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Research Article

Regularization Inertial Proximal Point Algorithm for Monotone Hemicontinuous Mapping and Inverse Strongly Monotone Mappings in Hilbert Spaces

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The purpose of this paper is to present a regularization variant of the inertial proximal point algorithm for finding a common element of the set of solutions for a variational inequality problem involving a hemicontinuous monotone mapping A and for a finite family of λ_i-inverse strongly monotone mappings $\{A_i\}_{i=1}^N$ from a closed convex subset K of a Hilbert space H into H.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let K be a closed convex subset of H. Denote the metric projection of H onto K by P_K, that is

$$P_K(x) = \min_{y \in K} \|x - y\| \tag{1.1}$$

for all $x \in H$.

Definition 1.1. A set $A \subset H \times H$ is called monotone on H if A has the following property:

$$\langle x' - y', x - y \rangle \geq 0, \quad \forall (x, x'), (y, y') \in A. \tag{1.2}$$

A monotone mapping A in H is said to be maximal monotone if it is not properly contained in any other monotone mapping on H. Equivalently, a monotone mapping A is maximal monotone if $R(I + tA) = H$ for all $t > 0$, where $R(A)$ denotes the range of A.

Definition 1.2. A mapping A is called hemicontinuous at a point x in K if

$$\lim_{t\to 0}\langle A(x+th), \mathrm{y}\rangle = \langle A(x), \mathrm{y}\rangle, \quad x+th \in K, y \in H. \tag{1.3}$$

Definition 1.3. A mapping A of K into H is called λ-inverse strongly monotone if there exists a positive number λ such that

$$\langle A(x) - A(y), x - y\rangle \geq \lambda \|A(x) - A(y)\|^2 \tag{1.4}$$

for all $x, y \in K$.

Definition 1.4. A mapping T of K into H is called Lipschitz continuous on K if there exists a positive number L, named Lipschitz constant, such that

$$\|Tx - Ty\| \leq L\|x - y\| \tag{1.5}$$

for all $x, y \in K$.

It is easy to see that any λ-inverse strongly monotone mapping A is monotone and Lipschitz continuous with Lipschitz constant $L = 1/\lambda$. When $L = 1$, T is said to be nonexpansive mapping. Note that a nonexpansive mapping in Hilbert space is $(1/2)$-inverse strongly monotone [1].

Definition 1.5. A mapping $T : K \to H$ is said to be strictly pseudocontractive, if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in K. \tag{1.6}$$

Clearly, when $k = 0$, T is nonexpansive. Therefore, the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings.

Definition 1.6. A mapping T from K into H is said to be demiclosed at a point v if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to v$, then $Tx = v$, where the symbols $\to$ and $\rightharpoonup$ denote the strong and weak convergences of any sequence, respectively.

The variational inequality problem is to find $u^* \in K$ such that

$$\langle A(u^*), x - u^*\rangle \geq 0 \tag{1.7}$$

for all $x \in K$. The set of solutions of the variational inequality problem (1.7) is denoted by $VI(K, A)$.

Let $\{A_i\}_{i=1}^N$ be a finite family of λ_i-inverse strongly monotone mappings from K into H with the set of solutions denoted by $S_i = \{x \in K : A_i(x) = 0\}$. And set

$$S = \bigcap_{i=1}^{N} S_i. \tag{1.8}$$

The problem which will be studied in this paper is to find an element

$$u^* \in VI(K, A) \cap S \tag{1.9}$$

with assumption $VI(K, A) \cap S \neq \emptyset$.

