= \exp\left(\operatorname{sgn}(t)\int_{-t}^t B(r)dr\right)\\&\quad\times\left[\int_0^t A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^s B(r)dr\right)ds + \int_{-t}^0 A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^s B(r)dr\right)ds\right]\\&= \exp\left(\operatorname{sgn}(t)\int_{-t}^t B(r)dr\right)\operatorname{sgn}(t)\int_{-t}^t A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^s B(r)dr\right).\end{aligned}\tag{2.35}$$

Applying (2.24), we obtain

$$u(t) \le \left[a(t) + b(t)\exp\left(\operatorname{sgn}(t)\int_{-t}^{t} B(r)dr\right) \times \operatorname{sgn}(t)\int_{-t}^{t} A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s} B(r)dr\right)ds\right]^{1/p}. \tag{2.36}$$

From (2.36), and (2.29) it follows (2.4) for $t \ge 0$. Let $t \le 0$; then

$$\begin{aligned} v(t) &= -\int_{-t}^{t} \left[g(s)u^{p}(s) + h(s)u(s)\right]ds, \\ -v'(t) &= g(t)u^{p}(t) + h(t)u(t) + g(-t)u^{p}(-t) + h(-t)u(-t). \end{aligned} \tag{2.37}$$

Using (2.24) and (2.25), we get

$$-v'(t) \le v(t)[B(t) + B(-t)] + A(t) + A(-t). \tag{2.38}$$

From that it follows that

$$\begin{aligned} &-\exp\left(\int_{t}^{s}[B(r) + B(-r)]dr\right)\left[v'(s) + v(s)(B(s) + B(-s))\right] \\ &\quad\le \exp\left(\int_{t}^{s}[B(r) + B(-r)]dr\right)(A(s) + A(-s)) \end{aligned} \tag{2.39}$$

for any $t \le s$. Integrating the last inequality from t to 0 and using $v(0) = 0$, we get

$$v(t) \le \int_{t}^{0}(A(s) + A(-s))\exp\left(\int_{t}^{s}[B(r) + B(-r)]dr\right)ds. \tag{2.40}$$

It is easy to see that

$$\int_{t}^{s}[B(r) + B(-r)]dr = \int_{t}^{-t} B(r)dr - \int_{s}^{-s} B(r)dr. \tag{2.41}$$

Then

$$v(t) \le \exp\left(\int_{t}^{-t} B(r)dr\right)\int_{t}^{0}(A(s) + A(-s))\exp\left(-\int_{s}^{-s} B(r)dr\right)ds. \tag{2.42}$$

Since $t \leq s \leq 0$, we have that

$$\begin{aligned}
v(t) &\leq \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} B(r)dr\right)\int_{t}^{0}(A(s)+A(-s))\exp\left(-\int_{s}^{-s} B(r)dr\right)ds \\
&= \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} B(r)dr\right) \\
&\quad\times\left[\int_{t}^{0} A(s)\exp\left(\int_{-s}^{s} B(r)dr\right)ds+\int_{t}^{0} A(-s)\exp\left(\int_{-s}^{s} B(r)dr\right)ds\right] \\
&= \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} B(r)dr\right) \\
&\quad\times\left[\int_{t}^{0} A(s)\exp\left(\int_{-s}^{s} B(r)dr\right)ds+\int_{0}^{-t} A(s)\exp\left(\int_{s}^{-s} B(r)dr\right)ds\right] \\
&= \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} B(r)dr\right)\int_{t}^{-t} A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s} B(r)dr ds\right) \\
&= \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} B(r)dr\right)\operatorname{sgn}(t)\int_{-t}^{t} A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s} B(r)dr\right)ds.
\end{aligned} \tag{2.43}$$

Applying (2.43) and (2.24), we obtain (2.36) for $t \leq 0$. Then from (2.36) and (2.29), (2.4) follows for $t \leq 0$.

(iii) Since $c(t)$ is a positive, continuous, and nondecreasing function for $t \in \mathbb{R}$, we have that

$$\left(\frac{u(t)}{c(t)}\right)^{p} \leq 1 + b(t)\operatorname{sgn}(t)\int_{-t}^{t}\left[g(s)\left(\frac{u(s)}{c(s)}\right)^{p} + h(s)c^{1-p}(s)\frac{u(s)}{c(s)}\right]ds. \tag{2.44}$$

Now the application of the inequality proven in (ii) yields the desired result in (2.6).

