Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 578310, 11 pages
doi:10.1155/2010/578310

Research Article

An Optimal Double Inequality for Means

Wei-Mao Qian and Ning-Guo Zheng

Huzhou Broadcast and TV University, Huzhou 313000, China

Correspondence should be addressed to Wei-Mao Qian, qwm661977@126.com

Received 3 September 2010; Accepted 27 September 2010

Academic Editor: Alberto Cabada

For $p \in \mathbb{R}$, the generalized logarithmic mean $L_p(a,b)$, arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ of two positive numbers a and b are defined by $L_p(a,b) = a$, $a = b$; $L_p(a,b) = [(a^{p+1} - b^{p+1})/((p+1)(a-b))]^{1/p}$, $p \neq 0$, $p \neq -1$, $a \neq b$; $L_p(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $p = 0$, $a \neq b$; $L_p(a,b) = (b-a)/(\ln b - \ln a)$, $p = -1$, $a \neq b$; $A(a,b) = (a+b)/2$ and $G(a,b) = \sqrt{ab}$, respectively. In this paper, we give an answer to the open problem: for $\alpha \in (0,1)$, what are the greatest value p and the least value q, such that the double inequality $L_p(a,b) \leq G^{\alpha}(a,b)A^{1-\alpha}(a,b) \leq L_q(a,b)$ holds for all $a, b > 0$?

1. Introduction

For $p \in \mathbb{R}$, the generalized logarithmic mean $L_p(a,b)$ of two positive numbers a and b is defined by

$$L_p(a,b) = \begin{cases} a, & a = b, \\ \left[\dfrac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right]^{1/p}, & p \neq 0,\ p \neq -1,\ a \neq b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0,\ a \neq b, \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1,\ a \neq b. \end{cases} \tag{1.1}$$

It is wellknown that $L_p(a,b)$ is continuous and increasing with respect to $p \in \mathbb{R}$ for fixed a and b. In the recent past, the generalized logarithmic mean has been the subject of intensive research. Many remarkable inequalities and monotonicity results can be found in the literature [1–9]. It might be surprising that the generalized logarithmic mean, has applications in physics, economics, and even in meteorology [10–13].

If we denote by $A(a,b) = (a+b)/2, I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}, L(a,b) = (b-a)/(\ln b - \ln a), G(a,b) = \sqrt{ab}$ and $H(a,b) = 2ab/(a+b)$ the arithmetic mean, identric mean, logarithmic mean, geometric mean and harmonic mean of two positive numbers a and b, respectively, then

$$\begin{aligned} \min\{a,b\} &\leq H(a,b) \leq G(a,b) = L_{-2}(a,b) \leq L(a,b) = L_{-1}(a,b) \\ &\leq I(a,b) = L_0(a,b) \leq A(a,b) = L_1(a,b) \leq \max\{a,b\}. \end{aligned} \tag{1.2}$$

For $p \in \mathbb{R}$, the pth power mean $M_p(a,b)$ of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.3}$$

In [14], Alzer and Janous established the following sharp double inequality (see also [15], Page 350):

$$M_{\log 2/\log 3}(a,b) \leq \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leq M_{2/3}(a,b) \tag{1.4}$$

for all $a, b > 0$.

For $\alpha \in (0,1)$, Janous [16] found the greatest value p and the least value q such that

$$M_p(a,b) \leq \alpha A(a,b) + (1-\alpha)G(a,b) \leq M_q(a,b) \tag{1.5}$$

for all $a,\ b > 0$.

In [17–19] the authors present bounds for $L(a,b)$ and $I(a,b)$ in terms of $G(a,b)$ and $A(a,b)$.

Theorem A. *For all positive real numbers a and b with $a \neq b$, one has*

$$\begin{aligned} &L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b), \\ &\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) < I(a,b). \end{aligned} \tag{1.6}$$

The proof of the following Theorem B can be found in [20].

Theorem B. *For all positive real numbers a and b with $a \neq b$, one has*

$$\sqrt{G(a,b)A(a,b)} < \sqrt{L(a,b)I(a,b)} < \frac{1}{2}(L(a,b)+I(a,b)) < \frac{1}{2}(G(a,b)+A(a,b)). \tag{1.7}$$

The following Theorems C–E were established by Alzer and Qiu in [21].

