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Research Article

On the System of Nonlinear Mixed Implicit Equilibrium Problems in Hilbert Spaces

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We use the Wiener-Hopf equations and the Yosida approximation notions to prove the existence theorem of a system of nonlinear mixed implicit equilibrium problems (SMIE) in Hilbert spaces. The algorithm for finding a solution of the problem (SMIE) is suggested; the convergence criteria and stability of the iterative algorithm are discussed. The results presented in this paper are more general and are viewed as an extension, refinement, and improvement of the previously known results in the literature.

1. Introduction and Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $F_1, F_2 : H \times H \to H$ be given two bi-functions satisfying $F_i(x, x) = 0$ for all $x \in H$ and $i = 1, 2$. Let $T : H \times H \to H$ be a nonlinear mapping. Let C be a nonempty closed convex subset of H. In this paper, we consider the following problem.

Find $x^*, y^* \in H$ such that

$$\begin{aligned} &F_1(x^*, z) + \langle T_1(x^*, y^*), z - x^* \rangle \geq 0, \quad \forall z \in C, \\ &F_2(y^*, z) + \langle T_2(x^*, y^*), z - y^* \rangle \geq 0, \quad \forall z \in C. \end{aligned} \tag{1.1}$$

The problem of type (1.1) is called the *system of nonlinear mixed implicit equilibrium problems*.

We denote by SMIE (F_1, F_2, T_1, T_2, C) the set of all solutions (x^*, y^*) of the problem (1.1).

Some examples of the problem (1.1) are as follows.

(I) If $F_i(x,z) = \sup_{\xi \in M_i(x)} \langle \xi, z-x \rangle$, where $M_i : C \to 2^H$ is a maximal monotone mapping for $i = 1,2$, then the problem (1.1) becomes the following problem.

Find $x^*, y^* \in H$ such that

$$0 \in T_1(x^*, y^*) + M_1(x^*),$$
$$0 \in T_2(x^*, y^*) + M_2(y^*), \tag{1.2}$$

which is called the system of *variational inclusion problems*. In particular, when $T_1 = T_2$ and $M_1 = M_2$, the problem (1.2) is reduced to the problem, so-called the *generalized variational inclusion problem*, which was studied by Kazmi and Bhat [1].

It is worth noting that the variational inclusions and related problems are being studied extensively by many authors and have important applications in operations research, optimization, mathematical finance, decision sciences, and other several branches of pure and applied sciences.

(II) If $F_i(x,z) = \varphi_i(z) - \varphi_i(x)$ for all $x, z \in H$, where $\varphi_i : H \to \mathbb{R}$ is a real valued function for each $i = 1,2$. Then the problem (1.1) reduces to the following problem.

Find $x^*, y^* \in H$ such that

$$\langle T_1(x^*, y^*), z - x^* \rangle + \varphi_1(z) - \varphi_1(x^*) \geq 0, \quad \forall z \in C,$$
$$\langle T_2(x^*, y^*), z - y^* \rangle + \varphi_2(z) - \varphi_2(y^*) \geq 0, \quad \forall z \in C. \tag{1.3}$$

Some corresponding results to the problem (1.3) were considered by Kassay and Kolumbán [2] when $\varphi_1 = \varphi_2 = 0$.

(III) For each $i = 1,2$, let $S_i : H \times H \to H$ be a nonlinear mapping and λ, η fixed positive real numbers. If $T_1(x,y) = \lambda S_1(y,x) + x - y$ and $T_2(x,y) = \eta S_2(x,y) + y - x$ for all $x, y \in H$, then the problem (1.3) reduces to the following problem.

