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Research Article

Oscillations of Second-Order Neutral Impulsive Differential Equations

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Necessary and sufficient conditions are established for oscillation of second-order neutral impulsive differential equation $[y(t)+py(t-\tau)]''+qy(t-\sigma)=0$, $t\neq t_k$, $\Delta(y'(t_k)+py'(t_k-\tau))+q_1y(t_k-\sigma)=0$, where the coefficients $q, q_1>0$; $p<0$, $\tau,\sigma<0$, and $\Delta y^{(i)}(t_k)=y^{(i)}(t_k^+)-y^{(i)}(t_k^-)$, $i=0,1$.

1. Introduction

Oscillation theory is one of the directions which initiated the investigations of the qualitative properties of differential equations. This theory started with the classical works of Sturm and Kneser, and still attracts the attention of many mathematicians as much for the interesting results obtained as for their various applications.

In 1989 the paper of Gopalsamy and Zhang [1] was published, where the first investigation on oscillatory properties of impulsive differential equations was carried out.

The monograph [2] is the first book to present systematically the results known up to 1998, and to demonstrate how well-know mathematical techniques and methods, after suitable modification, can be applied in proving oscillatory theorems for impulsive differential equations.

Recently, the oscillatory theory of differential equations has been the subject of intensive research [3–8]. In particular, many remarkable results for the oscillatory properties of various classes of impulsive differential equations can be found in the literature [2, 9–11].

The notion of characteristic system was first introduced by Bainov and Simeonov [2]; it can be used in obtaining of various necessary and sufficient conditions for oscillation of constant coefficients linear impulsive differential equations of first order with one or several deviating arguments.

As we know, on the case when we investigate constant coefficients linear neutral differential equations without impulse, it is very significant to obtain necessary and sufficient conditions for the oscillation corresponding to their characteristic equations; some necessary and sufficient conditions (in terms of the characteristic equation) for the oscillation of all solutions of first-or second-order neutral differential equations were established in [12–20]. However, the oscillation theory of the second order impulsive differential equations is not yet perfect compared to the second order differential equations with deviating argument (see [2]). For example, due to some obstacles of theoretical and technical character in handling with constant coefficients linear impulsive differential equations of second or higher order, there are no results which studied the necessary and sufficient conditions in monograph [2]. How to establish the necessary and sufficient conditions for second order constant coefficients linear impulsive differential equations corresponding to their characteristic systems? This is also an important open problem (see monograph [2]). In this paper, we study and solve this problem for a class of linear impulsive differential equations of second order with advanced argument.

We shall restrict ourselves to the studying of impulsive differential equations for which the impulse effects take part at fixed moments $\{t_k\}$. Considering the second order neutral impulsive differential equations with constant coefficients

$$\begin{gathered}[y(t)+py(t-\tau)]''+qy(t-\sigma)=0,\quad t\neq t_k,\\ \Delta(y'(t_k)+py'(t_k-\tau))+q_1y(t_k-\sigma)=0,\end{gathered}\tag{1.1}$$

where

$$q,q_1>0;\ p<0,\tau,\sigma<0,\quad \Delta y^{(i)}(t_k)=y^{(i)}(t_k^+)-y^{(i)}(t_k^-),\quad i=0,1.\tag{1.2}$$

Throughout this paper, we assume that the sequence $\{t_k\}$ of the moments of impulse effects has the following properties (H):

(H1.1) $0<t_1<t_2<\cdots$, and $\lim_{k\to\infty}t_k=+\infty$.

Here considering $|\tau|$-periodic and $|\sigma|$-periodic equation with $|\tau|,|\sigma|>0$, we suppose that the sequence $\{t_k\}$ satisfies the following conditions.

(H1.2) There exist nonnegative integers n_1 and n_2 such that

$$t_{k+n_1}=t_k+|\tau|,\quad t_{k+n_2}=t_k+|\sigma|,\quad k\in\mathbf{N}.\tag{1.3}$$

This condition is equivalent to the next one.

(H1.3) There exist nonnegative integers n_1 and n_2 such that

$$i[t,t+|\tau|)=n_1,\quad i[t,t+|\sigma|)=n_2,\quad t\in R^+.\tag{1.4}$$

Here $i[a,b)$ denotes the number of the points t_k, lying in the interval $[a,b)$.

As customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory. As usual, we use the term "finally" to mean "for sufficiently large t". For other related notions, see monographs [2, 9–11] for details.

We clearly see that (1.1) is an equation without impulse and $\Delta y(t_k) = \Delta y'(t_k) = 0$ together with $q_1 = 0$ if $n_1 = n_2 = 0$ in (H1.2) or (H1.3). In this case, (1.1) reduces to

$$\left[y(t) + py(t-\tau)\right]'' + qy(t-\sigma) = 0. \tag{1.5}$$

2. Asymptotic Behavior of the Solutions

In the sequence, we assume that conditions (H1.1)–(H1.3) hold, and

$$v(t) = -\left[y(t) + py(t-\tau)\right]. \tag{2.1}$$

Lemma 2.1. *Suppose that $v(t)$ is defined by* (2.1)*. If* (1.1) *has a finally positive solution $y(t)$, then*

(a) *$v(t)$ is also a solution of* (1.1)*, that is,*

$$\begin{gathered} \left[v(t) + pv(t-\tau)\right]'' + qv(t-\sigma) = 0, \quad t \neq t_k, \\ \Delta\left(v'(t_k) + pv'(t_k-\tau)\right) + q_1 v(t_k-\sigma) = 0. \end{gathered} \tag{2.2}$$

(b)

$$\begin{gathered} (-1)^\nu v^{(\nu)}(t) > 0, \quad \nu = 0,1,2, \qquad \lim_{t\to\infty} v^{(\nu)}(t) = 0, \quad \nu = 0,1, \\ \Delta v(t_k) < 0, \quad \Delta v'(t_k) > 0. \end{gathered} \tag{2.3}$$

or

$$\begin{gathered} v^{(\nu)}(t) > 0, \quad \nu = 0,1,2, \qquad \lim_{t\to\infty} v^{(\nu)}(t) = \infty, \quad \nu = 0,1,2, \\ \Delta v(t_k) > 0, \quad \Delta v'(t_k) > 0. \end{gathered} \tag{2.4}$$

(c) *If* (2.3) *holds, then $p < -1$.*

(d) *If $y(t)$ is a solution of* (1.1)*, then*

$$z(t) = y'(t), \quad t \neq t_k; z(t_k) = \frac{q}{q_1}\Delta y(t_k) \tag{2.5}$$

is also a solution of (1.1)*.*

Proof. (a) It follows from condition (H1.2) that if t_k is a moment of impulse effect, then $t_k - \tau$ is also a such moment. Thus, $y(t-\tau)$ is also a solution of (1.1) since (1.1) is a linear one with constant coefficients. Therefore, $v(t)$ is a solution of (1.1) which follows from the linear combination of solutions.

(b) We clearly see that

$$v''(t) = qy(t-\sigma) > 0, \quad t \neq t_k,$$
$$\Delta v'(t_k) = q_1 y(t_k - \sigma) > 0, \tag{2.6}$$

which imply that $v'(t)$ is strictly increasing and so either

$$\lim_{t\to\infty} v'(t) = \infty \tag{2.7}$$

or

$$\lim_{t\to\infty} v'(t) = l < \infty. \tag{2.8}$$

We clearly see that (2.7) implies inequality (2.4). Now, we assume that inequality (2.8) holds. We first prove that $l = 0$.

