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Research Article

A New Method for Solving Monotone Generalized Variational Inequalities

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We suggest new dual algorithms and iterative methods for solving monotone generalized variational inequalities. Instead of working on the primal space, this method performs a dual step on the dual space by using the dual gap function. Under the suitable conditions, we prove the convergence of the proposed algorithms and estimate their complexity to reach an ε-solution. Some preliminary computational results are reported.

1. Introduction

Let C be a convex subset of the real Euclidean space $\mathbb{R}^n$, F be a continuous mapping from C into $\mathbb{R}^n$, and φ be a lower semicontinuous convex function from C into $\mathbb{R}$. We say that a point x^* is a solution of the following generalized variational inequality if it satisfies

$$\langle F(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, \quad \forall x \in C, \tag{GVI}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard dot product in $\mathbb{R}^n$.

Associated with the problem (GVI), the dual form of this is expressed as following which is to find $y^* \in C$ such that

$$\langle F(x), x - y^* \rangle + \varphi(x) - \varphi(y^*) \geq 0, \quad \forall x \in C. \tag{DGVI}$$

In recent years, this generalized variational inequalities become an attractive field for many researchers and have many important applications in electricity markets, transportations, economics, and nonlinear analysis (see [1–9]).

It is well known that the interior quadratic and dual technique are powerfull tools for analyzing and solving the optimization problems (see [10–16]). Recently these techniques have been used to develop proximal iterative algorithm for variational inequalities (see [17–22]).

In addition Nesterov [23] introduced a dual extrapolation method for solving variational inequalities. Instead of working on the primal space, this method performs a dual step on the dual space.

In this paper we extend results in [23] to the generalized variational inequality problem (GVI) in the dual space. In the first approach, a gap function $g(x)$ is constructed such that $g(x) \geq 0$, for all $x^* \in C$ and $g(x^*) = 0$ if and only if x^* solves (GVI). Namely, we first develop a convergent algorithm for (GVI) with F being monotone function satisfying a certain Lipschitz type condition on C. Next, in order to avoid the Lipschitz condition we will show how to find a regularization parameter at every iteration k such that the sequence x^k converges to a solution of (GVI).

The remaining part of the paper is organized as follows. In Section 2, we present two convergent algorithms for monotone and generalized variational inequality problems with Lipschitzian condition and without Lipschitzian condition. Section 3 deals with some preliminary results of the proposed methods.

2. Preliminaries

First, let us recall the well-known concepts of monotonicity that will be used in the sequel (see [24]).

Definition 2.1. Let C be a convex set in $\mathbb{R}^n$, and $F : C \to \mathbb{R}^n$. The function F is said to be

(i) pseudomonotone on C if

$$\langle F(y), x - y \rangle \geq 0 \Longrightarrow \langle F(x), x - y \rangle \geq 0, \quad \forall x, y \in C, \tag{2.1}$$

(ii) monotone on C if for each $x, y \in C$,

$$\langle F(x) - F(y), x - y \rangle \geq 0, \tag{2.2}$$

(iii) strongly monotone on C with constant $\beta > 0$ if for each $x, y \in C$,

$$\langle F(x) - F(y), x - y \rangle \geq \beta \|x - y\|^2, \tag{2.3}$$

(iv) Lipschitz with constant $L > 0$ on C (shortly L-Lipschitz), if

$$\|F(x) - F(y)\| \leq L \|x - y\|, \quad \forall x, y \in C. \tag{2.4}$$

Note that when φ is differentiable on some open set containing C, then, since φ is lower semicontinuous proper convex, the generalized variational inequality (GVI) is equivalent to the following variational inequalities (see [25, 26]):

Find $x^* \in C$ such that

$$\langle F(x^*) + \nabla\varphi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{2.5}$$

Throughout this paper, we assume that:

(A_1) the interior set of C, $\operatorname{int} C$ is nonempty,

(A_2) the set C is bounded,

(A_3) F is upper semicontinuous on C, and φ is proper, closed convex and subdifferentiable on C,

(A_4) F is monotone on C.

In special case $\varphi = 0$, problem (GVI) can be written by the following.