Find $x^*, y^* \in H$ such that

$$\langle \lambda S_1(y^*, x^*) + x^* - y^*, z - x^* \rangle + \varphi_1(z) - \varphi_1(x^*) \geq 0, \quad \forall z \in C,$$
$$\langle \eta S_2(x^*, y^*) + y^* - x^*, z - y^* \rangle + \varphi_2(z) - \varphi_2(y^*) \geq 0, \quad \forall z \in C, \tag{1.4}$$

which is called the *system of nonlinear mixed variational inequalities problems*. A special case of the problem (1.4), when $S_1 = S_2$ and $\varphi_1 = \varphi_2$, has been studied by He and Gu [3].

(IV) If $\varphi_1(x) = \varphi_2 = \delta_C(x)$ for all $x \in C$, where δ_C is the indicator function of C defined by

$$\delta_K = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise}, \end{cases} \tag{1.5}$$

then the problem (1.4) reduces to the following problem.

Find $x^*, y^* \in C$ such that

$$\begin{aligned}&\langle \lambda S_1(y^*, x^*) + x^* - y^*, z - x^* \rangle \geq 0, \quad \forall z \in C,\\&\langle \eta S_2(x^*, y^*) + y^* - x^*, z - y^* \rangle \geq 0, \quad \forall z \in C,\end{aligned} \tag{1.6}$$

which is called the *system of nonlinear variational inequalities problems*. Some corresponding results to the problem (1.6) were studied by Agarwal et al. [4], Chang et al. [5], Cho et al. [6], J. K. Kim and D. S. Kim, [7] and Verma [8, 9].

For the recent trends and developments in the problem (1.6) and its special cases, see [3, 8–11] and the references therein, for examples.

(V) If $S_2 = 0$, and $S_1 : C \to H$ is a univariate mapping, then the problem (1.6) reduces to the following problem.

Find $x^* \in H$ such that

$$\langle S_1(x^*), z - x^* \rangle \geq 0, \quad \forall z \in C, \tag{1.7}$$

which is known as the *classical variational inequality* introduced and studied by Stampacchia [12] in 1964. This shows that a number of classes of variational inequalities and related optimization problems can be obtained as special cases of the system (1.1) of mixed equilibrium problems.

Inspired and motivated by the recent research going on in this area, in this paper, we use the Wiener-Hopf equations and the Yosida approximation notion to suggest and prove the existence and uniqueness of solutions for the problem (1.1). We also discuss the convergence criteria and stability of the iterative algorithm. The results presented in this paper improve and generalize many known results in the literature.

In the sequel, we need the following basic concepts and lemmas.

Definition 1.1 (Blum and Oettli [13]). A real valued bifunction $F : C \times C \to \mathbb{R}$ is said to be:

(1) *monotone* if

$$F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in C; \tag{1.8}$$

(2) *strictly monotone* if

$$F(x, y) + F(y, x) < 0, \quad \forall x, y \in C \quad (x \neq y); \tag{1.9}$$

(3) *upper-hemicontinuous* if

$$\limsup_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y), \quad \forall x, y, z \in C. \tag{1.10}$$

Definition 1.2. (1) A function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous* at x_0 if, for all $\alpha < f(x_0)$, there exists a constant $\beta > 0$ such that

$$\alpha \leq f(x), \quad \forall x \in B(x_0, \beta), \tag{1.11}$$

where $B(x_0, \beta)$ denotes the ball with center x_0 and radius β, that is,

$$B(x_0, \beta) = \{y : \|y - x_0\| \le \beta\}. \tag{1.12}$$

(2) The function f is said to be *lower semicontinuous* on H if it is lower semicontinuous at every point of H.

Lemma 1.3 (Combettes and Hirstoaga [14]). *Let C be a nonempty closed convex subset of H and F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the following conditions:*

(C1)*F is monotone and upper hemicontinuous;*

(C2) $F(x, \cdot)$ is convex and lower semi-continuous for all $x \in C$.

For all $\rho > 0$ and $x \in H$, define a mapping $T_\rho^F : H \to C$ as follows:

$$T_\rho^F(x) = \{w \in C : \rho F(w, z) + \langle w - x, z - w\rangle, \ \forall z \in C\}, \quad \forall x \in H. \tag{1.13}$$

Then T_ρ^F is a single-valued mapping.