(iv) We define a function $v(t)$ by

$$v(t) = \operatorname{sgn}(t)\int_{-t}^{t} k(t,s)\left[g(s)u^{p}(s) + h(s)u(s)\right]ds. \tag{2.45}$$

Evidently, the function $v(t)$ is a nonnegative, monotonic, and nondecreasing in t and $v(0) = 0$. We have that

$$u^{p}(t) \leq a(t) + b(t)v(t), \qquad u(t) \leq \left[a(t) + b(t)v(t)\right]^{1/p}. \tag{2.46}$$

Let $t \geq 0$. Then

$$v(t) = \int_{-t}^{t} k(t,s)\left[g(s)u^p(s) + h(s)u(s)\right]ds. \tag{2.47}$$

Differentiating (2.47), we get

$$\begin{aligned} v'(t) = {}& k(t,t)\left[g(t)u^p(t) + h(t)u(t)\right] + k(t,-t)\left[g(-t)u^p(-t) + h(-t)u(-t)\right] \\ &+ \int_{-t}^{t} \frac{\partial}{\partial t}k(t,s)\left[g(s)u^p(s) + h(s)u(s)\right]ds. \end{aligned} \tag{2.48}$$

Using (2.46) and Young's inequality, we obtain that

$$\begin{aligned} v'(t) \leq v(t)\Bigg[&k(t,t)b(t)\left(g(t) + \frac{1}{p}h(t)\right) + k(t,-t)b(-t)\left(g(-t) + \frac{1}{p}h(-t)\right) \\ &+ \int_{-t}^{t} \frac{\partial}{\partial t}k(t,s)b(s)\left(g(s) + \frac{1}{p}h(s)\right)ds\Bigg] \\ &+ k(t,t)\left[g(t)a(t) + h(t)\left(\frac{1}{p}a(t) + \frac{p-1}{p}\right)\right] \\ &+ k(t,-t)\left[g(-t)a(-t) + h(-t)\left(\frac{1}{p}a(-t) + \frac{p-1}{p}\right)\right] \\ &+ \int_{-t}^{t} \frac{\partial}{\partial t}k(t,s)\left[g(s)a(s) + h(s)\left(\frac{1}{p}a(s) + \frac{p-1}{p}\right)\right]ds. \end{aligned} \tag{2.49}$$

Using (2.29), we get

$$\begin{aligned} v'(t) \leq v(t)&\left(k(t,t)B(t) + k(t,-t)B(-t) + \int_{-t}^{t} \frac{\partial}{\partial t}k(t,s)B(s)ds\right) \\ &+ k(t,t)A(t) + k(t,-t)A(-t) + \int_{-t}^{t} \frac{\partial}{\partial t}k(t,s)A(s)ds. \end{aligned} \tag{2.50}$$

Applying the differential inequality, we get

$$\begin{aligned} v(t) \leq {}& \int_{0}^{t}\left(k(s,s)A(s) + k(s,-s)A(-s) + \int_{-s}^{s} \frac{\partial}{\partial s}k(s,r)A(r)dr\right) \\ &\times \exp\left[\int_{s}^{t}\left(k(r,r)B(r) + k(r,-r)B(-r) + \int_{-r}^{r} \frac{\partial}{\partial r}k(r,y)B(y)dy\right)dr\right]. \end{aligned} \tag{2.51}$$

Since $k(t,s) = k(-t,s)$, we have that

$$v(t) \leq \int_0^t \left(k(s,s)A(s) + k(-s,-s)A(-s) + \int_{-s}^{s} \frac{\partial}{\partial s} k(s,r)A(r)dr \right) \times \exp\left[\int_s^t \left(k(r,r)B(r) + k(-r,-r)B(-r) + \int_{-r}^{r} \frac{\partial}{\partial r} k(r,y)B(y)dy \right) dr \right] ds. \tag{2.52}$$

Using (2.33), we get

$$\begin{aligned} v(t) \leq{}& \exp\left(\int_{-t}^{t} k(r,r)B(r)dr \right) \\ &\times \Bigg[\int_0^t \int_{-s}^{s} \frac{\partial}{\partial s} k(s,r)A(r)dr \\ &\qquad \times \exp\left(-\int_{-s}^{s} k(r,r)B(r)dr + \int_s^t \int_{-r}^{r} \frac{\partial}{\partial r} k(r,y)B(y)dy\,dr \right) ds \\ &\quad + \int_0^t k(s,s)A(s)\exp\left(-\int_{-s}^{s} k(r,r)B(r)dr + \int_s^t \int_{-r}^{r} \frac{\partial}{\partial r} k(r,y)B(y)dy\,dr \right) ds \\ &\quad + \int_0^t k(-s,-s)A(-s)\exp\left(-\int_{-s}^{s} k(r,r)B(r)dr\,ds + \int_s^t \int_{-r}^{r} \frac{\partial}{\partial r} k(r,y)B(y)dy\,dr \right) ds \Bigg]. \end{aligned} \tag{2.53}$$