In fact, integrating (2.6) from t_0 to t and letting $t \to \infty$, we get

$$l - v'(t_0) = q\int_{t_0}^{\infty} y(s-\sigma)ds + \sum_{t_k \geq t_0} \Delta v'(t_k), \tag{2.9}$$

hence

$$\int_{t_0}^{\infty} y(s-\sigma)ds < \infty, \quad \sum_{t_k \geq t_0} \Delta v'(t_k) < \infty, \tag{2.10}$$

which implies that $y(t)$ is integrable in $[t_0, \infty)$,

$$y \in L^1[t_0, \infty). \tag{2.11}$$

Then from (2.1) we get

$$v \in L^1[t_0, \infty), \tag{2.12}$$

and so $l = 0$. (Otherwise, by L'Hospital's rule, we have $\lim_{t\to\infty} v(t)/t = \lim_{t\to\infty} v'(t) = l \neq 0$. $v(t) \to \pm\infty$ when $t \to \infty$, then $v(t) \notin L^1[t_0, \infty)$). Thus $v'(t)$ increases to zero, which implies that eventually

$$v'(t) < 0. \tag{2.13}$$

Then $v(t)$ decreases and in view of $v \in L^1[t_0, \infty)$, we have

$$\lim_{t\to\infty} v(t) = 0, \tag{2.14}$$

hence

$$v(t) > 0. \tag{2.15}$$

(c) For the sake of contradiction, assume that (2.3) holds and $p \geq -1$. From parts (a) and (b), we know that $w(t) = -[v(t) + pv(t-\tau)]$ is also a positive solution of (1.1), therefore,

$$-\left[v(t) + pv(t-\tau)\right] > 0, \tag{2.16}$$

which implies that

$$v(t) < -pv(t-\tau) \leq v(t-\tau); \tag{2.17}$$

this contradicts with the fact that $v(t)$ is a decreasing function.

(d) Let $y(t)$ be a solution of (1.1), and

$$\begin{aligned} z(t) &= y'(t), \quad t \neq t_k, \\ z(t_k) &= \frac{q}{q_1}\Delta y(t_k). \end{aligned} \tag{2.18}$$

We prove that $z(t)$ is also a solution of (1.1).

Clearly, $z(t)$ satisfies

$$\begin{aligned} &\left[z(t) + pz(t-\tau)\right]'' + qz(t-\sigma) = 0, \quad t \neq t_k, \\ &\Delta z'(t_k) = z'\left(t_k^+\right) - z'\left(t_k^-\right) = y''\left(t_k^+\right) - y''\left(t_k^-\right). \end{aligned} \tag{2.19}$$

Since $y(t)$ is a solution of (1.1), so

$$\begin{aligned} \Delta z'(t_k) + p\Delta z'(t_k-\tau) &= y''\left(t_k^+\right) - y''\left(t_k^-\right) + py''\left(t_k^+ - \tau\right) - py''\left(t_k^- - \tau\right) \\ &= -qy\left(t_k^+ - \sigma\right) + qy\left(t_k^- - \sigma\right) = -q\Delta y(t_k-\sigma) \\ &= -q\frac{q_1}{q}z(t_k-\sigma) = q_1 z(t_k-\sigma). \end{aligned} \tag{2.20}$$

That is,

$$\Delta\left(z'(t_k) + pz'(t_k-\tau)\right) + q_1 z(t_k-\sigma) = 0, \tag{2.21}$$

which implies that $z(t)$ is also a solution of (1.1).

Let W^- and W^+ be the set of all functions of the form

$$w(t) = -\left[v(t) + pv(t-\tau)\right] > 0, \tag{2.22}$$

where $v(t)$ is a solution of (1.1) which satisfies (2.3) and (2.4), respectively. In view of Lemma 2.1, either W^- or W^+ is nonempty. Also, an argument similar to that of Lemma 2.1 shows that each function $w \in W^- \bigcup W^+$ is a solution of (1.1), and satisfies

$$\begin{gathered} \left[w(t) + pw(t-\tau)\right]'' + qw(t-\sigma) = 0, \quad t \neq t_k, \\ \Delta\left(w'(t_k) + pw'(t_k - \tau)\right) + q_1 w(t_k - \sigma) = 0. \end{gathered} \tag{2.23}$$

Also, there is a solution v of (1.1) which satisfies (2.3) if $w \in W^-$ or (2.4) if $w \in W^-$ such that

$$\begin{gathered} w''(t) = qv(t-\sigma), \quad t \neq t_k, \\ \Delta w'(t_k) = q_1 v(t_k - \sigma). \end{gathered} \tag{2.24}$$

Clearly, every function $w \in W^-$ satisfies

$$\begin{gathered} (-1)^\nu w^{(\nu)}(t) > 0, \quad \nu = 0,1,2, \qquad \lim_{t \to \infty} w^{(\nu)}(t) = 0, \quad \nu = 0,1, \\ \Delta w(t_k) < 0, \quad \Delta w'(t_k) > 0, \end{gathered} \tag{2.25}$$

while every function $w \in W^+$ satisfies

$$\begin{gathered} w^{(\nu)}(t) > 0, \quad \nu = 0,1,2, \qquad \lim_{t \to \infty} \omega^{(\nu)}(t) = \infty, \quad \nu = 0,1,2. \\ \Delta w(t_k) > 0, \quad \Delta w'(t_k) > 0. \end{gathered} \tag{2.26}$$

Furthermore,

$$\begin{gathered} w(t) \in W^- \Longrightarrow -\left[w(t) + pw(t-\tau)\right] \in W^-, \\ w(t) \in W^+ \Longrightarrow -\left[w(t) + pw(t-\tau)\right] \in W^+. \end{gathered} \tag{2.27}$$

Finally, $w_1, w_2 \in W^-$ ($\in W^+$, resp.) and $a, b > 0 \Rightarrow aw_1 + bw_2 \in W^-$ ($\in W^+$, resp.). □

3. Oscillation of the Unbounded Solutions

First, we will assume that $W^- = \emptyset$ (i.e., $W^+ \neq \emptyset$).

The present section is devoted to the characterizing of the oscillatory properties of solutions of the periodic neutral impulsive differential equation (1.1).

We are looking for a positive solution of (1.1) having the form

$$y(t) = e^{\lambda t}(1+\mu)^{i[0,t)}, \tag{3.1}$$

where $\lambda, \mu > 0$ are constants.

Substituting (3.1) in (1.1), and from condition (H1.3), we obtain the characteristic system of (1.1) as follows:

$$\lambda^2 + p\lambda^2 e^{-\lambda\tau}(1+\mu)^{n_1} + qe^{-\lambda\sigma}(1+\mu)^{n_2} = 0,$$
$$\lambda\mu + p\lambda\mu e^{-\lambda\tau}(1+\mu)^{n_1} + qe^{-\lambda\sigma}(1+\mu)^{n_2} = 0. \tag{3.2}$$

As $q, q_1 > 0$, the characteristic system is equivalent to

$$\mu = \frac{q_1}{q}\lambda,$$
$$F(\lambda) \equiv \lambda^2 + p\lambda^2 e^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1} + qe^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2} = 0. \tag{3.3}$$

Equation (3.3) is called a characteristic equation, corresponding to the (1.1).

Theorem 3.1. *If the conditions (H1.1)–(H1.3) hold, then the following assertions are equivalent.*

(a) *Each unbounded regular solution of* (1.1) *is oscillatory.*

(b) *The characteristic equation* (3.3) *has no positive real roots.*

The proof of (a) ⇒ (b) is obvious. In fact, if the characteristic equation (3.3) has a real root $\lambda > 0$, then we clearly see that $\mu = (q_1/q)\lambda > 0$ and therefore the function

$$y(t) = e^{\lambda t}(1+\mu)^{i[0,t)} \tag{3.4}$$

is an unbounded positive solution of (1.1).

The proof of (b) ⇒ (a) is quite complicated and will be accomplished by establishing a series of lemmas.

We assume that (3.3) has no positive real roots and, for the sake of contradiction, we assume that (1.1) has an unbounded finally positive solution $y(t)$.

Lemma 3.2. *If* (1.1) *has an eventually positive solution, then*

(a) $\sigma < \tau$;

(b) *there exists a positive constant* m *such that*

$$F(\lambda) \equiv \lambda^2 + p\lambda^2 e^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1} + qe^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2} \geq m, \quad \forall\lambda \geq 0. \tag{3.5}$$

Proof. (a) Otherwise, $\sigma \geq \tau$, therefore $F(\infty) = -\infty$, but $F(0) = q > 0$ is impossible because the characteristic equation $F(\lambda) = 0$ has no positive real roots.