Definition 1.4. Let ρ be a positive number. For any bi-function $F : C \times C \to \mathbb{R}$, the associated Yosida approximation F_ρ over C and the corresponding regularized operator A_ρ^F are defined as follows:

$$F_\rho(x, z) = \frac{1}{\rho}\left\langle x - J_\rho^F(x), z - x\right\rangle, \qquad A_\rho^F(x) = \frac{1}{\rho}\left(x - J_\rho^F(x)\right), \tag{1.14}$$

in which $J_\rho^F(x) \in C$ is the unique solution of the following problem:

$$\rho F\left(J_\rho^F(x), z\right) + \left\langle J_\rho^F(x) - x, z - J_\rho^F(x)\right\rangle \ge 0, \quad \forall z \in C. \tag{1.15}$$

Remark 1.5. Definition 1.4 is an extension of the Yosida approximation notion introduced in [15]. The existence and uniqueness of the solution of the problem (1.15) follow from Lemma 1.3.

Definition 1.6. Let $M \subset H \times H$ be a set-valued mapping.

(1) M is said to be *monotone* if, for any $(x_1, y_1), (x_2, y_2) \in M$,

$$\langle y_1 - y_2, x_1 - x_2\rangle \ge 0. \tag{1.16}$$

(2) A monotone operator $M \subset H \times H$ is said to be *maximal* if M is not properly contained in any other monotone operators.

Example 1.7 (Huang et al. [16]). Let $F(x, z) = \sup_{\xi \in M(x)}\langle \xi, z - x\rangle$, where $M : H \to 2^H$ is a maximal monotone mapping. Then it directly follows that

$$J_\rho^F(x) = (I + \rho M)^{-1}(x), \qquad A_\rho^F(x) = M_\rho(x), \tag{1.17}$$

where $M_\rho := (1/\rho)(I - (I + \rho M)^{-1})$ is the Yosida approximation of M, and we recover the classical concepts.

Using the idea as in Huang et al. [16], we have the following result.

Lemma 1.8. *If $F : C \times C \to \mathbb{R}$ is a monotone function, then the operator J_ρ^F is a nonexpansive mapping, that is,*

$$\left\| J_\rho^F(x) - J_\rho^F(y) \right\| \le \|x - y\|, \quad \forall x, y \in H. \tag{1.18}$$

Proof. From (1.15), for all $x, y \in H$, we can obtain

$$\rho F\left(J_\rho^F(x), J_\rho^F(y)\right) + \left\langle J_\rho^F(x) - x, J_\rho^F(y) - J_\rho^F(x) \right\rangle \ge 0, \tag{1.19}$$

$$\rho F\left(J_\rho^F(y), J_\rho^F(x)\right) + \left\langle J_\rho^F(y) - y, J_\rho^F(x) - J_\rho^F(y) \right\rangle \ge 0. \tag{1.20}$$

By adding (1.19) with (1.20) and using the monotonicity of F, we have

$$\left\langle x - y - \left(J_\rho^F(x) - J_\rho^F(y)\right), J_\rho^F(x) - J_\rho^F(y) \right\rangle \ge 0, \tag{1.21}$$

and so

$$\begin{aligned} \left\| J_\rho^F(x) - J_\rho^F(y) \right\|^2 &\le \left\langle J_\rho^F(x) - J_\rho^F(y), J_\rho^F(x) - J_\rho^F(y) \right\rangle \\ &\le \left\langle x - y, J_\rho^F(x) - J_\rho^F(y) \right\rangle \\ &\le \|x - y\| \left\| J_\rho^F(x) - J_\rho^F(y) \right\|. \end{aligned} \tag{1.22}$$