A following example shows the fact that $VI(K, A) \cap S \neq \emptyset$. Consider the following case:

$$\begin{aligned} K &= \{(x_1, x_2) : -2 \leq x_1,\ x_2 \leq 2\}, \\ K_1 &= \{(x_1, x_2) : -2 \leq x_1 \leq 2,\ -1 \leq x_2 \leq 1\}, \\ K_2 &= \{(x_1, x_2) : -1 \leq x_1 \leq 1,\ -2 \leq x_2 \leq 2\}. \end{aligned} \tag{1.10}$$

$A_i := I - P_{K_i}$, $i = 1, 2$, where P_{K_i} denote the metric projections of K onto K_i, and the matrix A has the elements $a_{11} = a_{22} = 1$, $a_{12} = -2$, and $a_{21} = 0$. Then, A_i is $(1/2)$-inverse strongly monotone. Clearly, $A_i(x) = 0$ if and only if $x \in K_i$. It means that $S_i = K_i$. Consequently, $S = S_1 \cap S_2 = \{(x_1, x_2) : -1 \leq x_1,\ x_2 \leq 1\} \neq \emptyset$. It is easy to see that A is monotone, and for $y = (y_1, y_2)$, $Ay = 0$ if and only if $y_1 = y_2$. Therefore, $VI(K, A) \cap S \neq \emptyset$.

Since for a nonexpansive mapping T, the mapping $B := I - T$ is $(1/2)$-inverse strongly monotone, the problem of finding an element of $VI(K, A) \cap F(T)$, where $F(T)$ denotes the set of fixed points of the nonexpansive mapping T, is equivalent to that of finding an element of $VI(K, A) \cap S_B$, where S_B denotes the set of solutions of the mapping B, and contained in the class of problem (1.9).

The case, when A is λ-inverse strongly monotone and $A_1 = I - T$, where T is nonexpansive, is studied in [2].

Theorem 1.7 (see [2]). *Let K be a nonempty closed convex subset of a Hilbert space H. Let A be a λ-inverse strongly monotone mapping of K into H for $\lambda > 0$, and let T be a nonexpansive mapping of K into itself such that $VI(K, A) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} &x_0 \in K, \\ &x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_K (x_n - \alpha_n A(x_n)), \quad n \geq 0, \end{aligned} \tag{1.11}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\lambda)$ and $\{\alpha_n\} \subset (c, d)$ for some $c, d \in (0, 1)$. Then $\{x_n\}$ converges weakly to $z \in VI(K, A) \cap F(T)$, where

$$z = \lim_{n \to \infty} P_{VI(K,A) \cap F(T)}(x_n). \tag{1.12}$$

For finding an element of the set $VI(K, A) \cap F(T)$, one can use the extragradient method proposed in [3] for the case of finite-dimensional spaces. In the infinite-dimensional Hilbert spaces, the weak convergence result of the extragradient method was proved [1] and it was improved to the strong convergence in [4].

On the other hand, when $K \equiv H$, (1.7) is equivalent to the operator equation

$$A(u) = 0, \tag{1.13}$$

involving a maximal monotone A, since the domain of A is the whole space H, and A is hemicontinuous ([5, 6]). A zero element of (1.13) can be approximated by the inertial proximal point algorithm

$$c_n A(z_{n+1}) + z_{n+1} - z_n = \gamma_n (z_n - z_{n-1}), \quad z_0, z_1 \in H, \tag{1.14}$$

where $\{c_n\}$ and $\{\gamma_n\}$ are two sequences of positive numbers.

Note that the inertial proximal algorithm was proposed by Alvarez [7] in the context of convex minimization. Afterwards, Alvarez and Attouch [8] considered its extension to maximal monotone operators. Recently, Moudafi [9] applied this algorithm for variational inequalities; Moudafi and Elisabeth [10] studied the algorithm by using enlargement of a maximal monotone operator; Moudafi and Oliny [11] considered convergence of a splitting inertial proximal method. The main results in these papers are the weak convergence of the algorithm in Hilbert spaces.

In this paper, by introducing a regularization process we shall show that by adding the regularization term to the inertial proximal point algorithm, called regularization inertial proximal point algorithm, we obtain the strong convergence of the algorithm, and the strong convergence is proved for the general case $N > 1$; A_i, $i = 1, \ldots, N$, are λ_i-inverse strongly monotone nonself mappings of K into H; λ_i may not be $1/2$, A is monotone and hemicontinuous at each point $u \in K$.