Theorem C. *The inequalities*

$$\alpha A(a,b)+(1-\alpha)G(a,b) < I(a,b) < \beta A(a,b)+(1-\beta)G(a,b) \tag{1.8}$$

hold for all positive real numbers a and b with $a \neq b$ if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e = 0.73575\ldots$.

Theorem D. *Let a and b be real numbers with $a \neq b$. If $0 < a,\ b \leq e$, then*

$$[G(a,b)]^{A(a,b)} < [L(a,b)]^{I(a,b)} < [A(a,b)]^{G(a,b)}. \tag{1.9}$$

And if $a, b \geq e$, then

$$[A(a,b)]^{G(a,b)} < [I(a,b)]^{L(a,b)} < [G(a,b)]^{A(a,b)}. \tag{1.10}$$

Theorem E. *For all real numbers a and b with $a \neq b$, one has*

$$M_p(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) \tag{1.11}$$

with the best possible parameter $p = \log 2/(1+\log 2) = 0.40938\ldots$.

However, the following problem is still open: for $\alpha \in (0,1)$, what are the greatest value p and the least value q, such that the double inequality

$$L_p(a,b) \leq G^{\alpha}(a,b)A^{1-\alpha}(a,b) \leq L_q(a,b) \tag{1.12}$$

holds for all $a,\ b > 0$? The purpose of this paper is to give the solution to this open problem.

2. Lemmas

In order to establish our main result, we need two lemmas, which we present in this section.

Lemma 2.1. *If $t > 1$, then*

$$\frac{t}{t-1}\log t - \frac{1}{6}\log t - \frac{2}{3}\log\frac{1+t}{2} - 1 > 0. \tag{2.1}$$

Proof. Let $f(t) = (t/(t-1))\log t - (1/6)\log t - (2/3)\log((1+t)/2) - 1$, then simple computation yields

$$\lim_{t\to 1^+} f(t) = 0, \tag{2.2}$$

$$f'(t) = \frac{g(t)}{6t(t-1)^2(t+1)}, \tag{2.3}$$

where

$$\begin{gathered} g(t) = t^3 + 9t^2 - 9t - 6t(t+1)\log t - 1, \\ g(1) = 0, \\ g'(t) = 3t^2 + 12t - 6(2t+1)\log t - 15, \\ g'(1) = 0, \\ g''(t) = \frac{6}{t}h(t), \end{gathered} \tag{2.4}$$

where

$$\begin{gathered} h(t) = t^2 - 2t\log t - 1, \\ g''(1) = h(1) = 0, \end{gathered} \tag{2.5}$$

$$\begin{gathered} h'(t) = 2\left(t - \log t - 1\right), \\ h'(1) = 0, \end{gathered} \tag{2.6}$$

$$h''(t) = 2\left(1 - \frac{1}{t}\right). \tag{2.7}$$

If $t > 1$, then from (2.7) we clearly see that

$$h''(t) > 0. \tag{2.8}$$

Therefore, Lemma 2.1 follows from (2.3)–(2.6) and (2.8). □

Lemma 2.2. *If $t > 1$, then*

$$\log(t-1) - \log\left(\log t\right) - \frac{1}{3}\log\left(t^2 + t\right) + \frac{1}{3}\log 2 > 0. \tag{2.9}$$

Proof. Let $f(t) = \log(t-1) - \log(\log t) - (1/3)\log(t^2+t) + (1/3)\log 2$, then simple computation leads to

$$\lim_{t\to 1^+} f(t) = 0,$$
$$f'(t) = \frac{g(t)}{3t(t-1)(t+1)\log t}, \tag{2.10}$$

where

$$g(t) = \left(t^2+4t+1\right)\log t - 3t^2 + 3,$$
$$g(1) = 0, \tag{2.11}$$
$$g'(t) = \frac{h(t)}{t},$$