Since $0 \leq s \leq t$, we have that

$$\begin{aligned} v(t) \leq{}& \exp\left(\operatorname{sgn}(t) \int_{-t}^{t} k(r,r)B(r)dr \right) \\ &\times \Bigg[\operatorname{sgn}(t) \int_0^t \operatorname{sgn}(s) \int_{-s}^{s} \frac{\partial}{\partial s} k(s,r)A(r)dr \\ &\qquad \times \exp\left(-\int_{-s}^{s} k(r,r)B(r)dr + \int_s^t \int_{-r}^{r} \frac{\partial}{\partial r} k(r,y)B(y)dy\,dr \right) ds \\ &\quad + \int_0^t k(s,s)A(s)\exp\left(-\int_{-s}^{s} k(r,r)B(r)dr + \int_s^t \int_{-r}^{r} \frac{\partial}{\partial r} k(r,y)B(y)dy\,dr \right) ds \\ &\quad + \int_{-t}^{0} k(s,s)A(s)\exp\left(-\int_{s}^{-s} k(r,r)B(r)dr + \int_{-s}^t \int_{-r}^{r} \frac{\partial}{\partial r} k(r,y)B(y)dy\,dr \right) ds \Bigg]. \end{aligned} \tag{2.54}$$

Using (2.9) and (2.11), we get

$$\begin{aligned}
v(t) \le{}& \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} k(r,r)B(r)dr\right)\\
&\times\left[\operatorname{sgn}(t)\int_{0}^{t}\operatorname{sgn}(s)\int_{-s}^{s}\frac{\partial}{\partial s}k(s,r)A(r)dr\right.\\
&\qquad\times\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s}k(r,r)B(r)dr+\operatorname{sgn}(t)\int_{s}^{t}\operatorname{sgn}(r)\int_{-r}^{r}\frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds\\
&\qquad+\int_{0}^{t}k(s,s)A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s}k(r,r)B(r)dr\right)B_{k+}(t,s)ds\\
&\qquad\left.+\int_{-t}^{0}k(s,s)A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s}k(r,r)B(r)dr\right)B_{k+}(t,s)ds\right]\\
={}& \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} k(r,r)B(r)dr\right)\\
&\times\left[\operatorname{sgn}(t)\int_{0}^{t}\operatorname{sgn}(s)\int_{-s}^{s}\frac{\partial}{\partial s}k(s,r)A(r)dr\right.\\
&\qquad\times\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s}k(r,r)B(r)dr+\operatorname{sgn}(t)\int_{s}^{t}\operatorname{sgn}(r)\int_{-r}^{r}\frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds\\
&\qquad\left.+\operatorname{sgn}(t)\int_{-t}^{0}k(s,s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s}k(r,r)B(r)dr\right)A(s)B_{k}(t,s)ds\right].
\end{aligned}\tag{2.55}$$

Let $t \le 0$. Then

$$v(t) = -\int_{-t}^{t} k(t,s)\left[g(s)u^{p}(s)+h(s)u(s)\right]ds. \tag{2.56}$$

Differentiating (2.56), we get

$$\begin{aligned}
-v'(t) ={}& k(t,t)\left[g(t)u^{p}(t)+h(t)u(t)\right]+k(t,-t)\left[g(-t)u^{p}(-t)+h(-t)u(-t)\right]\\
&+\int_{-t}^{t}\frac{\partial}{\partial t}k(t,s)\left[g(s)u^{p}(s)+h(s)u(s)\right]ds.
\end{aligned}\tag{2.57}$$