This implies that J_ρ^F is a nonexpansive mapping. This completes the proof. □

Now, for solving the problem (1.1), we consider the following equation: let $(x, y) \in H \times H$ and ρ_1, ρ_2 be fixed positive real numbers. Find $(w_1, w_2) \in H \times H$ such that

$$\begin{aligned} &T_1(x, y) + A_{\rho_1}^{F_1}(w_1) = 0, \quad x = J_{\rho_1}^{F_1}(w_1), \\ &T_2(x, y) + A_{\rho_2}^{F_2}(w_2) = 0, \quad y = J_{\rho_2}^{F_2}(w_2). \end{aligned} \tag{1.23}$$

Lemma 1.9. *$(x, y) \in H \times H$ is a solution of the problem (1.1) if and only if the problem (1.23) has a solution (w_1, w_2), where*

$$\begin{aligned} &x = J_{\rho_1}^{F_1}(w_1), \quad w_1 = x - \rho_1 T_1(x, y), \\ &y = J_{\rho_2}^{F_2}(w_2), \quad w_2 = y - \rho_2 T_2(x, y), \end{aligned} \tag{1.24}$$

that is,

$$x = J_{\rho_1}^{F_1}[x - \rho_1 T_1(x,y)],$$
$$y = J_{\rho_2}^{F_2}[y - \rho_2 T_2(x,y)]. \tag{1.25}$$

Proof. The proof directly follows from the definitions of $J_{\rho_1}^{F_1}$ and $J_{\rho_2}^{F_2}$. □

In this paper, we are interested in the following class of nonlinear mappings.

Definition 1.10. (1) A mapping $T : H \to H$ is said to be *ν-strongly monotone* if there exists a constant $\nu > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \nu \|x - y\|^2, \quad \forall x, y \in H; \tag{1.26}$$

(2) A mapping $T : H \times H \to H$ is said to be *(τ, σ)-Lipschitz* if there exist constants $\tau, \sigma > 0$ such that

$$\|T(x_1, y_1) - T(x_2, y_2)\| \leq \tau \|x_1 - x_2\| + \sigma \|y_1 - y_2\|, \quad \forall x_1, x_2, y_1, y_2 \in H. \tag{1.27}$$

2. Existence of Solutions of the Problem (1.1)

In this section, we give an existence theorem of solutions for the problem (1.1). Firstly, in view of Lemma 1.9, we can obtain the following, which is an important tool, immediately.

Lemma 2.1. *Let $(x, y) \in H \times H$. Then $(x, y) \in \mathrm{SMIE}\ (F_1, F_2, T_1, T_2, C)$ if and only if there exist positive real numbers ρ_1, ρ_2 such that (x, y) is a fixed point of the mapping $G_{\rho_1,\rho_2} : H \times H \to H \times H$ defined by*

$$G_{\rho_1,\rho_2}(x, y) = (A_{\rho_1}(x, y), B_{\rho_2}(x, y)), \quad \forall (x, y) \in H \times H, \tag{2.1}$$

where $A_{\rho_1}, B_{\rho_2} : H \times H \to H$ are defined, respectively, by

$$A_{\rho_1}(x, y) = J_{\rho_1}^{F_1}[x - \rho_1 T_1(x, y)],$$
$$B_{\rho_2}(x, y) = J_{\rho_2}^{F_2}[y - \rho_2 T_2(x, y)]. \tag{2.2}$$

Now, we are in position to prove the existence theorem of solutions for the problem (1.1).

Theorem 2.2. *For each $i = 1, 2$, let $F_i : H \times H \to \mathbb{R}$ be a monotone bi-function. Let $T_1 : H \times H \to H$ be a ν_1-strongly monotone with respect to the first argument and (τ_1, σ_1)-Lipschitz mapping and*

$T_2 : H \times H \to H$ be a ν_2-strongly monotone with respect to the second argument and (τ_2, σ_2)-Lipschitz mapping. Suppose that there are positive real numbers ρ_1, ρ_2 such that

$$\begin{gathered}\left(1-2\rho_1\nu_1+\rho_1^2\tau_1^2\right)^{1/2}+\rho_2\tau_2<1,\\ \left(1-2\rho_2\nu_2+\rho_2^2\sigma_2^2\right)^{1/2}+\rho_1\sigma_1<1.\end{gathered} \tag{2.3}$$