Find $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{VI}$$

It is well known that the problem (VI) can be formulated as finding the zero points of the operator $T(x) = F(x) + N_C(x)$, where

$$N_C(x) = \begin{cases} \{y \in C : \langle y, z - x \rangle \leq 0, \ \forall z \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{2.6}$$

The dual gap function of problem (GVI) is defined as follows:

$$g(x) := \sup\{\langle F(y), x - y \rangle + \varphi(x) - \varphi(y) \mid y \in C\}. \tag{2.7}$$

The following lemma gives two basic properties of the dual gap function (2.7) whose proof can be found, for instance, in [6].

Lemma 2.2. *The function g is a gap function of* (GVI), *that is,*

(i) $g(x) \geq 0$ *for all* $x \in C$,

(ii) $x^* \in C$ *and* $g(x^*) = 0$ *if and only if* x^* *is a solution to* (DGVI). *Moreover, if F is pseudomonotone then x^* is a solution to* (DGVI) *if and only if it is a solution to* (GVI).

The problem $\sup\{\langle F(y), x - y \rangle + \varphi(x) - \varphi(y) \mid y \in C\}$ may not be solvable and the dual gap function g may not be well-defined. Instead of using gap function g, we consider a truncated dual gap function g_R. Suppose that $\overline{x} \in \operatorname{int} C$ fixed and $R > 0$. The truncated dual gap function is defined as follows:

$$g_R(x) := \max\{\langle F(y), x - y \rangle + \varphi(x) - \varphi(y) \mid y \in C, \|y - \overline{x}\| \leq R\}. \tag{2.8}$$

For the following consideration, we define $B_R(\overline{x}) := \{y \in \mathbb{R}^n \mid \|y - \overline{x}\| \leq R\}$ as a closed ball in $\mathbb{R}^n$ centered at $\overline{x}$ and radius R, and $C_R := C \cap B_R(\overline{x})$. The following lemma gives some properties for g_R.

Lemma 2.3. *Under assumptions* (A_1)*–*(A_4)*, the following properties hold.*

(i) *The function* $g_R(\cdot)$ *is well-defined and convex on* C.

(ii) *If a point* $x^* \in C \cap B_R(\overline{x})$ *is a solution to* (DGVI) *then* $g_R(x^*) = 0$.

(iii) *If there exists* $x^0 \in C$ *such that* $g_R(x^0) = 0$ *and* $\|x^0 - \overline{x}\| < R$*, and* F *is pseudomonotone, then* x^0 *is a solution to* (DGVI) *(and also* (GVI)*).*

Proof. (i) Note that $\langle F(y), x - y\rangle + \varphi(x) - \varphi(y)$ is upper semicontinuous on C for $x \in C$ and $B_R(\overline{x})$ is bounded. Therefore, the supremum exists which means that g_R is well-defined. Moreover, since φ is convex on C and g is the supremum of a parametric family of convex functions (which depends on the parameter x), then g_R is convex on C

(ii) By definition, it is easy to see that $g_R(x) \geq 0$ for all $x \in C \cap B_R(\overline{x})$. Let x^* be a solution of (DGVI) and $x^* \in B_R(\overline{x})$. Then we have

$$\langle F(y), x^* - y\rangle + \varphi(x^*) - \varphi(y) \leq 0 \quad \forall y \in C. \tag{2.9}$$

In particular, we have

$$\langle F(y), x^* - y\rangle + \varphi(y) - \varphi(x^*) \leq 0 \tag{2.10}$$

for all $y \in C \cap B_R(\overline{x})$. Thus

$$g_R(x^*) = \sup\{\langle F(y), x^* - y\rangle + \varphi(x^*) - \varphi(y) \mid y \in C \cap B_R(\overline{x})\} \leq 0, \tag{2.11}$$

this implies $g_R(x^*) = 0$.