(b) We have $F(0) = q > 0$, $F(\infty) = \infty$ and so $F(\lambda) > 0$ for all $\lambda \geq 0$.
Hence there exists a positive constant m such that

$$F(\lambda) \geq m, \quad \forall \lambda \geq 0, \tag{3.6}$$

which completes the proof of Lemma 3.2. □

For each function $w \in W^+$, define the set

$$\Lambda(w) = \left\{ \lambda \geq 0 \,\middle|\, \begin{array}{ll} -w''(t) + \lambda^2 w(t) \leq 0 & \text{finally} \\ -\Delta w'(t_k) + \dfrac{q_1}{q}\lambda^2 w(t_k) \leq 0 & \text{finally} \end{array} \right\}. \tag{3.7}$$

Clearly, $0 \in \Lambda(w)$ and if $\lambda \in \Lambda(w)$ then $[0, \lambda] \subseteq \Lambda(w)$. Therefore, $\Lambda(w)$ is a nonempty subinterval of $\mathbf{R}^+$.

Lemma 3.3. *For $w(t) \in W^+$ and $\Lambda(w)$, we have the folloeing.*

(a) *If $w \in W^+$ and $\lambda_0 \equiv (q/-p)^{1/2}$, then $\lambda_0 \in \Lambda(w)$.*

(b) *$\Lambda(w)$ is bounded above by a positive constant γ, for all $w \in W^+$.*

(c) *If $w \in W^+$ and $\lambda \in \Lambda(w)$, then*

$$\begin{gathered} w'(t) - \lambda w(t) \geq 0, \quad t \neq t_k, \\ \Delta w(t_k) - \frac{q_1}{q}\lambda w(t_k) \geq 0. \end{gathered} \tag{3.8}$$

Proof. (a) From (2.23), (2.26) and the fact that $w''(t) > 0$ and $\Delta w'(t_k) > 0$, we have

$$\begin{gathered} pw''(t-\tau) + qw(t-\sigma) < 0, \\ p\Delta w'(t_k - \tau) + q_1 w(t_k - \sigma) < 0. \end{gathered} \tag{3.9}$$

That is,

$$\begin{gathered} w''(t) + \frac{q}{p} w(t + (\tau - \sigma)) > 0, \\ \Delta w'(t_k) + \frac{q_1}{p} w(t_k + (\tau - \sigma)) < 0. \end{gathered} \tag{3.10}$$

The increasing nature of $w(t)$ and the fact that $\tau > \sigma$ imply that

$$\begin{gathered} w''(t) + \frac{q}{p} w(t) > 0, \quad t \neq t_k, \\ \Delta w'(t_k) + \frac{q_1}{p} w(t_k) < 0, \end{gathered} \tag{3.11}$$

which shows that

$$\lambda_0 \equiv \left(\frac{q}{-p}\right)^{1/2} \in \Lambda(w). \tag{3.12}$$

By integrating (3.10) from $t-\alpha$ to t with $\alpha > 0$, we get

$$w'(t) - w'(t-\alpha) + \frac{q}{p}\int_{t-\alpha}^{t} w(s+(\tau-\sigma))ds - \sum_{t\le t_k<t-\alpha} \Delta w'(t_k) > 0. \tag{3.13}$$

From the fact that $w'(t-\alpha) > 0$ and $\Delta w'(t_k) > 0$, we have

$$w'(t) + \frac{q}{p}\alpha w(t-\alpha+(\tau-\sigma)) > 0. \tag{3.14}$$

By integrating again from $t-\beta$ to t with $\beta > 0$ we get

$$w(t) - w(t-\beta) + \frac{q}{p}\int_{t-\beta}^{t} w(s-\alpha+(\tau-\sigma))ds - \sum_{t\le t_k<t-\beta} \Delta w(t_k) > 0, \tag{3.15}$$

from the fact that $w(t-\beta) > 0$ and $\Delta w(t_k) > 0$ together with $w'(t) > 0$, we have

$$w(t) + \frac{q}{p}\alpha\beta w\big(t-(\alpha+\beta)+(\tau-\sigma)\big) > 0. \tag{3.16}$$

Let $\alpha = \beta = 1/4(\tau-\sigma) > 0$, then

$$w(t) + \frac{q}{p}\frac{(\tau-\sigma)^2}{16} w\left(t+\frac{\tau-\sigma}{2}\right) > 0. \tag{3.17}$$

That is,

$$\left(t+\frac{\tau-\sigma}{2}\right) < Aw(t), \tag{3.18}$$

where $A = -16p/q(\tau-\sigma)^2 > 0$.

Now let $k \in \mathbf{N}$ such that $-\sigma \le (\tau-\sigma/2)k$, then (3.18) and the increasing nature of $w(t)$ imply that

$$w(t-\sigma) \le w\left(t+\frac{\tau-\sigma}{2}k\right) < Aw\left(t+\frac{\tau-\sigma}{2}(k-1)\right) < \cdots < A^k w(t). \tag{3.19}$$

By integrating (2.24) from $t-\alpha$ to t with $\alpha>0$, we get

$$\begin{aligned} w'(t) > w'(t) - w'(t-\alpha) &= \int_{t-\alpha}^{t} qv(s-\sigma)ds + \sum_{t-\alpha\le t_k<t} \Delta w'(t_k) \\ &> q\alpha v(t-\alpha-\sigma). \end{aligned} \tag{3.20}$$

By integrating again from $t-\beta$ to t with $\beta>0$, we have

$$\begin{aligned} w(t) > w(t) - w(t-\beta) &= \int_{t-\beta}^{t} q\alpha v(s-\alpha-\sigma)ds + \sum_{t-\beta\le t_k<t} \Delta w(t_k) \\ &> q\alpha\beta v(t-\alpha-\beta-\sigma). \end{aligned} \tag{3.21}$$

Let $\alpha=\beta=-\sigma/2>0$, then

$$w(t) > \frac{q\sigma^2}{4} v(t). \tag{3.22}$$

Combining (2.24), (3.19), and (3.22), we have

$$w''(t) = qv(t-\sigma) < \frac{4}{\sigma^2} w(t-\sigma) < \frac{4A^k}{\sigma^2} w(t), \tag{3.23}$$

which shows that

$$\gamma \equiv \frac{2A^{k/2}}{-\sigma} > 0 \tag{3.24}$$

and γ is not in the set $\Lambda(w)$ for all $w \in W^+$, that is, $\Lambda(w)$ is bounded above by the positive constant γ, for all $w \in W^+$.

(c) Let

$$\varphi(t) = e^{-\lambda t}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t)} w(t), \tag{3.25}$$

then

$$\begin{aligned}
\varphi'(t) &= -\lambda e^{-\lambda t}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t)} w(t) + e^{-\lambda t}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t)} w'(t) \\
&= e^{-\lambda t}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t)} (w'(t)-\lambda w(t)), \quad t \neq t_k, \\
\Delta\varphi(t_k) &= \Delta\left(e^{-\lambda t_k}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k)}\right) w(t_k) + e^{-\lambda t_k}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k)} \Delta w(t_k) \\
&= e^{-\lambda t_k}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k^+)}\left(\Delta w(t_k) - \frac{q_1}{q}\lambda w(t_k)\right), \\
\varphi''(t) &= e^{-\lambda t}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t)}\left(w''(t) - 2\lambda w'(t) + \lambda^2 w(t)\right), \quad t \neq t_k, \\
\Delta\varphi'(t_k) &= \Delta\left(e^{-\lambda t_k}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k)}\right)(w'(t_k)-\lambda w(t_k)) \\
&\quad + e^{-\lambda t_k}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k^+)} (\Delta w'(t_k) - \lambda\Delta w(t_k)) \\
&= e^{-\lambda t_k}\left(-\frac{q_1}{q}\lambda\right)\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k^+)}\left(\frac{q}{q_1}\Delta w(t_k) - \lambda w(t_k)\right) \\
&\quad + e^{-\lambda t_k}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k^+)} (\Delta w'(t_k) - \lambda\Delta w(t_k)) \\
&= e^{-\lambda t_k}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k)}\left(\Delta w'(t_k) - 2\lambda\Delta w(t_k) + \lambda^2\frac{q_1}{q}w(t_k)\right).
\end{aligned} \tag{3.26}$$