Then SMIE (F_1, F_2, T_1, T_2, C) *is a singleton.*

Proof. Notice that, in view of Lemma 2.1, it is sufficient to show that the mapping G_{ρ_1,ρ_2} defined in Lemma 2.1 has the unique fixed point. Since $J_{\rho_1}^{F_1}$ is nonexpansive, we have the following estimate:

$$\begin{aligned}&\left\|A_{\rho_1}(x_1,y_1)-A_{\rho_1}(x_2,y_2)\right\|\\ &\quad=\left\|J_{\rho_1}^{F_1}[x_1-\rho_1T_1(x_1,y_1)]-J_{\rho_1}^{F_1}[x_2-\rho_1T_1(x_2,y_2)]\right\|\\ &\quad\le\left\|[x_1-\rho_1T_1(x_1,y_1)]-[x_2-\rho_1T_1(x_2,y_2)]\right\|\\ &\quad\le\left\|x_1-x_2-\rho_1[T_1(x_1,y_1)-T_1(x_2,y_1)]\right\|+\rho_1\left\|T_1(x_2,y_1)-T_1(x_2,y_2)\right\|.\end{aligned} \tag{2.4}$$

Since $T_1 : H \times H \to H$ is a (τ_1, σ_1)-Lipschitz mapping and, for all $w \in H$, the mapping $T_1(\cdot, w) : H \to H$ is a ν_1-strongly monotone, we obtain

$$\begin{aligned}&\left\|x_1-x_2-\rho_1[T_1(x_1,y_1)-T_1(x_2,y_1)]\right\|^2\\ &\quad=\|x_1-x_2\|^2-2\rho_1\langle x_1-x_2,T_1(x_1,y_1)-T_1(x_2,y_1)\rangle\\ &\qquad+\rho_1^2\left\|T_1(x_1,y_1)-T_1(x_2,y_1)\right\|^2\\ &\quad\le\|x_1-x_2\|^2-2\rho_1\nu_1\|x_1-x_2\|^2+\rho_1^2\tau_1^2\|x_1-x_2\|^2\\ &\quad=\left(1-2\rho_1\nu_1+\rho_1^2\tau_1^2\right)\|x_1-x_2\|^2,\\ &\left\|T_1(x_2,y_1)-T_1(x_2,y_2)\right\|\le\sigma_1\left\|y_1-y_2\right\|.\end{aligned} \tag{2.5}$$

Consequently, from (2.4)-(2.5), it follows that

$$\left\|A_{\rho_1}(x_1,y_1)-A_{\rho_1}(x_2,y_2)\right\|\le\left(1-2\rho_1\nu_1+\rho_1^2\tau_1^2\right)^{1/2}\|x_1-x_2\|+\rho_1\sigma_1\left\|y_1-y_2\right\|. \tag{2.6}$$

Next, we have the following estimate:

$$\begin{aligned}
&\left\|B_{\rho_2}(x_1,y_1)-B_{\rho_2}(x_2,y_2)\right\| \\
&\quad=\left\|J_{\rho_2}^{F_2}[y_1-\rho_2T_2(x_1,y_1)]-J_{\rho_2}^{F_2}[y_2-\rho_2T_2(x_2,y_2)]\right\| \\
&\quad\le\left\|[y_1-\rho_2T_2(x_1,y_1)]-[y_2-\rho_2T_2(x_2,y_2)]\right\| \\
&\quad\le\left\|y_1-y_2-\rho_2[T_2(x_1,y_1)-T_2(x_1,y_2)]\right\|+\rho_2\left\|T_2(x_1,y_2)-T_2(x_2,y_2)\right\| \\
&\quad\le\left(1-2\rho_2\nu_2+\rho_2^2\sigma_2^2\right)^{1/2}\left\|y_1-y_2\right\|+\rho_2\tau_2\left\|x_1-x_2\right\|.
\end{aligned}\tag{2.7}$$