(iii) For some $x^0 \in C \cap \operatorname{int} B_R(\overline{x})$, $g_R(x^0) = 0$ means that x is a solution to (DGVI) restricted to $C \cap \operatorname{int} B_R(\overline{x})$. Since F is pseudomonotone, x^0 is also a solution to (GVI) restricted to $C \cap B_R(\overline{x})$. Since $x^0 \in \operatorname{int} B_R(\overline{x})$, for any $y \in C$, we can choose $\lambda > 0$ sufficiently small such that

$$y_\lambda := x^0 + \lambda\left(y - x^0\right) \in C \cap B_R(\overline{x}), \tag{2.12}$$

$$\begin{aligned}
0 &\leq \left\langle F\left(x^0\right), y_\lambda - x^0\right\rangle + \varphi(y_\lambda) - \varphi\left(x^0\right) \\
&= \left\langle F\left(x^0\right), x^0 + \lambda\left(y - x^0\right) - x^0\right\rangle + \varphi\left(x^0 + \lambda\left(y - x^0\right)\right) - \varphi\left(x^0\right) \\
&\leq \lambda\langle F(x^0), y - x^0\rangle + \lambda\varphi(y) + (1 - \lambda)\varphi(x^0) - \varphi(x^0) \\
&= \lambda\left(\left\langle F\left(x^0\right), y - x^0\right\rangle + \varphi(y) - \varphi\left(x^0\right)\right),
\end{aligned} \tag{2.13}$$

where (2.13) follows from the convexity of $\varphi(\cdot)$. Since $\lambda > 0$, dividing this inequality by λ, we obtain that x^0 is a solution to (GVI) on C. Since F is pseudomonotone, x^0 is also a solution to (DGVI). □

Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed convex set and $x \in \mathbb{R}^n$. Let us denote $d_C(x)$ the Euclidean distance from x to C and $Pr_C(x)$ the point attained this distance, that is,

$$d_C(x) := \min_{y \in C} \|y - x\|, \qquad Pr_C(x) := \arg\min_{y \in C} \|y - x\|. \tag{2.14}$$

As usual, Pr_C is referred to the Euclidean projection onto the convex set C. It is well-known that Pr_C is a nonexpansive and co-coercive operator on C (see [27, 28]).

The following lemma gives a tool for the next discussion.

Lemma 2.4. *For any $x, y, z \in \mathbb{R}^n$ and for any $\beta > 0$, the function d_C and the mapping Pr_C defined by* (2.14) *satisfy*

$$\langle Pr_C(x) - x, y - Pr_C(x)\rangle \geq 0, \quad \forall y \in C, \tag{2.15}$$

$$d_C^2(x + y) \geq d_C^2(x) + d_C^2(Pr_C(x) + y) - 2\langle y, Pr_C(x) - x\rangle, \tag{2.16}$$

$$\left\| x - Pr_C\left(x + \frac{1}{\beta} y\right)\right\|^2 \leq \frac{1}{\beta^2}\|y\|^2 - d_C^2\left(x + \frac{1}{\beta} y\right), \quad \forall x \in C. \tag{2.17}$$

Proof. Inequality (2.15) is obvious from the property of the projection Pr_C (see [27]). Now, we prove the inequality (2.16). For any $v \in C$, applying (2.15) we have

$$\begin{aligned} \|v - (x + y)\|^2 &= \|v - (Pr_C(x) + y) + (Pr_C(x) - x)\|^2 \\ &= \|v - (Pr_C(x) + y)\|^2 + 2\langle v - (Pr_C(x) + y), Pr_C(x) - x\rangle + \|Pr_C(x) - x\|^2 \\ &= \|v - (Pr_C(x) + y)\|^2 + 2\langle Pr_C(x) - x, v - Pr_C(x)\rangle \\ &\quad - 2\langle y, Pr_C(x) - x\rangle + \|Pr_C(x) - x\|^2 \\ &\geq \|v - (Pr_C(x) + y)\|^2 - 2\langle y, Pr_C(x) - x\rangle + \|Pr_C(x) - x\|^2. \end{aligned} \tag{2.18}$$

Using the definition of $d_C(\cdot)$ and noting that $d_C^2(x) = \|Pr_C(x) - x\|^2$ and taking minimum with respect to $v \in C$ in (2.18), then we have

$$d_C^2(x + y) \geq d_C^2(Pr_C(x) + y) + d_C^2(x) - 2\langle y, Pr_C(x) - x\rangle, \tag{2.19}$$

which proves (2.16).