Therefore,

$$\varphi''(t) + 2\lambda\varphi'(t) = e^{-\lambda t}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t)}\left(w''(t) - \lambda^2 w(t)\right) \geq 0, \quad t \neq t_k, \tag{3.27}$$

$$\Delta\varphi'(t_k) + 2\lambda\Delta\varphi(t_k) = e^{-\lambda t_k}\left(1+\frac{q_1}{q}\lambda\right)^{-i[0,t_k^+)}\left(\Delta w'(t_k) - \lambda^2\frac{q_1}{q}w(t_k)\right) \geq 0. \tag{3.28}$$

We clearly see that $\varphi'(t)e^{2\lambda t}$ is a nondecreasing function and so if the conclusion in part (c) was false, then

$$\varphi'(t) < 0, \quad \Delta\varphi(t_k) < 0. \tag{3.29}$$

From (3.27), (3.28), and (3.29), we know that

$$\varphi''(t) > 0, \quad \Delta\varphi'(t_k) > 0, \tag{3.30}$$

and so

$$\begin{aligned} &w''(t) - 2\lambda w'(t) + \lambda^2 w(t) > 0, \quad t \neq t_k, \\ &\Delta w'(t_k) - 2\lambda \Delta w(t_k) + \lambda^2 \frac{q_1}{q} w(t_k) > 0, \end{aligned} \tag{3.31}$$

which together with the hypothesis yield that

$$\begin{aligned} &w''(t) - \lambda^2 w(t) \geq 0, \quad t \neq t_k, \\ &\Delta w'(t_k) - \lambda^2 \frac{q_1}{q} w(t_k) \geq 0. \end{aligned} \tag{3.32}$$

Hence

$$w''(t) - \lambda w'(t) > 0, \quad t \neq t_k, \tag{3.33}$$

$$\Delta w'(t_k) - \lambda \Delta w(t_k) > 0. \tag{3.34}$$

Let

$$\begin{aligned} u(t) &= -\left[w'(t) - \lambda w(t)\right], \quad t \neq t_k, \\ u(t_k) &= -\left[w'(t_k) - \lambda w(t_k)\right] = -\left[\frac{q}{q_1} \Delta w(t_k) - \lambda w(t_k)\right], \end{aligned} \tag{3.35}$$

Then by Lemma 2.1(d) and the linear combination of solutions, we know that $u(t)$ is a solution of (1.1). From (3.29), (3.33), and (3.34), we have

$$u(t) > 0, \quad u'(t) < 0, \quad \Delta u(t) < 0. \tag{3.36}$$

Now using u instead of y in (2.1) and the hypothesis that $W^- = \emptyset$ together with a similar argument as in (2.4), we get

$$\lim_{t \to \infty} \left[-\left(u(t) + pu(t - \tau)\right)\right] = \infty. \tag{3.37}$$

But (3.36) implies that

$$\lim_{t \to \infty} u(t) \in \mathbf{R}, \tag{3.38}$$

and the contradiction completes the proof of Lemma 3.3.

By integrating both sides of (2.23) from $t_0+\sigma$ to t, we get

$$w'(t) + pw'(t-\tau) - \left(w'(t_0+\sigma) + pw'(t_0+\sigma-\tau)\right) + q\int_{t_0+\sigma}^{t} w(s-\sigma)ds - \sum_{t_0+\sigma\le t_k<t} \Delta w'(t_k) - \sum_{t_0+\sigma\le t_k<t} p\Delta w'(t_k-\tau) = 0. \tag{3.39}$$

That is,

$$\begin{aligned}-\left(w'(t) + pw'(t-\tau)\right) &= c + q\int_{t_0+\sigma}^{t} w(s-\sigma)ds + \sum_{t_0+\sigma\le t_k<t} q_1 w(t_k-\sigma)\\ &= c + q\int_{t_0}^{t-\sigma} w(s)ds + \sum_{t_0\le t_k<t-\sigma} q_1 w(t_k),\end{aligned} \tag{3.40}$$

where

$$c = -\left(w'(t_0+\sigma) + pw'(t_0+\sigma-\tau)\right). \tag{3.41}$$

As $w'(t)$ is a solution of (1.1) and $w(t)$ satisfying (2.24), it follows from (3.40) that if $w \in W^+$, then

$$c + q\int_{t_0}^{t-\sigma} w(s)ds + \sum_{t_0\le t_k<t-\sigma} q_1 w(t_k) \in W^+, \tag{3.42}$$

where c is the constant given by (3.41). □

Lemma 3.4. *Suppose that m and γ are the constants defined in Lemma 3.2(b) and Lemma 3.3(b), respectively. If $w \in W^+$, $\lambda \in \Lambda(w)$, and*

$$N = \frac{m}{2\left(e^{-\gamma\sigma}\left(1 + (q_1/q)\gamma\right)^{n_2} - pe^{-\gamma\tau}\left(1 + (q_1/q)\gamma\right)^{n_1}\right)} > 0, \tag{3.43}$$

then

$$\left(\lambda^2 + N\right)^{1/2} \in \Lambda(z), \tag{3.44}$$

where

$$z(t) = -\left(w(t) + pw(t-\tau)\right) + \lambda\int_{t_0}^{t-\sigma} w(s)ds + \frac{q_1}{q}\lambda\sum_{t_0\le t_k<t-\sigma} w(t_k) + \frac{c\lambda}{q}. \tag{3.45}$$

Proof. Clearly $z(t)$ is an element of W^+. From Lemma 3.3(c), we have

$$\begin{aligned} &w'(t) - \lambda w(t) \geq 0, \quad t \neq t_k, \\ &\Delta w(t_k) - \frac{q_1}{q}\lambda w(t_k) \geq 0. \end{aligned} \tag{3.46}$$

Then from (3.45) we get

$$\begin{aligned} z''(t) &= qw(t-\sigma) + \lambda w'(t-\sigma) \geq \left(q + \lambda^2\right) w(t-\sigma), \quad t \neq t_k, \\ \Delta z'(t_k) &= z'\left(t_k^+\right) - z'\left(t_k^-\right) = q_1 w(t_k - \sigma) + \lambda \Delta w(t_k - \sigma) \geq \left(q_1 + \frac{q_1}{q}\lambda^2\right) w(t_k - \sigma). \end{aligned} \tag{3.47}$$

By integrating (3.46) from t_0 to t, we obtain

$$\begin{aligned} 0 &\leq w(t) - \lambda \int_{t_0}^{t} w(s)ds - \sum_{t_0 \leq t_k < t} \Delta w(t_k) \leq w(t) - \lambda \int_{t_0}^{t} w(s)ds - \sum_{t_0 \leq t_k < t} \frac{q_1}{q}\lambda w(t_k) \\ &= w(t) - \lambda \int_{t_0}^{t-\sigma} w(s)ds - \sum_{t_0 \leq t_k < t-\sigma} \frac{q_1}{q}\lambda w(t_k) + \lambda \int_{t}^{t-\sigma} w(s)ds + \sum_{t \leq t_k < t-\sigma} \frac{q_1}{q}\lambda w(t_k) \end{aligned} \tag{3.48}$$

and so

$$-w(t) + \lambda \int_{t_0}^{t-\sigma} w(s)ds + \sum_{t_0 \leq t_k < t-\sigma} \frac{q_1}{q}\lambda w(t_k) \leq \lambda \int_{t}^{t-\sigma} w(s)ds + \sum_{t \leq t_k < t-\sigma} \frac{q_1}{q}\lambda w(t_k). \tag{3.49}$$

Let

$$\theta(t) = e^{\lambda t}\left(1 + \frac{q_1}{q}\lambda\right)^{i[0,t)}; \tag{3.50}$$

then

$$\begin{aligned} &\theta'(t) = \lambda\theta(t), \quad t \neq t_k, \\ &\Delta\theta(t_k) = \frac{q_1}{q}\lambda\theta(t_k). \end{aligned} \tag{3.51}$$

By integrating it from t to $t-\sigma$, we have

$$\theta(t-\sigma) - \theta(t) = \lambda \int_{t}^{t-\sigma} \theta(s)ds + \sum_{t \leq t_k < t-\sigma} \frac{q_1}{q}\lambda\theta(t_k). \tag{3.52}$$

Set

$$\varphi(t) = w(t)e^{-\lambda t}\left(1 + \frac{q_1}{q}\lambda\right)^{-i[0,t)}, \tag{3.53}$$

as defined in the proof of Lemma 3.3(c).