2. Main Results

Let F be an equilibrium bifunction from $K \times K$ to $\mathbf{R}$, that is $F(u,u) = 0$ for every $u \in K$. In addition, assume that $F(u,v)$ is convex and lower semicontinuous in the variable v for each fixed $u \in K$.

The equilibrium problem for F is to find $u^* \in K$ such that

$$F(u^*, v) \geq 0 \quad \forall v \in K. \tag{2.1}$$

First, we recall several well-known facts in [12, 13] which are necessary in the proof of our results.

The equilibrium bifunction F is said to be

(i) monotone, if for all $u, v \in K$, we have

$$F(u,v) + F(v,u) \leq 0, \tag{2.2}$$

(ii) strongly monotone with constant τ, if, for all $u, v \in K$, we have

$$F(u,v) + F(v,u) \leq -\tau \|u - v\|^2, \tag{2.3}$$

(iii) hemicontinuous in the variable u for each fixed v, if

$$\lim_{t \to +0} F(u + t(z-u), v) = F(u,v) \quad \forall (u,z,v) \in K \times K \times K. \tag{2.4}$$

We can get the following proposition from the above definitions.

Proposition 2.1. (i) *If $F(u,v)$ is hemicontinuous in the first variable for each fixed $v \in K$ and F is monotone, then $U^* = V^*$, where U^* is the solution set of* (2.1)*, V^* is the solution set of $F(u,v^*) \leq 0$ for all $u \in K$, and they are closed and convex.*

(ii) *If $F(u,v)$ is hemicontinuous in the first variable for each $v \in K$ and F is strongly monotone, then U^* is a nonempty singleton.*

Lemma 2.2 (see [14]). *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be the sequences of positive numbers satisfying the conditions:*

(i) $a_{n+1} \leq (1 - b_n) a_n + c_n$, $b_n < 1$,

(ii) $\sum_{n=0}^{\infty} b_n = +\infty$, $\lim_{n \to +\infty} (c_n / b_n) = 0$.

Then, $\lim_{n \to +\infty} a_n = 0$.

Lemma 2.3 (see [15]). *Let K be a nonempty closed convex subset of a Hilbert space H and $T : K \to K$ a strictly pseudocontractive mapping. Then $I - T$ is demiclosed at zero.*

We construct a regularization solution u_α for (1.9) by solving the following variational inequality problem: find $u_\alpha \in K$ such that

$$\left\langle A(u_\alpha) + \alpha^\mu \sum_{i=1}^{N} A_i(u_\alpha) + \alpha u_\alpha, v - u_\alpha \right\rangle \geq 0 \quad \forall v \in K, \ 0 < \mu < 1, \tag{2.5}$$

where $\alpha \searrow 0$, is the regularization parameter.

We have the following result.

Theorem 2.4. *Let K be a nonempty closed convex subset of H. Let A_i be a λ_i-inverse strongly monotone mapping of K into H, and let A be a monotone hemicontinuous mapping of K into H such that $VI(K,A) \cap F(T) \neq \emptyset$. Then, we have*

(i) *For each $\alpha > 0$, the problem* (2.5) *has a unique solution u_α;*

(ii) *If $\lim_{\alpha \to +0} u_\alpha = u^*$, then $u^* \in VI(K,A) \cap S$ and $\|u^*\| \leq \|y\|$ for all $y \in VI(K,A) \cap S$;*

(iii)

$$\|u_\alpha - u_\beta\| \leq M \frac{|\alpha - \beta|}{\alpha}, \quad \alpha, \beta > 0, \tag{2.6}$$

where M is a positive constant.