From the definition of d_C, we have

$$\begin{aligned}
d_C^2\left(x+\frac{1}{\beta}y\right) &= \left\|Pr_C\left(x+\frac{1}{\beta}y\right)-x-\frac{1}{\beta}y\right\|^2 \\
&= \frac{1}{\beta^2}\left\|y^2\right\| - \left\|x+\frac{1}{\beta}y-Pr_C\left(x+\frac{1}{\beta}y\right)-\left(x-Pr_C\left(x+\frac{1}{\beta}y\right)\right)\right\|^2 \\
&\quad + \left\|Pr_C\left(x+\frac{1}{\beta}y\right)-x-\frac{1}{\beta}y\right\|^2 \\
&= \frac{1}{\beta^2}\left\|y^2\right\| - \left\|x-Pr_C\left(x+\frac{1}{\beta}y\right)\right\|^2 \\
&\quad + 2\left\langle x+\frac{1}{\beta}y-Pr_C\left(x+\frac{1}{\beta}y\right), x-Pr_C\left(x+\frac{1}{\beta}y\right)\right\rangle.
\end{aligned} \tag{2.20}$$

Since $x \in C$, applying (2.15) with $Pr_C(x+(1/\beta)y)$ instead of $Pr_C(x)$ and $y = x$ for (2.20), we obtain the last inequality in Lemma 2.4. □

For a given integer number $m \geq 0$, we consider a finite sequence of arbitrary points $\{x^k\}_{k=0}^m \subset C$, a finite sequence of arbitrary points $\{w^k\}_{k=0}^m \subset \mathbb{R}^n$ and a finite positive sequence $\{\lambda_k\}_{k=0}^m \subseteq (0,+\infty)$. Let us define

$$\overline{w}^m = \sum_{k=0}^m \lambda_k w^k, \quad \overline{\lambda}_m = \sum_{k=0}^m \lambda_k, \quad \overline{x}^m = \frac{1}{\overline{\lambda}_m}\sum_{k=0}^m \lambda_k x^k. \tag{2.21}$$

Then upper bound of the dual gap function g_R is estimated in the following lemma.

Lemma 2.5. *Suppose that Assumptions (A_1)–(A_4) are satisfied and*

$$w^k \in -F\left(x^k\right) - \partial\varphi\left(x^k\right). \tag{2.22}$$

Then, for any $\beta > 0$,

(i) $\max\{\langle w, y-\overline{x}\rangle \mid y \in C_R\} \leq (1/2\beta)\|w\|^2 - (\beta/2)d_C^2(\overline{x}+(1/\beta)w) + \beta R^2/2$, *for all* $\overline{x} \in C$, $w \in \mathbb{R}^n$.

(ii) $g_R(\overline{x}^m) \leq (1/\overline{\lambda}_m)(\sum_{k=0}^m \lambda_k\langle w^k, \overline{x}-x^k\rangle + (1/2\beta)\|\overline{w}^m\|^2 - (\beta/2)d_C^2(\overline{x}+(1/\beta)\overline{w}^m) + \beta R^2/2)$.

Proof. (i) We define $L(x,\rho) = \langle w, y-\overline{x}\rangle + (\rho/2)(R^2 - \|y-\overline{x}\|^2)$ as the Lagrange function of the maximizing problem $\max\{\langle w, y-\overline{x}\rangle \mid y \in C_R\}$. Using duality theory in convex optimization, then we have

$$\begin{aligned}
\max\{\langle w, y-\overline{x}\rangle \mid y \in C_R\} &= \max\left\{\langle w, y-\overline{x}\rangle \mid y \in C, \|y-\overline{x}\|^2 \le R^2\right\} \\
&= \max_{y\in C}\min_{\rho\ge 0}\left\{\langle w, y-\overline{x}\rangle + \rho\left(R^2 - \|y-\overline{x}\|^2\right)\right\} \\
&= \min_{\rho\ge 0}\left\{\max_{y\in C}\left\{\langle w, y-\overline{x}\rangle - \frac{\rho}{2}\|y-\overline{x}\|^2\right\} + \frac{\rho}{2}R^2\right\} \\
&= \min_{\rho\ge 0}\left\{\frac{1}{2\rho}\max_{y\in C}\left\{\|w\|^2 - \rho^2\left\|y-\overline{x}-\frac{1}{\rho}w\right\|^2\right\} + \frac{\rho}{2}R^2\right\} \\
&\le \frac{1}{2\beta}\left[\|w\|^2 - \beta^2\min_{y\in C}\left\|y-\overline{x}-\frac{1}{\beta}w\right\|^2\right] + \frac{\beta R^2}{2} \\
&= \frac{1}{2\beta}\|w\|^2 - \frac{\beta}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}w\right) + \frac{\beta R^2}{2}.
\end{aligned} \tag{2.23}$$