We clearly see that $\varphi(t)$ is a nondecreasing function, therefore,

$$\begin{aligned}
\lambda\int_t^{t-\sigma} w(s)ds + \sum_{t\le t_k<t-\sigma}\frac{q_1}{q}\lambda w(t_k) &= \lambda\int_t^{t-\sigma}\varphi(s)\theta(s)dt + \sum_{t\le t_k<t-\sigma}\frac{q_1}{q}\lambda\varphi(t_k)\theta(t_k)\\
&\le \varphi(t-\sigma)\left(\lambda\int_t^{t-\sigma}\theta(s)ds + \sum_{t\le t_k<t-\sigma}\frac{q_1}{q}\lambda\theta(t_k)\right)\\
&= \varphi(t-\sigma)(\theta(t-\sigma) - \theta(t)).
\end{aligned} \tag{3.54}$$

Let

$$c_1 = -\left(\lambda^2 + N\right)\frac{c\lambda}{q}. \tag{3.55}$$

Using (3.45), (3.47), (3.49), (3.54), the increasing nature of $w(t)$, $\varphi(t)$, and the fact that

$$t_0 < t < t-\tau < t-\sigma, \tag{3.56}$$

we have

$$\begin{aligned}
&z''(t) - \left(\lambda^2 + N\right)z(t)\\
&\quad\ge \left(\lambda^2 + q\right)w(t-\sigma) - \left(\lambda^2 + N\right)\left[\lambda\int_t^{t-\sigma} w(s)ds + \sum_{t\le t_k<t-\sigma}\frac{q_1}{q}\lambda w(t_k) - pw(t-\tau)\right] + c_1\\
&\quad\ge \left(\lambda^2 + q\right)\varphi(t-\sigma)e^{\lambda(t-\sigma)}\left(1 + \frac{q_1}{q}\lambda\right)^{i[0,t-\sigma)}\\
&\qquad - \left(\lambda^2 + N\right)\left[\varphi(t-\sigma)(\theta(t-\sigma) - \theta(t)) - p\varphi(t-\sigma)e^{\lambda(t-\tau)}\left(1 + \frac{q_1}{q}\lambda\right)^{i[0,t-\tau)}\right] + c_1\\
&\quad= \varphi(t-\sigma)e^{\lambda(t)}\left(1 + \frac{q_1}{q}\lambda\right)^{i[0,t)}
\end{aligned}$$

$$\times\left[\left(\lambda^2+q\right)e^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2}-\left(\lambda^2+N\right)\right.$$
$$\left.\times\left(e^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2}-1\right)+\left(\lambda^2+N\right)pe^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1}+\frac{c_1e^{-\lambda\sigma}(1+(q_1/q)\lambda)^{n_2}}{w(t-\sigma)}\right]$$
$$=\varphi(t-\sigma)e^{\lambda(t)}\left(1+\frac{q_1}{q}\lambda\right)^{i[0,t)}$$
$$\times\left(()\lambda^2+pe^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1}+qe^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2}-Ne^{-\lambda\sigma}\right.$$
$$\left.\times\left(1+\frac{q_1}{q}\lambda\right)^{n_2}+N+Npe^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1}+\frac{c_1e^{-\lambda\sigma}(1+(q_1/q)\lambda)^{n_2}}{w(t-\sigma)}\right)$$
$$\geq\varphi(t-\sigma)e^{\lambda(t)}\left(1+\frac{q_1}{q}\lambda\right)^{i[0,t)}$$
$$\times\left(\lambda^2+pe^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1}+qe^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2}\right.$$
$$\left.+\frac{c_1e^{-\lambda\sigma}(1+(q_1/q)\lambda)^{n_2}}{w(t-\sigma)}-N\left(e^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2}-pe^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1}\right)\right)$$
$$\geq\varphi(t-\sigma)e^{\lambda(t)}\left(1+\frac{q_1}{q}\lambda\right)^{i[0,t)}$$
$$\times\left(m+\frac{c_1e^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2}}{w(t-\sigma)}-N\left(e^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2}-pe^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1}\right)\right),\quad t\neq t_k. \tag{3.57}$$

As

$$\lim_{t\to\infty}w(t)=\infty, \tag{3.58}$$

we see that for sufficiently large t,

$$m+\frac{c_1e^{-\lambda\sigma}(1+(q_1/q)\lambda)^{n_2}}{w(t-\sigma)}\geq\frac{m}{2}, \tag{3.59}$$

and as $\gamma\geq\lambda$, so

$$e^{-\lambda\sigma}\left(1+\frac{q_1}{q}\lambda\right)^{n_2}-pe^{-\lambda\tau}\left(1+\frac{q_1}{q}\lambda\right)^{n_1}\leq e^{-\gamma\varrho}\left(1+\frac{q_1}{q}\gamma\right)^{n_2}-pe^{-\gamma\tau}\left(1+\frac{q_1}{q}\gamma\right)^{n_1}=\frac{m}{2N}. \tag{3.60}$$

Then

$$z''(t) - \left(\lambda^2 + N\right)z(t) \geq \varphi(t-\sigma)e^{\lambda(t)}\left(1 + \frac{q_1}{q}\lambda\right)^{i[0,t)}\left(\frac{m}{2} - \frac{m}{2}\right) = 0, \quad t \neq t_k. \tag{3.61}$$

Analogously,

$$\begin{aligned}
&\Delta z'(t_k) - \frac{q_1}{q}\left(\lambda^2 + N\right)z(t_k) \\
&\quad \geq \left(q_1 + \frac{q_1}{q}\lambda^2\right)w(t_k - \sigma) - \frac{q_1}{q}\left(\lambda^2 + N\right)\left[\lambda\int_{t_k}^{t_k-\sigma} w(s)ds + \frac{q_1}{q}\sum_{t_k \leq t_l < t_k - \sigma}\frac{q_1}{q}\lambda w(t_l) - pw(t_k - \tau)\right] \\
&\quad + \frac{q_1}{q}c_1 \geq \frac{q_1}{q}\left(\lambda^2 + q\right)\varphi(t_k - \sigma)e^{\lambda(t_k-\sigma)}\left(1 + \frac{q_1}{q}\lambda\right)^{i[0,t_k-\sigma)} \\
&\quad - \frac{q_1}{q}\left(\lambda^2 + N\right)\left[\varphi(t_k - \sigma)(\theta(t_k - \sigma) - \theta(t_k)) - p\varphi(t_k - \sigma)e^{\lambda(t_k-\tau)}\left(1 + \frac{q_1}{q}\lambda\right)^{i[0,t_k-\tau)}\right] + c_1 \geq 0.
\end{aligned} \tag{3.62}$$

Therefore, the proof of Lemma 3.4 is completed. □

Now, considering the sequence of functions

$$z_n(t) = -\left(z_{n-1}(t) + pz_{n-1}(t-\tau)\right) + \lambda_n\int_{t_0}^{t-\sigma} w(s)ds + \frac{q_1}{q}\lambda_n\sum_{t_0 \leq t_k < t-\sigma} w(t_k) + \frac{c_n\lambda_n}{q}, \tag{3.63}$$

$$n = 1, 2, \ldots,$$

where $z_0(t)$ is the function defined by (3.45), λ_0 is the number defined in Lemma 3.3(a),

$$\begin{gathered}
N = \frac{m}{2\left(e^{-\gamma\varrho}\left(1 + (q_1/q)\gamma\right)^{n_2} - pe^{-\gamma\tau}\left(1 + (q_1/q)\gamma\right)^{n_1}\right)} > 0, \\
\lambda_n = \left(\lambda_{n-1}^2 + N\right)^{1/2}, \\
c_n = -\left(z_n'(t_0 + \sigma) + pz_n'(t_0 + \sigma - \tau)\right).
\end{gathered} \tag{3.64}$$

The repeated applications of Lemma 3.4 lead to

$$\lambda_n \in \Lambda(z_{n-1}) \quad \text{for } n = 1, 2, \ldots. \tag{3.65}$$

Clearly

$$\lim_{n\to\infty}\lambda_n = \infty, \tag{3.66}$$

which contradicts with the fact proved in Lemma 3.3(b) that

$$\lambda_n \leq \gamma \quad \text{for } n = 1, 2, \ldots. \tag{3.67}$$

Therefore, the proof of Theorem 3.1 is completed.