From (2.6) and (2.7), we have

$$\begin{aligned}
&\left\|A_{\rho_1}(x_1,y_1)-A_{\rho_1}(x_2,y_2)\right\|+\left\|B_{\rho_2}(x_1,y_1)-B_{\rho_2}(x_2,y_2)\right\| \\
&\quad\le\max\{\kappa_1,\kappa_2\}\left(\left\|x_1-x_2\right\|+\left\|y_1-y_2\right\|\right),
\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}
\kappa_1&=l_1+\rho_2\tau_2, &\qquad \kappa_2&=l_2+\rho_1\sigma_1,\\
l_1&:=\left(1-2\rho_1\nu_1+\rho_1^2\tau_1^2\right)^{1/2}, &\qquad l_2&:=\left(1-2\rho_2\nu_2+\rho_2^2\sigma_2^2\right)^{1/2}.
\end{aligned}\tag{2.9}$$

Now, define the norm $\|\cdot\|^+$ on $H\times H$ by

$$\left\|(x,y)\right\|^+=\|x\|+\|y\|,\quad \forall(x,y)\in H\times H.\tag{2.10}$$

Notice that $(H\times H,\|\cdot\|^+)$ is a Banach space and

$$\left\|G_{\rho_1,\rho_2}(x_1,y_1)-G_{\rho_1,\rho_2}(x_2,y_2)\right\|^+\le\max\{\kappa_1,\kappa_2\}\left\|(x_1,y_1)-(x_2,y_2)\right\|^+.\tag{2.11}$$

By the condition (2.3), we have $\max\{\kappa_1,\kappa_2\}<1$, which implies that G_{ρ_1,ρ_2} is a contraction mapping. Hence, by Banach contraction principle, there exists a unique $(x,y)\in H\times H$ such that $G_{\rho_1,\rho_2}(x,y)=(x,y)$. This completes the proof. □

3. Convergence and Stability Analysis

In view of Lemma 2.1, for the fixed point formulation of the problem (2.1), we suggest the following iterative algorithm.

3.1. Mann Type Perturbed Iterative Algorithm (MTA)

For any $(x_0, y_0) \in H \times H$, compute approximate solution $(x_n, y_n) \in H \times H$ given by the iterative schemes:

$$
\begin{aligned}
&x_0 \in H, \\
&x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_{\rho_1}^{F_1}\left[x_n - \rho_1 T_1(x_n, y_n)\right], \\
&y_{n+1} = (1-\alpha_n)y_n + \alpha_n J_{\rho_2}^{F_2}\left[y_n - \rho_2 T_2(x_n, y_n)\right], \quad \forall n \geq 0,
\end{aligned}
\tag{3.1}
$$

where $\{\alpha_n\}$ is a sequence of real numbers such that $\alpha_n \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

In order to consider the convergence theorem of the sequences generated by the algorithm (MTA), we need the following lemma.

Lemma 3.1. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following conditions. There exists a positive integer n_0 such that*

$$
a_{n+1} \leq (1-\lambda_n)a_n + b_n, \quad \forall n \geq n_0,
\tag{3.2}
$$

where $\{\lambda_n\} \subset (0,1)$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $b_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

Now, we prove the convergence theorem for a solution for the problem (1.1).

Theorem 3.2. *If all the conditions of the Theorem 2.2 hold, then the sequence $\{(x_n, y_n)\}$ in $H \times H$ generated by the algorithm* (3.1) *converges strongly to the unique solution for the problem* (1.1).