(ii) From the monotonicity of F and (2.22), we have

$$\begin{aligned}
\sum_{k=0}^{m}\lambda_k\left(\left\langle F(y), x^k - y\right\rangle + \varphi\left(x^k\right) - \varphi(y)\right) &\le -\sum_{k=0}^{m}\lambda_k\left(\left\langle F\left(x^k\right), y - x^k\right\rangle + \varphi(y) - \varphi\left(x^k\right)\right) \\
&\le \sum_{k=0}^{m}\lambda_k\left\langle w^k, y - x^k\right\rangle \\
&\le \sum_{k=0}^{m}\lambda_k\langle w^k, y-\overline{x}\rangle + \sum_{k=0}^{m}\lambda_k\left\langle w^k, \overline{x} - x^k\right\rangle \\
&= \langle \overline{w}^m, y-\overline{x}\rangle + \sum_{k=0}^{m}\lambda_k\left\langle w^k, \overline{x} - x^k\right\rangle.
\end{aligned} \tag{2.24}$$

Combining (2.24), Lemma 2.5(i) and

$$\begin{aligned}
g_R(\overline{x}^m) &= \max\{\langle F(y), \overline{x}^m - y\rangle + \varphi(\overline{x}^m) - \varphi(y) \mid y \in C_R\} \\
&= \max\left\{\left\langle F(y), \frac{1}{\overline{\lambda}_m}\sum_{k=0}^{m}\lambda_k x^k - y\right\rangle + \varphi\left(\frac{1}{\overline{\lambda}_m}\sum_{k=0}^{m}\lambda_k x^k\right) - \varphi(y) \mid y \in C_R\right\} \\
&\le \max\left\{\frac{1}{\overline{\lambda}_m}\sum_{k=0}^{m}\lambda_k\left(\left\langle F(y), x^k - y\right\rangle + \varphi\left(x^k\right) - \varphi(y)\right) \mid y \in C_R\right\} \\
&= \frac{1}{\overline{\lambda}_m}\max\left\{\sum_{k=0}^{m}\lambda_k\left(\left\langle F(y), x^k - y\right\rangle + \varphi\left(x^k\right) - \varphi(y)\right) \mid y \in C_R\right\},
\end{aligned} \tag{2.25}$$

we get

$$
\begin{aligned}
g_R(\bar{x}^m) &\le \frac{1}{\bar{\lambda}_m} \max\{\langle \bar{w}^m, y - \bar{x}\rangle \mid y \in C_R\} + \sum_{k=0}^{m} \lambda_k \left\langle w^k, \bar{x} - x^k \right\rangle \\
&\le \frac{1}{\bar{\lambda}_m} \left(\frac{1}{2\beta} \|\bar{w}^m\|^2 - \frac{\beta}{2} d_C^2\left(\bar{x} + \frac{1}{\beta}\bar{w}^m\right) + \frac{\beta R^2}{2} + \sum_{k=0}^{m} \lambda_k \left\langle w^k, \bar{x} - x^k \right\rangle \right).
\end{aligned}
\tag{2.26}
$$

□

3. Dual Algorithms

Now, we are going to build the dual interior proximal step for solving (GVI). The main idea is to construct a sequence $\{\bar{x}^k\}$ such that the sequence $g_R(\bar{x}^k)$ tends to 0 as $k \to \infty$. By virtue of Lemma 2.5, we can check whether $\bar{x}^k$ is an ε-solution to (GVI) or not.