4. Oscillation of the Bounded Solutions

In Section 2, we complete the case of $W^- = \emptyset$. Now we consider the conditions ensuring oscillation of the case of $W^- \neq \emptyset$, that is, considering the conditions ensuring oscillation of the bounded solutions of (1.1). Then in view of Lemma 2.1(c), $p < -1$.

We are looking for a positive solution of (1.1) with the form

$$y(t) = e^{-\lambda t}(1-\mu)^{i[0,t)}, \tag{4.1}$$

where $\lambda, \mu > 0$ are constants.

Substituting (4.1) in (1.1), just like in Section 2, we can obtain the characteristic equation of (1.1) as follows:

$$F(\lambda) \equiv \lambda^2 + p\lambda^2 e^{\lambda\tau}\left(1 - \frac{q_1}{q}\lambda\right)^{n_1} + qe^{\lambda\sigma}\left(1 - \frac{q_1}{q}\lambda\right)^{n_2} = 0, \tag{4.2}$$

where

$$i[t, t-\tau) = n_1, \quad i[t, t-\sigma) = n_2. \tag{4.3}$$

Theorem 4.1. *If the condition (H) holds, then the following assertions are equivalent.*

(a) *Each bounded regular solution of the equation* (1.5) *is oscillatory.*

(b) *The characteristic equation* (4.2) *has no real roots* $\lambda \in [0, q/q_1]$.

The proof of (a) $\Rightarrow$ (b) is obvious. In fact, if the characteristic equation (4.2) has a real root $\lambda \in [0, q/q_1]$, then $\mu = (q_1/q)\lambda \geq 0$ and $0 < 1 - \mu \leq 1$, therefore the function

$$y(t) = e^{-\lambda t}(1-\mu)^{i[0,t)} \tag{4.4}$$

is a bounded positive solution of (1.1).

The proof of (b) $\Rightarrow$ (a) is quite complicated and will be accomplished by establishing a series of lemmas.

As in Section 3 we assume, without further mention, that (1.2) holds. We also assume that (4.2) has no real roots $\lambda \in [0, q/q_1]$. For the sake of contradiction, we assume that (1.1) has a bounded finally positive solution $y(t)$.

Also like the case in Section 3, for each function $w \in W^-$, define the set

$$\Lambda(w) = \left\{ \lambda \geq 0 \left| \begin{array}{ll} -w''(t) + \lambda^2 w(t) \leq 0 & \text{finally} \\ -\Delta w'(t_k) + \dfrac{q_1}{q}\lambda^2 w(t_k) \leq 0 & \text{finally} \end{array} \right. \right\}. \tag{4.5}$$

Clearly, $0 \in \Lambda(w)$ and if $\lambda \in \Lambda(w)$, then $[0, \lambda] \subseteq \Lambda(w)$. That is, $\Lambda(w)$ is a nonempty subinterval of $\mathbf{R}^+$.

Lemma 4.2. *There exists a positive constant m such that*

$$F(\lambda) \equiv \lambda^2 + p\lambda^2 e^{\lambda\tau}\left(1 - \frac{q_1}{q}\lambda\right)^{n_1} + qe^{\lambda\sigma}\left(1 - \frac{q_1}{q}\lambda\right)^{n_2} \geq m \quad \forall \lambda \in \left[0, \frac{q}{q_1}\right]. \tag{4.6}$$

Proof. We clearly see that

$$F(0) = q > 0, \quad F\left(\frac{q^-}{q_1}\right) = \left(\frac{q}{q_1}\right)^2 > 0 \tag{4.7}$$

and so there exists a positive constant m such that

$$F(\lambda) \geq m \quad \forall \lambda \in \left[0, \frac{q}{q_1}\right]. \tag{4.8}$$

□

Lemma 4.3. *(a) If $w \in W^-$ and $l \in \mathbf{N}$ with $-l\tau > \tau - \sigma$, then*

$$\lambda_1 \equiv \left(\frac{q}{(-p)^{k+1}}\right)^{1/2} \in \Lambda(w). \tag{4.9}$$

(b) If $w \in W^-$ and $\lambda \in \Lambda(w)$, then

$$\begin{aligned} &w'(t) + \lambda w(t) \leq 0, \quad t \neq t_k, \\ &\Delta w(t_k) + \frac{q_1}{q}\lambda w(t_k) \leq 0. \end{aligned} \tag{4.10}$$

(c) $\Lambda(w)$ is bounded above by a positive constant λ_2 for any $w \in W^-$.

Proof. (a) Let $l \in \mathbf{N}$ with $-l\tau > \tau - \sigma$.

For $w \in W^-$ we have

$$-\left(w(t) + \mathrm{p}w(t-\tau)\right) > 0 \tag{4.11}$$

and so

$$w(t) < -pw(t-\tau) < (-p)^l w(t - l\tau) < (-p)^l w(t + (\tau - \sigma)). \tag{4.12}$$

That is,

$$w(t + (\tau - \sigma)) > \frac{w(t)}{(-p)^l}, \tag{4.13}$$

as

$$\begin{aligned} &w''(t) + \frac{q}{p} w(t + (\tau - \sigma)) > 0, \quad t \neq t_k, \\ &\Delta w'(t_k) + \frac{q_1}{p} w(t_k + (\tau - \sigma)) > 0. \end{aligned} \tag{4.14}$$

So

$$\begin{aligned} &w''(t) - \frac{q}{(-p)^{k+1}} w(t) > 0, \quad t \neq t_k, \\ &\Delta w'(t_k) - \frac{q_1}{q}\left(\frac{q}{(-p)^{k+1}} w(t_k)\right) > 0. \end{aligned} \tag{4.15}$$

That is,

$$\left(\frac{q}{(-p)^{k+1}}\right)^{1/2} \in \Lambda(w). \tag{4.16}$$

(b) Let

$$\begin{aligned} \varphi(t) &= w'(t) + \lambda w(t), \quad t \neq t_k, \\ \varphi(t_k) &= \frac{q_1}{q} \Delta w(t_k) + \lambda w(t_k), \end{aligned} \tag{4.17}$$

then

$$\begin{aligned} \varphi'(t) - \lambda\varphi(t) &= w''(t) - \lambda^2 w(t) \geq 0, \quad t \neq t_k, \\ \Delta\varphi(t_k) - \frac{q_1}{q}\lambda\varphi(t_k) &= \Delta w'(t_k) - \frac{q_1}{q}\lambda^2 w(t_k) \geq 0. \end{aligned} \tag{4.18}$$

Let

$$u(t) = \varphi(t)e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)}, \tag{4.19}$$

then

$$\begin{aligned} u'(t) &= e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)} \left(\varphi'(t) - \lambda\varphi(t)\right) \geq 0, \quad t \neq t_k, \\ \Delta u(t_k) &= e^{-\lambda t_k}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t_k)} \left(\Delta\varphi(t) - \frac{q_1}{q}\lambda\varphi(t_k)\right) \geq 0. \end{aligned} \tag{4.20}$$

Therefore, $u(t)$ is a nondecreasing function.