Proof. It follows from Theorem 2.2 that there exists $(x^*, y^*) \in H \times H$ which is the unique solution for the problem (1.1). Moreover, in view of Lemma 2.1, we have

$$
\begin{aligned}
x^* &= J_{\rho_1}^{F_1}\left[x^* - \rho_1 T_1(x^*, y^*)\right], \\
y^* &= J_{\rho_2}^{F_2}\left[y^* - \rho_2 T_2(x^*, y^*)\right].
\end{aligned}
\tag{3.3}
$$

Since $J_{\rho_1}^{F_1}$ is nonexpansive, from the iterative sequences (3.1) and (3.3), it follows that

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad = \left\|(1-\alpha_n)x_n + \alpha_n J_{\rho_1}^{F_1}\left[x_n - \rho_1 T_1(x_n, y_n)\right] - x^*\right\| \\
&\quad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \left\|J_{\rho_1}^{F_1}\left[x_n - \rho_1 T_1(x_n, y_n)\right] - J_{\rho_1}^{F_1}\left[x^* - \rho_1 T_1(x^*, y^*)\right]\right\| \\
&\quad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \left\|x_n - x^* - \rho_1\left[T_1(x_n, y_n) - T_1(x^*, y^*)\right]\right\| \\
&\quad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \left\|x_n - x^* - \rho_1\left[T_1(x_n, y_n) - T_1(x^*, y_n)\right]\right\| \\
&\qquad + \alpha_n \rho_1 \left\|T_1(x^*, y_n) - T_1(x^*, y^*)\right\|.
\end{aligned}
\tag{3.4}
$$

Next, we have the following estimate:

$$\begin{aligned}
&\|x_n - x^* - \rho_1[T_1(x_n, y_n) - T_1(x^*, y_n)]\| + \rho_1\|T_1(x^*, y_n) - T_1(x^*, y^*)\| \\
&\quad \le \left(1 - 2\rho_1\nu_1 + \rho_1^2\tau_1^2\right)^{1/2}\|x_n - x^*\| + \rho_1\sigma_1\|y_n - y^*\|.
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) yields that

$$\|x_{n+1} - x^*\| \le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\left(1 - 2\rho_1\nu_1 + \rho_1^2\tau_1^2\right)^{1/2}\|x_n - x^*\| + \alpha_n\rho_1\sigma_1\|y_n - y^*\|. \tag{3.6}$$

Similarly, we have

$$\|y_{n+1} - y^*\| \le (1-\alpha_n)\|y_n - y^*\| + \alpha_n\left(1 - 2\rho_2\nu_2 + \rho_2^2\sigma_2^2\right)^{1/2}\|y_n - y^*\| + \alpha_n\rho_2\tau_2\|x_n - x^*\|. \tag{3.7}$$

Thus, from (3.6) and (3.7), we have

$$\begin{aligned}
&\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|^+ \\
&\quad \le (1-\alpha_n)\|(x_n, y_n) - (x^*, y^*)\|^+ + \alpha_n \max\{\kappa_1, \kappa_2\}\|(x_n, y_n) - (x^*, y^*)\|^+ \\
&\quad = [1 - \alpha_n(1 - \max\{\kappa_1, \kappa_2\})]\|(x_n, y_n) - (x^*, y^*)\|^+,
\end{aligned} \tag{3.8}$$

where κ_1 and κ_2 are given in (2.9). Setting

$$\begin{gathered}
a_n = \|(x_n, y_n) - (x^*, y^*)\|^+, \\
\lambda_n = \alpha_n(1 - \max\{\kappa_1, \kappa_2\}), \quad b_n = 0, \quad \forall n \ge 1.
\end{gathered} \tag{3.9}$$

From the condition (2.3), it follows that $\max\{\kappa_1, \kappa_2\} < 1$ and so $\{\lambda_n\} \subset [0,1]$. Moreover, since $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have $\sum_{n=0}^{\infty} \lambda_n = \infty$. Hence all the conditions of Lemma 3.1 are satisfied and so $a_n \to 0$ as $n \to \infty$, that is,

$$\|(x_n, y_n) - (x^*, y^*)\|^+ \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.10}$$

Thus the sequence $\{(x_n, y_n)\}$ in $H \times H$ converges strongly to a solution (x^*, y^*) for the problem (1.1). This completes the proof. □

3.2. Stability of the Algorithm (MTA)

Consider the following definition as an extension of the concept of stability of the iterative procedure given by Harder and Hicks [17].