The dual interior proximal step $(u^k, x^k, \bar{w}^k, w^k)$ at the iteration $k \ge 0$ is generated by using the following scheme:

$$
\begin{aligned}
u^k &:= Pr_C\left(\bar{x} + \frac{1}{\beta}\bar{w}^{k-1}\right), \\
x^k &:= \arg\min\left\{ \left\langle F\left(u^k\right), y - u^k \right\rangle + \varphi(y) - \varphi\left(u^k\right) + \frac{\beta\rho_k}{2}\left\|y - u^k\right\|^2 \mid y \in C \right\}, \\
\bar{w}^k &:= \bar{w}^{k-1} + \frac{1}{\rho_k} w^k,
\end{aligned}
\tag{3.1}
$$

where $\rho_k > 0$ and $\beta > 0$ are given parameters, $w^k \in \mathbb{R}^n$ is the solution to (2.22).

The following lemma shows an important property of the sequence (u^k, x^k, s^k, w^k).

Lemma 3.1. *The sequence $(u^k, x^k, \bar{w}^k, w^k)$ generated by scheme (3.1) satisfies*

$$
\begin{aligned}
d_C^2\left(\bar{x} + \frac{1}{\beta}\bar{w}^k\right) \ge{}& d_C^2\left(\bar{x} + \frac{1}{\beta}\bar{w}^{k-1}\right) + \left\|x^k - u^k\right\|^2 + \left\|\pi_C^k - x^k\right\|^2 \\
&- \frac{2}{\beta\rho_k}\left\langle \pi_C^k - x^k, \xi^k + w^k \right\rangle \\
&+ \frac{1}{\beta^2\rho_k^2}\left\|w^k\right\|^2 + \frac{2}{\beta\rho_k}\left\langle w^k, \bar{x} - x^k + \frac{1}{\beta}\bar{w}^{k-1} \right\rangle,
\end{aligned}
\tag{3.2}
$$

where $\eta^k \in \partial\varphi(x^k)$, $\xi^k = \eta^k + F(u^k)$ and $\pi_C^k = Pr_C(x^k + (1/\beta\rho_k)(\xi^k + w^k))$. As a consequence, we have

$$
\begin{aligned}
d_C^2\left(\bar{x} + \frac{1}{\beta}\bar{w}^k\right) - d_C^2\left(\bar{x} + \frac{1}{\beta}\bar{w}^{k-1}\right) \ge{}& \frac{2}{\beta\rho_k}\left\langle w^k, \bar{x} - x^k \right\rangle + \frac{1}{\beta^2}\left\|\bar{w}^k\right\|^2 - \frac{1}{\beta^2}\left\|\bar{w}^{k-1}\right\|^2 \\
&- \frac{1}{\beta^2\rho_k^2}\left\|\xi^k + w^k\right\|^2.
\end{aligned}
\tag{3.3}
$$

Proof. We replace x by $x + (1/\beta)y$ and y by $(1/\beta)z$ into (2.16) to obtain

$$\begin{aligned} d_C^2\left(x + \frac{1}{\beta}(y+z)\right) &\geq d_C^2\left(x + \frac{1}{\beta}y\right) + d_C^2\left(Pr_C\left(x + \frac{1}{\beta}y\right) + \frac{1}{\beta}z\right) \\ &\quad - \frac{2}{\beta}\left\langle z, Pr_C\left(x + \frac{1}{\beta}y\right) - \left(x + \frac{1}{\beta}y\right)\right\rangle. \end{aligned} \tag{3.4}$$

Using the inequality (3.4) with $x = \overline{x}$, $y = \overline{w}^{k-1}$, $z = (1/\rho_k)w^k$ and noting that $u^k = Pr_C(\overline{x} + (1/\beta)\overline{w}^{k-1})$, we get

$$\begin{aligned} d_C^2\left(\overline{x} + \frac{1}{\beta}\overline{w}^{k-1} + \frac{1}{\beta\rho_k}w^k\right) &\geq d_C^2\left(\overline{x} + \frac{1}{\beta}\overline{w}^{k-1}\right) + d_C^2\left(Pr_C\left(\overline{x} + \frac{1}{\beta}\overline{w}^{k-1}\right) + \frac{1}{\beta\rho_k}w^k\right) \\ &\quad - \frac{2}{\beta\rho_k}\left\langle w^k, Pr_C\left(\overline{x} + \frac{1}{\beta}\overline{w}^{k-1}\right) - \overline{x} - \frac{1}{\beta}\overline{w}^{k-1}\right\rangle. \end{aligned} \tag{3.5}$$