Proof. (i) Let

$$F_0(u,v) = \langle A(u), v-u\rangle,$$
$$F_i(u,v) = \langle A_i(u), v-u\rangle, \quad i = 1,\dots,N. \tag{2.7}$$

Then, problem (2.5) has the following form: find $u_\alpha \in K$ such that

$$\mathbf{F}_\alpha(u_\alpha, v) \geq 0 \quad \forall v \in K, \tag{2.8}$$

where

$$\mathbf{F}_\alpha(u,v) = F_0(u,v) + \alpha^\mu \sum_{i=1}^{N} F_i(u,v) + \alpha\langle u, v-u\rangle. \tag{2.9}$$

It is not difficult to verify that F_i, $i = 0,\dots,N$, are the monotone bifunctions, and for each fixed $v \in K$, they are hemicontinuous in the variable u. Therefore, $\mathbf{F}_\alpha(u,v)$ also is monotone hemicontinuous in the variable u for each fixed $v \in K$. Moreover, it is strongly monotone with constant $\alpha > 0$. Hence, (2.8) (consequently (2.5)) has a unique solution u_α for each $\alpha > 0$.

(ii) Now we prove that

$$\|u_\alpha\| \leq \|y\|, \quad \forall y \in VI(K,A) \cap S. \tag{2.10}$$

Since $y \in VI(K,A) \cap S, F_0(y,u_\alpha) \geq 0$ and $A_i(y) = 0$, $i = 1,\dots,N$, $F_i(y,u_\alpha) = 0$, $i = 1,\dots,N$, and

$$F_0(y,u_\alpha) + \alpha^\mu \sum_{i=1}^{N} F_i(y,u_\alpha) \geq 0, \quad \forall y \in VI(K,A) \cap S. \tag{2.11}$$

By adding the last inequality to (2.8) in which v is replaced by y and using the properties of F_i, we obtain

$$\langle u_\alpha, y - u_\alpha\rangle \geq 0, \quad \forall y \in VI(K,A) \cap S, \tag{2.12}$$

that implies (2.10). It means that $\{u_\alpha\}$ is bounded. Let $u_{\alpha_k} \rightharpoonup u^* \in H$, as $k \to +\infty$. Since K is closed and convex, K is weakly closed. Hence $u^* \in K$. We prove that $u^* \in VI(K,A)$. From the monotone property of F_i, $i = 0,\dots,N$ and (2.8), it follows

$$F_0(v,u_{\alpha_k}) + \alpha_k^\mu \sum_{i=1}^{N} F_i(v,u_{\alpha_k}) \leq \alpha_k\langle v, v - u_{\alpha_k}\rangle, \quad \forall v \in K. \tag{2.13}$$

Letting $k \to \infty$, we obtain $F_0(v,u^*) \leq 0$ for any $v \in K$. By virtue of Proposition 2.1, we have $u^* \in VI(K,A)$.

Now we show that $u^* \in S_i$, for all $i = 1, \ldots, N$. From (2.8), $F_0(y, u_{\alpha_k}) \geq 0$ for any $y \in VI(K, A) \cap S$, and the monotone property of F_0, it implies that

$$\sum_{i=1}^{N} F_i(u_{\alpha_k}, y) + \alpha_k^{1-\mu} \langle u_{\alpha_k}, y - u_{\alpha_k} \rangle \geq 0, \quad \forall y \in VI(K, A) \cap S. \tag{2.14}$$

On the base of λ_l-inverse strongly monotone property of A_l, the monotone property of A_i, $i \neq l$, $A_i(y) = 0$, for all $y \in VI(K, A) \cap S$, $i = 1, \ldots, N$. From the last inequality, we have

$$\begin{aligned}
\lambda_l \| A_l(u_{\alpha_k}) - A_l(y) \|^2 &\leq \langle A_l(u_{\alpha_k}), u_{\alpha_k} - y \rangle \\
&\leq \sum_{i=1}^{N} \langle A_i(u_{\alpha_k}), u_{\alpha_k} - y \rangle \\
&\leq \alpha_k^{1-\mu} \langle u_{\alpha_k}, y - u_{\alpha_k} \rangle \\
&\leq 2\alpha_k^{1-\mu} \| y \|^2.
\end{aligned} \tag{2.15}$$