where

$$h(t) = 2t(t+2)\log t - 5t^2 + 4t + 1,$$
$$g'(1) = h(1) = 0,$$
$$h'(t) = 4(t+1)\log t - 8t + 8, \tag{2.12}$$
$$h'(1) = 0,$$
$$h''(t) = \frac{4}{t}p(t),$$

where

$$p(t) = t\log t - t + 1,$$
$$h''(1) = p(1) = 0, \tag{2.13}$$

$$p'(t) = \log t. \tag{2.14}$$

If $t > 1$, then from (2.14) we clearly see that

$$p'(t) > 0. \tag{2.15}$$

From (2.10)–(2.13) and (2.15) we know that $f(t) > 0$ for $t > 1$. □

3. Main Results

Theorem 3.1. *If $\alpha \in (0,1)$, then $G^{\alpha}(a,b)A^{1-\alpha}(a,b) \le L_{1-3\alpha}(a,b)$ for all $a,b > 0$, with equality if and only if $a = b$, and the constant $1-3\alpha$ in $L_{1-3\alpha}(a,b)$, cannot be improved.*

Proof. If $a = b$, then we clearly see that $G^{\alpha}(a,b)A^{1-\alpha}(a,b) = L_{1-3\alpha}(a,b) = a$.

If $a \neq b$, without loss of generality, we assume that $a > b$. Let $t = (a/b) > 1$ and

$$f(t) = \log L_{1-3\alpha}(a,b) - \log\left[G^{\alpha}(a,b)A^{1-\alpha}(a,b)\right]. \tag{3.1}$$

Firstly, we prove $G^{\alpha}(a,b)A^{1-\alpha}(a,b) < L_{1-3\alpha}(a,b)$. The proof is divided into three cases.

Case 1. $\alpha = 1/3$. We note that (1.1) leads to the following identity:

$$f(t) = \frac{t}{t-1}\log t - \frac{1}{6}\log t - \frac{2}{3}\log\frac{1+t}{2} - 1. \tag{3.2}$$

From (3.2) and Lemma 2.1 we clearly see that $L_{1-3\alpha}(a,b) > G^{\alpha}(a,b)A^{1-\alpha}(a,b)$ for $\alpha = 1/3$ and $a \neq b$.

Case 2. $\alpha = 2/3$. Equation (1.1) leads to the following identity:

$$f(t) = \log(t-1) - \log(\log t) - \frac{1}{3}\log\left(t^2 + t\right) + \frac{1}{3}\log 2. \tag{3.3}$$

From (3.3) and Lemma 2.2 we clearly see that $L_{1-3\alpha}(a,b) > G^{\alpha}(a,b)A^{1-\alpha}(a,b)$ for $\alpha = 2/3$ and $a \neq b$.

Case 3. $\alpha \in (0,1) \setminus \{1/3, 2/3\}$. From (1.1) we have the following identity:

$$f(t) = \frac{1}{1-3\alpha}\log\frac{t^{2-3\alpha}-1}{(2-3\alpha)(t-1)} - \frac{\alpha}{2}\log t - (1-\alpha)\log\frac{1+t}{2}. \tag{3.4}$$