Note that

$$\lim_{t\to\infty} u(t) = 0. \tag{4.21}$$

We know that $u(t) \leq 0$ and so

$$\varphi(t) \leq 0, \quad \varphi(t_k) \leq 0. \tag{4.22}$$

(c) Otherwise,

$$\lambda_2 \equiv \frac{1}{-\tau}\ln(-p) \in \Lambda(w) \quad \text{for some } w \in W^-. \tag{4.23}$$

Then from part (b)

$$w'(t) + \lambda_2 w(t) \leq 0, \tag{4.24}$$

we know that the function $w(t)e^{\lambda_2 t}$ is nonincreasing and

$$w(t)e^{\lambda_2 t} \geq w(t-\tau)e^{\lambda_2(t-\tau)} \tag{4.25}$$

or

$$w(t) \geq w(t-\tau)e^{-\lambda_2\tau} = -pw(t-\tau). \tag{4.26}$$

This contradicts with(3.54); therefore, The proof of Lemma 4.3 is completed. □

Lemma 4.4. *Suppose that m is a constant defined in Lemma 4.2 and $T > \sigma$. If $w \in W^-$, $\lambda = \sup\Lambda(w)$, and*

$$N = \frac{m}{e^{-\lambda T}\left(1-(q_1/q)\lambda\right)^{-i[t-T,t)} + (q/\lambda^2)e^{\lambda T}\left(1-(q_1/q)\lambda\right)^{-i[t-T,t)} - pe^{\lambda T}\left(1-(q_1/q)\gamma\right)^{n_1}} > 0, \tag{4.27}$$

then

$$\left(\lambda^2 + N\right)^{1/2} \in \Lambda(z), \tag{4.28}$$

where

$$\begin{aligned} z(t) = {} & g'(t) + pg'(t-\tau) + q\int_{t-T}^{t-\sigma} g(\xi)d\xi + q_1 \sum_{t-T\leq t_k<t-\sigma} g(t_k) \\ & + \lambda^2\int_{t-T}^{\infty} g(\xi)d\xi + \lambda^2\frac{q_1}{q}\sum_{t_k\geq t-T} g(t_k) \end{aligned} \tag{4.29}$$

and

$$g(t) = w''(t) - \lambda w'(t). \tag{4.30}$$

Proof. Clearly, $z(t)$ is an element of W^{-}. If

$$\phi(t) = e^{\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[0,t)}\left(-g'(t)\right) > 0, \tag{4.31}$$

then

$$\begin{aligned}
\phi'(t) &= e^{\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[0,t)}\left(-w^{(4)}(t) + \lambda^2 w''(t)\right) < 0, \\
-g'(t) &= e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)}\phi(t), \quad t \neq t_k, \\
-\Delta g(t_k) &= -\frac{q_1}{q}g'(t_k) = \frac{q_1}{q}e^{-\lambda t_k}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t_k)}\phi(t_k).
\end{aligned} \tag{4.32}$$

Let

$$\theta(t) = e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)}, \tag{4.33}$$

then

$$\begin{aligned}
\theta'(t) &= -\lambda\theta(t), \quad t \neq t_k, \\
\Delta\theta(t_k) &= -\frac{q_1}{q}\lambda\theta(t_k).
\end{aligned} \tag{4.34}$$

By integrating (4.34) from t to t_1, we get

$$\theta(t_1) - \theta(t) = -\lambda\int_t^{t_1}\theta(\xi)d\xi - \sum_{t \le t_k < t_1}\Delta\theta(t_k). \tag{4.35}$$

Let $t_1 \to \infty$; then

$$-\theta(t) = -\lambda \int_t^{\infty} \theta(\xi)d\xi - \frac{q_1}{q}\lambda \sum_{t_k \geq t} \theta(t_k). \tag{4.36}$$

Integrating (4.32) from t to ∞, we get

$$\begin{aligned}
g(t) &= \int_t^{\infty} e^{-\lambda\xi}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,\xi)} \phi(\xi)d\xi - \sum_{t_k \geq t} \Delta g(t_k) \\
&= \int_t^{\infty} e^{-\lambda\xi}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,\xi)} \phi(\xi)d\xi + \frac{q_1}{q}\sum_{t_k \geq t} e^{-\lambda t_k}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t_k)} \phi(t_k) \\
&\leq \phi(t)\left(\int_t^{\infty} \theta(\xi)d\xi + \frac{q_1}{q}\sum_{t_k \geq t} \theta(t_k)\right) \\
&= \frac{1}{\lambda}\phi(t)\theta(t) = \frac{1}{\lambda}\phi(t)e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)} = -\frac{1}{\lambda}g'(t).
\end{aligned} \tag{4.37}$$

That is,

$$g'(t) + \lambda g(t) \leq 0. \tag{4.38}$$

By integrating again from t to ∞, we find

$$\begin{aligned}
0 &\geq -g(t) + \lambda \int_t^{\infty} g(\xi)d\xi + \frac{q_1}{q}\lambda \sum_{t_k \geq t} g(t_k) \geq \frac{1}{\lambda}g'(t) + \lambda \int_t^{\infty} g(\xi)d\xi + \frac{q_1}{q}\lambda \sum_{t_k \geq t} g(t_k) \\
&= \frac{1}{\lambda}g'(t) + \lambda \int_{t-T}^{\infty} g(\xi)d\xi + \frac{q_1}{q}\lambda \sum_{t_k \geq t-T} g(t_k) - \lambda \int_{t-T}^{t} g(\xi)d\xi - \frac{q_1}{q}\lambda \sum_{t-T \leq t_k < t} g(t_k).
\end{aligned} \tag{4.39}$$

Then from (4.35) we have

$$\begin{aligned}
g'(t) + \lambda^2 \int_{t-T}^{\infty} g(\xi)d\xi + \frac{q_1}{q}\lambda^2 \sum_{t_k \geq t-T} g(t_k) &\leq \lambda^2 \int_{t-T}^{t} g(\xi)d\xi + \frac{q_1}{q}\lambda^2 \sum_{t-T \leq t_k < t} g(t_k) \\
&\leq \lambda^2 \int_{t-T}^{t} \frac{1}{\lambda}\theta(\xi)\phi(\xi)d\xi + \frac{q_1}{q}\lambda^2 \sum_{t-T \leq t_k < t} \frac{1}{\lambda}\theta(t_k)\phi(t_k) \\
&\leq \phi(t-T)(\theta(t-T) - \theta(t))
\end{aligned} \tag{4.40}$$

and

$$q\int_{t-T}^{t-\sigma} g(\xi)d\xi + q_1 \sum_{t-T\le t_k<t-\sigma} g(t_k) \le \frac{q}{\lambda^2}\phi(t-T)(\theta(t-T)-\theta(t-\sigma)). \tag{4.41}$$

Using (4.29) and (4.32) together with (4.38)–(4.41), we see that

$$\begin{aligned}
z''(t) &= \left(-q-\lambda^2\right)g'(t-T)\\
&= \left(\lambda^2+q\right)e^{-\lambda(t-T)}\left(1-\frac{q_1}{q}\lambda\right)^{i[0,t-T)}\phi(t-T),\\
z(t) &= g'(t)+pg'(t-\tau)+q\int_{t-T}^{t-\sigma} g(\xi)d\xi + q_1\sum_{t-T\le t_k<t-\sigma} g(t_k) + \lambda^2\int_{t-T}^{\infty} g(\xi)d\xi + \lambda^2\frac{q_1}{q}\sum_{t_k\ge t-T} g(t_k)\\
&\le \phi(t-T)(\theta(t-T)-\theta(t)) + \frac{q}{\lambda^2}\phi(t-T)(\theta(t-T)-\theta(t-\sigma)) + pg'(t-\tau).
\end{aligned} \tag{4.42}$$

Therefore,

$$\begin{aligned}
&z''(t) - \left(\lambda^2+N\right)z(t)\\
&\quad\ge \left(\lambda^2+q\right)e^{-\lambda(t-T)}\left(1-\frac{q_1}{q}\lambda\right)^{i[0,t-T)}\phi(t-T)\\
&\qquad - \left(\lambda^2+N\right)\left[\phi(t-T)(\theta(t-T)-\theta(t)) + \frac{q}{\lambda^2}\phi(t-T)(\theta(t-T)-\theta(t-\sigma)) + pg'(t-\tau)\right]\\
&\quad= \left(\lambda^2+q\right)e^{-\lambda(t-T)}\left(1-\frac{q_1}{q}\lambda\right)^{i[0,t-T)}\phi(t-T)\\
&\qquad - \left(\lambda^2+N\right)\left[\phi(t-T)\left(e^{-\lambda(t-T)}\left(1-\frac{q_1}{q}\lambda\right)^{i[0,t-T)} - e^{-\lambda t}\left(1-\frac{q_1}{q}\lambda\right)^{i[0,t)}\right)\right.\\
&\qquad\qquad + \phi(t-T)\frac{q}{\lambda^2}\left(e^{-\lambda(t-T)}\left(1-\frac{q_1}{q}\lambda\right)^{i[0,t-T)} - e^{-\lambda(t-\sigma)}\left(1-\frac{q_1}{q}\lambda\right)^{i[0,t-\sigma)}\right)\\
&\qquad\qquad \left. -p\phi(t-\tau)e^{-\lambda(t-\tau)}\left(1-\frac{q_1}{q}\lambda\right)^{i[0,t-\tau)}\right].
\end{aligned} \tag{4.43}$$