Tending $k \to +\infty$ in the last inequality, we obtain

$$\lim_{k \to +\infty} \| A_l(u_{\alpha_k}) - A_l(y) \| = 0. \tag{2.16}$$

Since A_l is λ_l-inverse strongly monotone, the mapping $T_l := I - A_l$ satisfies (1.6), where $\lambda_l = (1 - k_l)/2$. Because $0 < \lambda_l < 1$, we have $-1 < k_l < 1$. When $k_l < 0$, this inequality will not be changed if k_l is replaced by $-k_l$. Thus, T_l is strictly pseudocontractive. Applying Lemma 2.2, we can conclude that $A_l(u^*) = A_l(y) = 0$. It means that $u^* \in S_l$. It is well known that the sets $VI(K, A), S_i$ are closed and convex. Therefore, $VI(K, A) \cap S$ is also closed and convex. Then, from (2.10) it implies that u^* is the unique element in $VI(K, A) \cap S$ having a minimal norm. Consequently, we have

$$\lim_{\alpha \to +0} u_\alpha = u^*. \tag{2.17}$$

(iii) From (2.8) and the properties of $F_i(u, v)$, for each $\alpha, \beta > 0$, it follows

$$(\alpha^\mu - \beta^\mu) \sum_{i=1}^{N} F_i(u_\alpha, u_\beta) + \alpha \langle u_\alpha, u_\beta - u_\alpha \rangle + \beta \langle u_\beta, u_\alpha - u_\beta \rangle \geq 0 \tag{2.18}$$

or

$$\| u_\alpha - u_\beta \| \leq \frac{|\alpha - \beta|}{\alpha} \| u_\beta \| + \frac{|\alpha^\mu - \beta^\mu|}{\alpha} \sum_{i=1}^{N} |F_i(u_\alpha, u_\beta)|. \tag{2.19}$$

For each $i = 1, \ldots, N, F_i$ is bounded since the operator A_i is Lipschitzian with Lipschitz constants $L_i = 1/\lambda_i$. Using (2.10), the boundedness of F_i and the Lagrange's mean-value theorem for the function $\alpha(t) = t^{-\mu}$, $0 < \mu < 1$, $t \in [1, +\infty)$, on $[\alpha, \beta]$ if $\alpha < \beta$ or $[\beta, \alpha]$ if $\beta < \alpha$, we have conclusion (iii). This completes the proof. $\square$

Remark 2.5. Obviously, if $u_{\alpha_k} \to \tilde{u}$, where u_{α_k} is the solution of (2.8) with $\alpha = \alpha_k \to 0$, as $k \to +\infty$, then $VI(K, A) \cap S \neq \emptyset$.

Now, we consider the regularization inertial proximal point algorithm

$$\begin{gathered}\left\langle c_n \left[A(z_{n+1}) + \alpha_n^{\mu} \sum_{i=1}^{N} A_i(z_{n+1}) + \alpha_n z_{n+1} \right] + z_{n+1} - z_n, v - z_{n+1} \right\rangle \\ \geq \gamma_n \langle z_n - z_{n-1}, v - z_{n+1} \rangle, \quad \forall v \in K,\ z_0, z_1 \in K.\end{gathered} \tag{2.20}$$

Clearly,

$$\mathbf{F}_n(u, v) := \left\langle c_n \left[A(u) + \alpha_n^{\mu} \sum_{i=1}^{N} A_i(u) + \alpha_n u \right] + u - z_n, v - u \right\rangle - \gamma_n \langle z_n - z_{n-1}, v - u \rangle \tag{2.21}$$

is a bifunction. Moreover, it is strongly monotone with $\tau = c_n \alpha_n + 1$. By Proposition 2.1, there exists a unique element z_{n+1} satisfying (2.20).