Using (2.46) and Young's inequality, we obtain that

$$\begin{aligned}
-v'(t) \le v(t)\Bigg[&k(t,t)b(t)\left(g(t)+\frac{1}{p}h(t)\right)+k(t,-t)b(-t)\left(g(-t)+\frac{1}{p}h(-t)\right)\\
&+\int_{-t}^{t}\frac{\partial}{\partial t}k(t,s)b(s)\left(g(s)+\frac{1}{p}h(s)\right)ds\Bigg]\\
&+k(t,t)\left[g(t)a(t)+h(t)\left(\frac{1}{p}a(t)+\frac{p-1}{p}\right)\right]\\
&+k(t,-t)\left[g(-t)a(-t)+h(-t)\left(\frac{1}{p}a(-t)+\frac{p-1}{p}\right)\right]\\
&+\int_{t}^{-t}\frac{\partial}{\partial t}k(t,s)\left[g(s)a(s)+h(s)\left(\frac{1}{p}h(s)+\frac{p-1}{p}\right)\right]ds.
\end{aligned} \tag{2.58}$$

Using (2.29), we get

$$\begin{aligned}
-v'(t) \le v(t)&\left(k(t,t)B(t)+k(t,-t)B(-t)+\int_{t}^{-t}\frac{\partial}{\partial t}k(t,s)B(s)ds\right)\\
&+k(t,t)A(t)+k(t,-t)A(-t)+\int_{t}^{-t}\frac{\partial}{\partial t}k(t,s)A(s)ds.
\end{aligned} \tag{2.59}$$

Applying the differential inequality, we get

$$\begin{aligned}
v(t) \le \int_{t}^{0}&\left(k(s,s)A(s)+k(s,-s)A(-s)+\int_{s}^{-s}\frac{\partial}{\partial s}k(s,r)A(r)dr\right)\\
&\times\exp\left[\int_{t}^{s}\left(k(r,r)B(r)+k(r,-r)B(-r)+\int_{r}^{-r}\frac{\partial}{\partial r}k(r,y)B(y)dy\right)dr\right]ds.
\end{aligned} \tag{2.60}$$

Since $k(t,s)=k(-t,s)$, we have that

$$\begin{aligned}
v(t) \le \int_{t}^{0}&\left(k(s,s)A(s)+k(-s,-s)A(-s)+\int_{s}^{-s}\frac{\partial}{\partial s}k(s,r)A(r)dr\right)\\
&\times\exp\left[\int_{t}^{s}\left(k(r,r)B(r)+k(-r,-r)B(-r)+\int_{r}^{-r}\frac{\partial}{\partial r}k(r,y)B(y)dy\right)dr\right]ds.
\end{aligned} \tag{2.61}$$

Using (2.41), we get

$$v(t) \le \exp\left(\int_t^{-t} k(r,r)B(r)dr\right)$$
$$\times \left[\int_t^0 \int_s^{-s} \frac{\partial}{\partial s}k(s,r)A(r)dr \right.$$
$$\times \exp\left(-\int_s^{-s} k(r,r)B(r)dr + \int_t^s \int_r^{-r} \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds$$
$$+\int_t^0 k(s,s)A(s)\exp\left(-\int_s^{-s} k(r,r)B(r)dr + \int_t^s \int_r^{-r} \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds$$
$$\left. +\int_t^0 k(-s,-s)A(-s)\exp\left(-\int_s^{-s} k(r,r)B(r)dr\right)ds + \int_t^s \int_r^{-r} \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\,ds\right]. \tag{2.62}$$

Since $t \le s \le 0$, we have that

$$v(t) \le \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} k(r,r)B(r)dr\right)$$
$$\times \left[\operatorname{sgn}(t)\int_0^t \operatorname{sgn}(s)\int_{-s}^{s} \frac{\partial}{\partial s}k(s,r)A(r)dr \right.$$
$$\times \exp\left(-\operatorname{sgn}(s)\int_{-s}^{s} k(r,r)B(r)dr + \int_t^s \int_r^{-r} \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds$$
$$+\int_t^0 k(s,s)A(s)\exp\left(-\int_s^{-s} k(r,r)B(r)dr + \int_t^s \int_r^{-r} \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds$$
$$\left. +\int_0^{-t} k(s,s)A(s)\exp\left(-\int_{-s}^{s} k(r,r)B(r)dr + \int_t^{-s} \int_r^{-r} \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds\right]. \tag{2.63}$$