For $T > 0$ and $\tau < 0$, we have $t - T < t - \tau$. From the fact that $\phi(t)$ is nonincreasing, we clearly see that $\phi(t-T) > \phi(t-\tau)$. Therefore,

$$
\begin{aligned}
&z''(t) - \left(\lambda^2 + N\right)z(t) \\
&\quad \geq \phi(t-T)e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)} \\
&\qquad \times \left\{ \left(\lambda^2 + q\right)e^{\lambda T}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[t-T,t)} - \left(\lambda^2 + N\right) \right. \\
&\qquad\quad \times \left[e^{\lambda T}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[t-T,t)} - 1 + \frac{q}{\lambda^2}\left(e^{\lambda T}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[t-T,t)} - e^{\lambda\sigma}\left(1 - \frac{q_1}{q}\lambda\right)^{n_2} \right) \right. \\
&\qquad\qquad \left.\left. - pe^{\lambda\tau}\left(1 - \frac{q_1}{q}\lambda\right)^{n_1} \right] \right\} \\
&\quad = \phi(t-T)e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)} \\
&\qquad \times \left\{ \lambda^2 + p\lambda^2 e^{\lambda\tau}\left(1 - \frac{q_1}{q}\lambda\right)^{n_1} + qe^{\lambda\sigma}\left(1 - \frac{q_1}{q}\lambda\right)^{n_2} \right. \\
&\qquad\quad - N\left[e^{\lambda T}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[t-T,t)} - 1 + \frac{q}{\lambda^2}\left(e^{\lambda T}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[t-T,t)} - e^{\lambda\sigma}\left(1 - \frac{q_1}{q}\lambda\right)^{n_2} \right) \right. \\
&\qquad\qquad \left.\left. - pe^{\lambda\tau}\left(1 - \frac{q_1}{q}\lambda\right)^{n_1} \right] \right\} \\
&\quad \geq \phi(t-T)e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)} \\
&\qquad \times \left\{ \lambda^2 + p\lambda^2 e^{\lambda\tau}\left(1 - \frac{q_1}{q}\lambda\right)^{n_1} + qe^{\lambda\sigma}\left(1 - \frac{q_1}{q}\lambda\right)^{n_2} - N \right. \\
&\qquad\quad \left. \times \left[e^{\lambda T}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[t-T,t)} + \frac{q}{\lambda^2}e^{\lambda T}\left(1 - \frac{q_1}{q}\lambda\right)^{-i[t-T,t)} - pe^{\lambda\tau}\left(1 - \frac{q_1}{q}\lambda\right)^{n_1} \right] \right\} \\
&\quad \geq \phi(t-T)e^{-\lambda t}\left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t)}(m - m) = 0.
\end{aligned}
\tag{4.44}
$$

Analogously, we have

$$\begin{aligned}\Delta z'(t_k) &= z'(t_k^+) - z'(t_k^-)\\ &= g''(t_k^+) + pg''(t_k^+ - \tau) + q(g(t_k^+ - \sigma) - g(t_k^+ - T)) - \lambda^2 g(t_k^+ - T)\\ &\quad - \left(g''(t_k^-) + pg''(t_k^- - \tau) + q(g(t_k^- - \sigma) - g(t_k^- - T\ )) - \lambda^2 g(t_k^- - T)\right)\\ &= \Delta g''(t_k) + p\Delta g''(t_k - \tau) + q(\Delta g(t_k - \sigma) - \Delta g(t_k - T)) - \lambda^2 \Delta g(t_k - T)\\ &= -q_1 g'(t_k - \sigma) + q(\Delta g(t_k - \sigma) - \Delta g(t_k - T)) - \lambda^2 \Delta g(t_k - T)\\ &= -q\Delta g(t_k - T) - \lambda^2 \Delta g(t_k - T)\\ &= \frac{q_1}{q}\left(\lambda^2 + q\right) e^{-\lambda(t_k - T)} \left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t_k - T)} \phi(t_k - T).\end{aligned} \tag{4.45}$$

So

$$\begin{aligned}&\Delta z'(t_k) - \frac{q_1}{q}\left(\lambda^2 + N\right) z(t)\\ &\geq \frac{q_1}{q}\Bigg\{\left(\lambda^2 + q\right) e^{-\lambda(t_k - T)} \left(1 - \frac{q_1}{q}\lambda\right)^{i[0,t_k - T)} \phi(t_k - T) - \left(\lambda^2 + N\right)\\ &\qquad \times\left[\phi(t_k - T)(\theta(t_k - T) - \theta(t_k)) + \frac{q}{\lambda^2}\phi(t_k - T)(\theta(t_k - T) - \theta(t_k - \sigma)) + pg'(t_k - \tau)\right]\Bigg\}\\ &\geq 0.\end{aligned} \tag{4.46}$$

Therefore, the proof of Lemma 4.4 is completed. □

Now, we consider the sequence of functions

$$\begin{aligned}z_n(t) &= z'_{n-1}(t) + pz'_{n-1}(t - \tau) + q\int_{t-T}^{t-\sigma} z_{n-1}(\xi)d\xi + q_1 \sum_{t-T\leq t_k < t-\sigma} z_{n-1}(t_k)\\ &\quad + \lambda^2 \int_{t-T}^{\infty} z_{n-1}(\xi)d\xi + \lambda^2 \frac{q_1}{q} \sum_{t_k \geq t-T} z_{n-1}(t_k), \quad n = 1, 2, \ldots,\end{aligned} \tag{4.47}$$

where, $z_0(t)$ is the function $z(t)$ defined in (4.29),

$$\lambda_0 = \sup \Lambda(w),$$

$$\lambda_n = \left(\lambda_{n-1}^2 + N\right)^{1/2},$$

$$N = \frac{m}{e^{-\lambda T}(1-(q_1/q)\lambda)^{-i[t-T,t)} + \frac{q}{\lambda^2}e^{\lambda T}(1-(q_1/q)\lambda)^{-i[t-T,t)} - pe^{\lambda T}(1-(q_1/q)\gamma)^{n_1}} > 0. \tag{4.48}$$

The repeated applications of Lemma 4.4 lead to

$$\lambda_n \in \Lambda(z_{n-1}) \quad \text{for } n = 1, 2, \ldots. \tag{4.49}$$

Clearly,

$$\lim_{n\to\infty} \lambda_n = \infty, \tag{4.50}$$

which contradicts with the fact proved in Lemma 4.3(c) that $\Lambda(w)$ is bounded above for any $w \in W^-$. Therefore, the proof of Theorem 4.1 is completed.

Remark 4.5. Let $n_1 = n_2 = 0, q_1 = 0$, then (1.1) reduces to (1.5), and (3.3) and (4.2) reduce to

$$F(\lambda) \equiv \lambda^2 + p\lambda^2 e^{\lambda\tau} + qe^{\lambda\sigma} = 0. \tag{4.51}$$

From our Theorems 3.1 and 4.1, we get the following well-known results:

Corollary 4.6. *The following assertions are equivalent.*

(a) *Each unbounded regular solution of the equation* (1.5) *is oscillatory.*

(b) *The characteristic equation* (4.51) *has no positive real roots.*

Corollary 4.7. *The following assertions are equivalent.*

(a) *Each bounded regular solution of the equation* (1.5) *is oscillatory.*

(b) *The characteristic equation* (4.51) *has no real root* $\lambda \in [0, \infty)$.

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