Equation (3.4) and elementary computation yields

$$\lim_{t\to 1^+} f(t) = 0, \tag{3.5}$$

$$f'(t) = \frac{1}{t(t^2-1)(t^{2-3\alpha}-1)}g(t), \tag{3.6}$$

where

$$
\begin{aligned}
g(t) &= \frac{\alpha}{2}t^{4-3\alpha} - \frac{\alpha(4-3\alpha)}{1-3\alpha}t^{3-3\alpha} - \frac{(1-\alpha)(4-3\alpha)}{2(1-3\alpha)}t^{2-3\alpha} \\
&\quad + \frac{(1-\alpha)(4-3\alpha)}{2(1-3\alpha)}t^{2} + \frac{\alpha(4-3\alpha)}{1-3\alpha}t - \frac{\alpha}{2}. \\
g(1) &= 0, \\
g'(t) &= \frac{\alpha(4-3\alpha)}{2}t^{3-3\alpha} - \frac{3\alpha(4-3\alpha)(1-\alpha)}{1-3\alpha}t^{2-3\alpha} \\
&\quad - \frac{(1-\alpha)(4-3\alpha)(2-3\alpha)}{2(1-3\alpha)}t^{1-3\alpha} + \frac{(1-\alpha)(4-3\alpha)}{1-3\alpha}t + \frac{\alpha(4-3\alpha)}{1-3\alpha}, \\
g'(1) &= 0, \\
g''(t) &= \frac{3\alpha(4-3\alpha)(1-\alpha)}{2}t^{2-3\alpha} - \frac{3\alpha(4-3\alpha)(2-3\alpha)(1-\alpha)}{1-3\alpha}t^{1-3\alpha} \\
&\quad - \frac{(1-\alpha)(4-3\alpha)(2-3\alpha)}{2}t^{-3\alpha} + \frac{(1-\alpha)(4-3\alpha)}{1-3\alpha}, \\
g''(1) &= 0,
\end{aligned}
\tag{3.7}
$$

$$
g'''(t) = \frac{3\alpha(4-3\alpha)(1-\alpha)(2-3\alpha)}{2t^{3\alpha+1}}(t-1)^2. \tag{3.8}
$$

If $\alpha \in (0,1) \setminus \{1/3, 2/3\}$, then (3.8) implies

$$
g'''(t) > 0 \tag{3.9}
$$

for $t > 1$. Therefore, $f(t) > 0$ follows from (3.5)–(3.7) and (3.9).

If $\alpha \in (2/3, 1)$, then (3.8) leads to

$$
g'''(t) < 0 \tag{3.10}
$$

for $t > 1$. Therefore, $f(t) > 0$ follows from (3.5)–(3.7) and (3.10).

Next, we prove that the constant $1-3\alpha$ in the inequality $G^{\alpha}(a,b)A^{1-\alpha}(a,b) \le L_{1-3\alpha}(a,b)$ cannot be improved. The proof is divided into five cases.

Case 1. $\alpha = 1/3$. For any $\epsilon \in (0,1)$, let $x \in (0,1)$, then (1.1) leads to

$$
\left[G^{1/3}(1,1+x)A^{2/3}(1,1+x)\right]^{\epsilon} - \left[L_{-\epsilon}(1,1+x)\right]^{\epsilon} = \frac{f_1(x)}{(1+x)^{1-\epsilon}-1}, \tag{3.11}
$$

where $f_1(x) = (1+x)^{(1/6)\epsilon}(1+x/2)^{(2/3)\epsilon}[(1+x)^{1-\epsilon}-1] - (1-\epsilon)x$.

Making use of Taylor expansion we get

$$
\begin{aligned}
f_1(x) &= \left[1+\frac{\epsilon}{6}x-\frac{\epsilon(6-\epsilon)}{72}x^2+o\left(x^2\right)\right]\left[1+\frac{\epsilon}{3}x-\frac{\epsilon(3-2\epsilon)}{36}x^2+o\left(x^2\right)\right] \\
&\quad\times(1-\epsilon)x\left[1-\frac{\epsilon}{2}x+\frac{\epsilon(1+\epsilon)}{6}x^2+o\left(x^2\right)\right]-(1-\epsilon)x \\
&= \frac{\epsilon^2(1-\epsilon)}{24}x^3+o\left(x^3\right).
\end{aligned}
\tag{3.12}
$$

Case 2. $\alpha = 2/3$. For any $\epsilon > 0$, let $x \in (0,1)$, then

$$
\left[G^{2/3}(1,1+x)A^{1/3}(1,1+x)\right]^{1+\epsilon}-\left[L_{-1-\epsilon}(1,1+x)\right]^{1+\epsilon}=\frac{f_2(x)}{(1+x)^{\epsilon}-1},
\tag{3.13}
$$

where $f_2(x) = [(1+x)^{\epsilon}-1](1+x)^{(1+\epsilon)/3}(1+x/2)^{(1+\epsilon)/3}-\epsilon x(1+x)^{\epsilon}$.