Using (2.9) and (2.10), we get

$$v(t) \le \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} k(r,r)B(r)dr\right)$$
$$\times \left[\operatorname{sgn}(t)\int_0^t \operatorname{sgn}(s)\int_{-s}^{s} \frac{\partial}{\partial s}k(s,r)A(r)dr \right.$$
$$\times \exp\left(-\operatorname{sgn}(s)\int_{-s}^{s} k(r,r)B(r)dr + \operatorname{sgn}(t)\int_s^t \operatorname{sgn}(r)\int_{-r}^{r} \frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds$$
$$+\int_t^0 k(s,s)A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s} k(r,r)B(r)dr\right)B_{k-}(t,s)ds$$
$$\left. +\int_0^{-t} k(s,s)A(s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s} k(r,r)B(r)dr\right)B_{k-}(t,s)ds\right]$$

$$
\begin{aligned}
&= \exp\left(\operatorname{sgn}(t)\int_{-t}^{t} k(r,r)B(r)dr\right) \\
&\quad\times\left[\operatorname{sgn}(t)\int_{0}^{t}\operatorname{sgn}(s)\int_{-s}^{s}\frac{\partial}{\partial s}k(s,r)A(r)dr\right. \\
&\qquad\times\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s}k(r,r)B(r)dr+\operatorname{sgn}(t)\int_{s}^{t}\operatorname{sgn}(r)\right. \\
&\qquad\qquad\left.\times\int_{-r}^{r}\frac{\partial}{\partial r}k(r,y)B(y)dy\,dr\right)ds \\
&\quad\left.+\operatorname{sgn}(t)\int_{0}^{-t}k(s,s)\exp\left(-\operatorname{sgn}(s)\int_{-s}^{s}k(r,r)B(r)dr\right)A(s)B_k(t,s)ds\right].
\end{aligned}
\tag{2.64}
$$

The inequality (2.8) follows from (2.29), (2.55), and (2.64). Theorem 2.1 is proved. □

3. An Application

In this section, we indicate an application of Theorem 2.1 (part (ii)) to obtain the explicit bound on the solution of the following boundary value problem for one dimensional partial differential equations:

$$
\begin{aligned}
v_{tt}^{p}(t,x)-\left(a(x)v_{x}^{p}(t,x)\right)_{x}+\delta v^{p}(t,x)&=F(t,x;v(t,x)),\quad t\in\mathbb{R},0<x<l,\\
v(t,0)=v(t,l),\quad v_x(t,0)&=v_x(t,l),\quad t\in\mathbb{R},\\
v(0,x)=\varphi(x),\quad v_t(0,x)&=\psi(x),\quad 0\le x\le l,
\end{aligned}
\tag{3.1}
$$

where $p>1$ is a fixed real number and $\delta=\text{const}>0$. Let $F(t,x;v(t,x))$, $t\in\mathbb{R}$, $x\in(0,l)$, $a(x)\ge a>0$, $x\in(0,l),\varphi(x),\psi(x)$, $x\in[0,l]$ be smooth functions and problem (3.1) has a unique smooth solution $v(t,x)$. Assume that

$$
\left(\int_0^l F^2(t,x;v(t,x))dx\right)^{1/2}\le g(t)\left(\int_0^l v^{2p}(t,x)dx\right)^{1/2}+h(t)\left(\int_0^l v^2(t,x)dx\right)^{1/2}
\tag{3.2}
$$

for all $t\in\mathbb{R}$. Here $g(t)$ and $h(t)$ are real-valued nonnegative continuous functions defined on $\mathbb{R}$.

This allows us to reduce the nonlocal boundary-value (3.1) to the initial-value problem

$$
\begin{aligned}
v_{tt}^{p}(t)+Av^{p}(t)&=F(t,v(t)),\quad t\in\mathbb{R},\\
v(0)=\varphi,\qquad v_t(0)&=\psi
\end{aligned}
\tag{3.3}
$$

in a Hilbert space $H = L_2[0,l]$ with a self-adjoint positive definite operator A defined by the formula $Au(x) = -(a(x)u_x(x))_x + \delta u(x)$, with the domain $D(A) = \{u(x) : u''(x) \in L_2[0,l], u(0) = u(l), u'(0) = u'(l)\}$ (see, e.g., [15, 16]).

Let us give a corollary of Theorem 2.1.