Theorem 2.6. *Let K be a nonempty closed convex subset of a Hilbert space H. Let A_i be a λ_i-inverse strongly monotone mapping of K into H, and let A be a monotone hemicontinuous mapping of K into H such that $VI(K, A) \cap F(T) \neq \emptyset$. Assume that the parameters c_n, γ_n, and α_n are chosen such that*

(i) $0 < c_0 < c_n < C_0$, $0 \leq \gamma_n < \gamma_0$, $1 \geq \alpha_n \searrow 0$,

(ii) $\sum_{n=1}^{\infty} b_n = +\infty$, $b_n = c_n \alpha_n / (1 + c_n \alpha_n)$,

(iii) $\sum_{n=1}^{\infty} \gamma_n b_n^{-1} \| z_n - z_{n-1} \| < +\infty$,

(iv)

$$\lim_{n \to \infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n b_n} = 0. \tag{2.22}$$

Then the sequence $\{z_n\}$ defined by (2.20) *converges strongly to the element u^* as $n \to +\infty$.*

Proof. From (2.20) it follows

$$\begin{gathered}\left\langle \mu_n \left[A(z_{n+1}) + \alpha_n^{\mu} \sum_{i=1}^{N} A_i(z_{n+1}) \right] + z_{n+1}, v - z_{n+1} \right\rangle \geq \beta_n \langle z_n, v - z_{n+1} \rangle + \beta_n \gamma_n \langle z_n - z_{n-1}, v - z_{n+1} \rangle, \\ \mu_n = c_n \beta_n, \quad \beta_n = \frac{1}{(1 + c_n \alpha_n)}.\end{gathered} \tag{2.23}$$

By the similar argument, from (2.5) it implies

$$\left\langle \mu_n \left[A(u_n) + \alpha_n^{\mu} \sum_{i=1}^{N} A_i(u_n) \right] + u_n, v - u_n \right\rangle \geq \beta_n \langle u_n, v - u_n \rangle, \tag{2.24}$$

where u_n is the solution of (2.5) when α is replaced by α_n. By setting $v = u_n$ in (2.23) and $v = z_{n+1}$ in (2.24) and adding one obtained result to the other, we have,

$$\begin{aligned} &\mu_n \left\langle A(z_{n+1}) - A(u_n) + \alpha_n^{\mu} \sum_{i=1}^{N} (A_i(z_{n+1}) - A_i(u_n)), u_n - z_{n+1} \right\rangle + \langle z_{n+1} - u_n, u_n - z_{n+1} \rangle \\ &\quad \geq \beta_n \langle z_n - u_n, u_n - z_{n+1} \rangle + \beta_n \gamma_n \langle z_n - z_{n-1}, u_n - z_{n+1} \rangle. \end{aligned} \tag{2.25}$$

Therefore, from the monotone property of the mappings A, A_i, $i = 1, \ldots, N$, it follows

$$\|z_{n+1} - u_n\| \leq \beta_n \|z_n - u_n\| + \beta_n \gamma_n \|z_n - z_{n-1}\|. \tag{2.26}$$

From (2.23), (2.5) with $y = u^*$ and $\beta_n < 1$, we have

$$\begin{aligned} \|z_{n+1} - u_{n+1}\| &\leq \|z_{n+1} - u_n\| + \|u_{n+1} - u_n\| \\ &\leq \beta_n \|z_n - u_n\| + \gamma_n \|z_n - z_{n-1}\| + M \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} \\ &\leq (1 - b_n) \|z_n - x_n\| + d_n, \end{aligned} \tag{2.27}$$

where

$$d_n = \gamma_n \|z_n - z_{n-1}\| + M \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}. \tag{2.28}$$

Since the seris in (iii) is convergent, $\lim_{n \to \infty} \gamma_n \|z_n - z_{n-1}\| / b_n = 0$. Consequently, $\lim_{n \to \infty} d_n / b_n = 0$. From Lemma 2.2, it follows that $\|z_{n+1} - x_{n+1}\| \to 0$ as $n \to +\infty$.

On the other hand, $\|x_n - u^*\| \to 0$ as $n \to +\infty$. Therefore, we have $z_n \to u^*$, as $n \to +\infty$. This completes the proof. □

Remark 2.7. The sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ which are defined by

$$\alpha_n = (1 + n)^{-p}, \quad 0 < 2p < 1, \quad \gamma_n = (1 + n)^{-\tau} \frac{\|z_n - z_{n-1}\|}{1 + \|z_n - z_{n-1}\|^2} \tag{2.29}$$

with $\tau > 1 + p$ satisfy all conditions in Theorem 2.6.

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