Using Taylor expansion we have

$$
\begin{aligned}
f_2(x) &= \epsilon x\Bigg\{\left[1-\frac{1-\epsilon}{2}x+\frac{(1-\epsilon)(2-\epsilon)}{6}x^2+o\left(x^2\right)\right] \\
&\qquad\times\left[1+\frac{1+\epsilon}{3}x-\frac{(1+\epsilon)(2-\epsilon)}{18}x^2+o\left(x^2\right)\right] \\
&\qquad\times\left[1+\frac{1+\epsilon}{6}x-\frac{(1+\epsilon)(2-\epsilon)}{72}x^2+o\left(x^2\right)\right] \\
&\qquad-\left[1+\epsilon x-\frac{\epsilon(1-\epsilon)}{2}x^2+o\left(x^2\right)\right]\Bigg\} \\
&= \frac{\epsilon^2(1+\epsilon)}{24}x^3+o\left(x^3\right).
\end{aligned}
\tag{3.14}
$$

Case 3. $\alpha \in (0,1/3)$. For any $\epsilon \in (0,1-3\alpha)$, let $x \in (0,1)$, then

$$
\left[G^{\alpha}(1,1+x)A^{1-\alpha}(1,1+x)\right]^{1-3\alpha-\epsilon}-\left[L_{1-3\alpha-\epsilon}(1,1+x)\right]^{1-3\alpha-\epsilon}=\frac{f_3(x)}{(2-3\alpha-\epsilon)x},
\tag{3.15}
$$

where $f_3(x) = (2-3\alpha-\epsilon)x(1+x)^{\alpha(1-3\alpha-\epsilon)/2}(1+x/2)^{(1-\alpha)(1-3\alpha-\epsilon)}-[(1+x)^{2-3\alpha-\epsilon}-1]$.

Making use of Taylor expansion and elaborated calculation we have

$$
f_3(x) = \frac{\epsilon}{24}(1-3\alpha-\epsilon)(2-3\alpha-\epsilon)x^3+o\left(x^3\right).
\tag{3.16}
$$

Case 4. $\alpha \in (1/3, 2/3)$. For any $\epsilon \in (0, 2-3\alpha)$, let $x \in (0,1)$, then

$$\left[G^{\alpha}(1,1+x)A^{1-\alpha}(1,1+x)\right]^{3\alpha+\epsilon-1} - \left[L_{1-3\alpha-\epsilon}(1,1+x)\right]^{3\alpha+\epsilon-1} = \frac{f_4(x)}{(1+x)^{2-3\alpha-\epsilon}-1}, \tag{3.17}$$

where $f_4(x) = [(1+x)^{2-3\alpha-\epsilon}-1](1+x)^{\alpha(3\alpha+\epsilon-1)/2}(1+x/2)^{(1-\alpha)(3\alpha+\epsilon-1)} - (2-3\alpha-\epsilon)x$.

Using Taylor expansion and elaborated calculation we have

$$f_4(x) = \frac{\epsilon}{24}(3\alpha+\epsilon-1)(2-3\alpha-\epsilon)x^3 + o\left(x^3\right). \tag{3.18}$$

Case 5. $\alpha \in (2/3, 1)$. For any $\epsilon > 0$, let x $(0,1)$, then

$$\left[G^{\alpha}(1,1+x)A^{1-\alpha}(1,1+x)\right]^{3\alpha+\epsilon-1} - \left[L_{1-3\alpha-\epsilon}(1,1+x)\right]^{3\alpha+\epsilon-1} = \frac{f_5(x)}{(1+x)^{3\alpha+\epsilon-2}-1}, \tag{3.19}$$

where $f_5(x) = [(1+x)^{3\alpha+\epsilon-2}-1](1+x)^{\alpha(3\alpha+\epsilon-1)/2}(1+x/2)^{(1-\alpha)(3\alpha+\epsilon-1)} - (3\alpha+\epsilon-2)x(1+x)^{3\alpha+\epsilon-2}$.