Theorem 3.1. *The solution of problem* (3.1) *satisfies the estimates*

$$\left(\int_0^l v^{2p}(t,x)dx\right)^{1/2p} \leq \Bigg\{M + \frac{1}{\sqrt{\delta}}\exp\left[\operatorname{sgn}(t)\int_{-t}^{t}\frac{1}{\sqrt{\delta}}\left(g(r)+\frac{1}{p}l^{1-1/p}h(r)\right)dr\right]$$
$$\times \operatorname{sgn}(t)\int_{-t}^{t}\left[M\left(g(s)+\frac{1}{p}l^{1-1/p}h(s)\right)+\frac{p-1}{p}l^{1-1/p}h(s)\right]$$
$$\times \exp\left[-\operatorname{sgn}(s)\int_{-s}^{s}\frac{1}{\sqrt{\delta}}\left(g(r)+\frac{1}{p}l^{1-1/p}h(r)\right)dr\right]ds\Bigg\}^{1/p} \tag{3.4}$$

for all $t \in \mathbb{R}$. *Here* $M = (\int_0^l \varphi^{2p}(x)dx)^{1/2} + ((p/\delta)\int_0^l \varphi^{2(p-1)}(x)\psi^2(x)dx)^{1/2}$.

Proof. It is known thatthe formula (see, e.g., [15, 16])

$$v^p(t) = c(t)v^p(0) + s(t)(v^p)'(0) + \int_0^t s(t-s)F(s,v(s))ds \tag{3.5}$$

gives a solution of problem (3.3). Here

$$c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}, \qquad s(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}. \tag{3.6}$$

Applying the triangle inequality, condition (3.2), formula (3.5), and estimates (see, e.g., [17])

$$\|c(t)\|_{H\to H} \leq 1, \qquad \left\|A^{1/2}s(t)\right\|_{H\to H} \leq 1, \qquad \left\|A^{-1/2}\right\|_{H\to H} \leq \frac{1}{\sqrt{\delta}}, \tag{3.7}$$

we get

$$\|v^p(t)\|_H \leq \|v^p(0)\|_H + \frac{1}{\sqrt{\delta}}\left\|(v^p)'(0)\right\|_H + \frac{1}{\sqrt{\delta}}\int_0^t (g(s)\|v^p(s)\|_H + h(s)\|v(s)\|_H)ds. \tag{3.8}$$

Since

$$\|v^p(0)\|_H + \frac{1}{\sqrt{\delta}}\left\|(v^p)'(0)\right\|_H = \left(\int_0^l \varphi^{2p}(x)dx\right)^{1/2} + \left(\frac{p}{\delta}\int_0^l \varphi^{2(p-1)}(x)\psi^2(x)dx\right)^{1/2}, \qquad (3.9)$$
$$\|v(s)\|_H \le l^{1-1/p}\|v^p(s)\|_H^{1/p}$$

we have that

$$\|v^p(t)\|_H \le M + \frac{1}{\sqrt{\delta}}\,\mathrm{sgn}(t)\int_{-t}^{t}\left(g(s)\|v^p(s)\|_H + l^{1-1/p}h(s)\|v^p(s)\|_H^{1/p}\right)ds. \qquad (3.10)$$

Denote that $u(t) = \|v^p(t)\|_H^{1/p}$. Then

$$u^p(t) \le M + \frac{1}{\sqrt{\delta}}\,\mathrm{sgn}(t)\int_{-t}^{t}\left[g(s)u^p(s) + l^{1-1/p}h(s)u(s)\right]ds \qquad (3.11)$$

for $t \in \mathbb{R}$. Applying the integral inequality (2.4), we get

$$\begin{aligned} u(t) \le \Bigg\{ M + \frac{1}{\sqrt{\delta}}\exp\left[\mathrm{sgn}(t)\int_{-t}^{t}\frac{1}{\sqrt{\delta}}\left(g(r) + \frac{1}{p}l^{1-1/p}h(r)\right)dr\right] \\ \times\,\mathrm{sgn}(t)\int_{-t}^{t}\left[M\left(g(s) + \frac{1}{p}l^{1-1/p}h(s)\right) + \frac{p-1}{p}l^{1-1/p}h(s)\right] \\ \times\,\exp\left[-\,\mathrm{sgn}(s)\int_{-s}^{s}\frac{1}{\sqrt{\delta}}\left(g(r) + \frac{1}{p}l^{1-1/p}h(r)\right)dr\right]ds\Bigg\}^{1/p}. \end{aligned} \qquad (3.12)$$

We have that

$$u(t) = \|v^p(t)\|_H^{1/p} = \left(\int_0^l v^{2p}(t,x)dx\right)^{1/2p}. \qquad (3.13)$$

Therefore, the inequality (3.4) follows from the last inequality. Theorem 3.1 is proved. □

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