Definition 3.3 (Kazmi and Khan [18]). Let H be a Hilbert space and $A, B : H \times H \to H$ be nonlinear mappings. Let $G : H \times H \to H \times H$ be defined as $G(x,y) = (A(x,y), B(x,y))$ for all $(x,y) \in H \times H$ and $(x_0, y_0) \in H \times H$. Assume that $(x_{n+1}, y_{n+1}) = f(G, x_n, y_n)$ defines an iterative procedure which yields a sequence $\{(x_n, y_n)\}$ in $H \times H$. Suppose that $F(G) = \{(x,y) \in H \times H : G(x,y) = (x,y)\} \neq \emptyset$ and the sequence $\{(x_n, y_n)\}$ converges to some $(x^*, y^*) \in F(G)$. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $H \times H$ and

$$\delta_n = \left\| (u_n, v_n) - f(G, x_n, y_n) \right\|, \quad \forall n \geq 0. \tag{3.11}$$

If $\lim_{n\to\infty}\delta_n = 0$ implies that $\lim_{n\to\infty}(u_n, v_n) = (x^*, y^*)$, then the iterative procedure $\{(x_n, y_n)\}$ is said to be *G-stable* or *stable* with respect to G.

Theorem 3.4. *Assume that all the conditions of Theorem 2.2 hold. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $H \times H$ and define $\{\delta_n\} \subset [0, \infty)$ by*

$$\delta_n = \| (u_{n+1}, v_{n+1}) - (C_n, D_n) \|^{+}, \tag{3.12}$$

where

$$\begin{aligned} C_n &= (1-\alpha_n)x_n + \alpha_n J_{\rho_1}^{F_1}\left[x_n - \rho_1 T_1(x_n, y_n)\right], \\ D_n &= (1-\alpha_n)y_n + \alpha_n J_{\rho_2}^{F_2}\left[y_n - \rho_2 T_2(x_n, y_n)\right], \end{aligned} \tag{3.13}$$

where $\{(x_n, y_n)\}$ is a sequence defined in (3.1). *If G_{ρ_1,ρ_2} is defined as in* (2.1), *then the iterative procedure generated by* (3.1) *is G_{ρ_1,ρ_2}-stable.*

Proof. Assume that $\lim_{n\to\infty}\delta_n = 0$. Let (x^*, y^*) be the unique fixed point of the mapping G_{ρ_1,ρ_2}. This means that

$$\begin{aligned} x^* &= J_{\rho_1}^{F_1}\left[x^* - \rho_1 T_1(x^*, y^*)\right], \\ y^* &= J_{\rho_2}^{F_2}\left[y^* - \rho_2 T_2(x^*, y^*)\right]. \end{aligned} \tag{3.14}$$

Now, from (3.12) and (3.13), it follows that

$$\left\| (u_{n+1}, v_{n+1}) - (x^*, y^*) \right\|^{+} \leq \delta_n + \| C_n - x^* \| + \left\| D_n - y^* \right\|. \tag{3.15}$$

Notice that $(C_n, D_n) = \{(x_{n+1}, y_{n+1})\}$ for each $n \geq 1$, which implies that

$$\lim_{n\to\infty} C_n = x^*, \qquad \lim_{n\to\infty} D_n = y^*. \tag{3.16}$$

Using (3.16) and the assumption $\lim_{n\to\infty}\delta_n = 0$, it follows from (3.15) that

$$\lim_{n\to\infty} (u_{n+1}, v_{n+1}) = (x^*, y^*). \tag{3.17}$$

This completes the proof. □

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