Using Taylor expansion and elaborated calculation we get

$$f_5(x) = \frac{\epsilon}{24}(3\alpha+\epsilon-1)(3\alpha+\epsilon-2)x^3 + o\left(x^3\right). \tag{3.20}$$

Cases 1–5 show that for any $\alpha \in (0,1)$, there exists $\epsilon_0 = \epsilon_0(\alpha) > 0$, for any $\epsilon \in (0, \epsilon_0)$ there exists $\delta = \delta(\alpha, \epsilon) > 0$ such that $L_{1-3\alpha-\epsilon}(1,1+x) < G^{\alpha}(1,1+x)A^{(1-\alpha)}(1,1+x)$ for $x \in (0, \delta)$. □

Theorem 3.2. *If $\alpha \in (0,1)$, then $G^{\alpha}(a,b)A^{1-\alpha}(a,b) \geq L_{2/(\alpha-2)}(a,b)$ for all $a, b > 0$, with equality if and only if $a = b$, and the constant $2/(\alpha-2)$ in $L_{2/(\alpha-2)}(a,b)$ cannot be improved.*

Proof. If $a = b$, then we clearly see that $G^{\alpha}(a,b)A^{1-\alpha}(a,b) = L_{2/(\alpha-2)}(a,b) = a$.

If $a \neq b$, without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and

$$f(t) = \log L_{2/(\alpha-2)}(a,b) - \log\left[G^{\alpha}(a,b)A^{1-\alpha}(a,b)\right]. \tag{3.21}$$

Firstly, we prove $f(t) < 0$ for $t = (a/b) > 1$. Simple computation leads to

$$\begin{gathered} f(t) = \frac{\alpha-2}{2}\log\frac{t^{\alpha/(\alpha-2)}-1}{(\alpha/(\alpha-2))(t-1)} - \frac{\alpha}{2}\log t - (1-\alpha)\log\frac{1+t}{2}, \\ \lim_{t\to 1^+} f(t) = 0, \\ f'(t) = \frac{g(t)}{t(t-1)(t+1)\left(t^{\alpha/(\alpha-2)}-1\right)}, \end{gathered} \tag{3.22}$$

where

$$\begin{gathered} g(t)=\frac{\alpha}{2}t^{(3\alpha-4)/(\alpha-2)}+\frac{4-3\alpha}{2}t^{2(\alpha-1)/(\alpha-2)}-\frac{4-3\alpha}{2}t-\frac{\alpha}{2}. \\ g(1)=0, \\ g'(t)=\frac{\alpha(3\alpha-4)}{2(\alpha-2)}t^{2(\alpha-1)/(\alpha-2)}+\frac{(\alpha-1)(4-3\alpha)}{\alpha-2}t^{\alpha/(\alpha-2)}-\frac{4-3\alpha}{2}, \\ g'(1)=0, \\ g''(t)=\frac{\alpha(4-3\alpha)(1-\alpha)}{(\alpha-2)^2}t^{2/(\alpha-2)}(t-1)>0 \end{gathered} \tag{3.23}$$

for $t>1$ and $\alpha\in(0,1)$.

From (3.23) we clearly see that

$$g(t)>0 \tag{3.24}$$

for $t>1$.

Since $\alpha/(\alpha-2)<0$, we have $t(t-1)(t+1)(t^{\alpha/(\alpha-2)}-1)<0$ for $t\in(1,+\infty)$. Therefore, $f(t)<0$ follows from (3.22) and (3.24).

Next, we prove that the constant $2/(\alpha-2)$ cannot be improved.

For any $\epsilon\in(0,\alpha/(2-\alpha))$, we have

$$\begin{aligned} &\left[L_{2/(\alpha-2)+\epsilon}(1,t)\right]^{2/(2-\alpha)-\epsilon}-\left[G^{\alpha}(1,t)A^{1-\alpha}(1,t)\right]^{2/(2-\alpha)-\epsilon} \\ &\quad=t\left[\frac{(\alpha/(2-\alpha)-\epsilon)(1-(1/t))}{1-t^{-(\alpha/(2-\alpha)-\epsilon)}}-t^{-\epsilon(2-\alpha)/2}\left(\frac{1+(1/t)}{2}\right)^{(1-\alpha)(2/(2-\alpha)-\epsilon)}\right], \\ &\lim_{t\to+\infty}\left[\frac{(\alpha/(2-\alpha)-\epsilon)(1-(1/t))}{1-t^{-(\alpha/(2-\alpha)-\epsilon)}}-t^{-\epsilon(2-\alpha)/2}\left(\frac{1+(1/t)}{2}\right)^{(1-\alpha)(2/(2-\alpha)-\epsilon)}\right]=\frac{\alpha}{2-\alpha}-\epsilon. \end{aligned} \tag{3.25}$$

Equation (3.25) imply that for any $\epsilon\in(0,\alpha/(2-\alpha))$ there exists $T=T(\epsilon,\alpha)>1$, such that $L_{2/(\alpha-2)+\epsilon}(1,t)>G^{\alpha}(1,t)A^{1-\alpha}(1,t)$ for $t\in(T,\infty)$. □

Acknowledgment

This work was supported by the Natural Science Foundation of Zhejiang Broadcast and TV University (Grant no. XKT-09G21).

References

[1] K. B. Stolarsky, "The power and generalized logarithmic means," *The American Mathematical Monthly*, vol. 87, no. 7, pp. 545–548, 1980.

[2] C. E. M. Pearce and J. Pečarić, "Some theorems of Jensen type for generalized logarithmic means," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 40, no. 9-10, pp. 789–795, 1995.

[3] B. Mond, C. E. M. Pearce, and J. Pečarić, "The logarithmic mean is a mean," *Mathematical Communications*, vol. 2, no. 1, pp. 35–39, 1997.

[4] Ch.-P. Chen and F. Qi, "Monotonicity properties for generalized logarithmic means," *The Australian Journal of Mathematical Analysis and Applications*, vol. 1, no. 2, article 2, pp. 1–4, 2004.

[5] W.-F. Xia, Y.-M. Chu, and G.-D. Wang, "The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means," *Abstract and Applied Analysis*, vol. 2010, Article ID 604804, 9 pages, 2010.

[6] B.-Y. Long and Y.-M. Chu, "Optimal inequalities for generalized logarithmic, arithmetic, and geometric means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 806825, 10 pages, 2010.

[7] Y.-M. Chu and W.-F. Xia, "Inequalities for generalized logarithmic means," *Journal of Inequalities and Applications*, vol. 2009, Article ID 763252, 7 pages, 2009.

[8] Ch.-P. Chen, "The monotonicity of the ratio between generalized logarithmic means," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 86–89, 2008.

[9] F. Qi, X.-A. Li, and S.-X. Chen, "Refinements, extensions and generalizations of the second Kershaw's double inequality," *Mathematical Inequalities & Applications*, vol. 11, no. 3, pp. 457–465, 2008.

[10] P. Kahlig and J. Matkowski, "Functional equations involving the logarithmic mean," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 76, no. 7, pp. 385–390, 1996.

[11] A. O. Pittenger, "The logarithmic mean in n variables," *The American Mathematical Monthly*, vol. 92, no. 2, pp. 99–104, 1985.

[12] N. S. Nadirashvili, "New isoperimetric inequalities in mathematical physics," in *Partial Differential Equations of Elliptic Type (Cortona, 1992)*, Sympos. Math., XXXV, pp. 197–203, Cambridge University Press, Cambridge, UK, 1994.

[13] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, NJ, USA, 1951.

[14] H. Alzer and W. Janous, "Solution of problem 8*," *Crux Mathematicorum*, vol. 13, pp. 173–178, 1987.

[15] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, vol. 31 of *Mathematics and Its Applications (East European Series)*, D. Reidel, Dordrecht, The Netherlands, 1988.

[16] W. Janous, "A note on generalized Heronian means," *Mathematical Inequalities & Applications*, vol. 4, no. 3, pp. 369–375, 2001.

[17] B. C. Carlson, "The logarithmic mean," *The American Mathematical Monthly*, vol. 79, pp. 615–618, 1972.

[18] E. B. Leach and M. C. Sholander, "Extended mean values. II," *Journal of Mathematical Analysis and Applications*, vol. 92, no. 1, pp. 207–223, 1983.

[19] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.

[20] H. Alzer, "Ungleichungen für Mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.

[21] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.