This implies that

$$d_C^2\left(\overline{x} + \frac{1}{\beta}\overline{w}^k\right) \geq d_C^2\left(\overline{x} + \frac{1}{\beta}\overline{w}^{k-1}\right) + d_C^2\left(u^k + \frac{1}{\beta\rho_k}w^k\right) - \frac{2}{\beta\rho_k}\left\langle w^k, u^k - \overline{x} - \frac{1}{\beta}\overline{w}^{k-1}\right\rangle. \tag{3.6}$$

From the subdifferentiability of the convex function φ to scheme (3.1), using the first-order necessary optimality condition, we have

$$\left\langle F\left(u^k\right) + \eta^k + \beta\rho_k\left(x^k - u^k\right), v - x^k\right\rangle \geq 0, \quad \forall v \in C, \tag{3.7}$$

for all $\eta^k \in \partial\varphi(x^k)$. This inequality implies that

$$x^k = Pr_C\left(u^k - \frac{1}{\beta\rho_k}\xi^k\right), \tag{3.8}$$

where $\xi^k = \eta^k + F(u^k)$.

We apply inequality (3.4) with $x = u^k$, $y = -(1/\rho_k)\xi^k$ and $z = (1/\rho_k)(\xi^k + w^k)$ and using (3.8) to obtain

$$\begin{aligned}
d_C^2\left(u^k + \frac{1}{\beta\rho_k}w^k\right) &\geq d_C^2\left(u^k - \frac{1}{\beta\rho_k}\xi^k\right) + d_C^2\left(x^k + \frac{1}{\beta\rho_k}\left(\xi^k + w^k\right)\right) \\
&\quad - \frac{2}{\beta\rho_k}\left\langle \xi^k + w^k, x^k - u^k + \frac{1}{\beta\rho_k}\xi^k \right\rangle \\
&= \left\| Pr_C\left(u^k - \frac{1}{\beta\rho_k}\xi^k\right) - u^k + \frac{1}{\beta\rho_k}\xi^k \right\|^2 \\
&\quad + d_C^2\left(x^k + \frac{1}{\beta\rho_k}\left(\xi^k + w^k\right)\right) - \frac{2}{\beta\rho_k}\left\langle \xi^k + w^k, x^k - u^k + \frac{1}{\beta\rho_k}\xi^k \right\rangle \\
&= \left\| x^k - u^k + \frac{1}{\beta\rho_k}\xi^k \right\|^2 + d_C^2\left(x^k + \frac{1}{\beta\rho_k}\left(\xi^k + w^k\right)\right) \\
&\quad + \frac{2}{\beta\rho_k}\left\langle \xi^k + w^k, u^k - \frac{1}{\beta\rho_k}\xi^k - x^k \right\rangle \\
&= \left\| x^k - u^k \right\|^2 + \frac{1}{\beta^2\rho_k^2}\left\|\xi^k\right\|^2 + \frac{2}{\beta\rho_k}\left\langle \xi^k, x^k - u^k \right\rangle \\
&\quad + d_C^2\left(x^k + \frac{1}{\beta\rho_k}\left(\xi^k + w^k\right)\right) + \frac{2}{\beta\rho_k}\left\langle \xi^k + w^k, u^k - \frac{1}{\beta\rho_k}\xi^k - x^k \right\rangle.
\end{aligned} \tag{3.9}$$

Combine this inequality and (3.6), we get

$$\begin{aligned}
d_C^2\left(\overline{x} + \frac{1}{\beta}\overline{w}^k\right) &\geq d_C^2\left(\overline{x} + \frac{1}{\beta}\overline{w}^{k-1}\right) - \frac{2}{\beta\rho_k}\left\langle w^k, u^k - \overline{x} - \frac{1}{\beta}\overline{w}^{k-1} \right\rangle \\
&\quad + \left\| x^k - u^k \right\|^2 + \frac{1}{\beta^2\rho_k^2}\left\|\xi^k\right\|^2 + \frac{2}{\beta\rho_k}\left\langle \xi^k, x^k - u^k \right\rangle \\
&\quad + d_C^2\left(x^k + \frac{1}{\beta\rho_k}\left(\xi^k + w^k\right)\right) + \frac{2}{\beta\rho_k}\left\langle \xi^k + w^k, u^k - \frac{1}{\beta\rho_k}\xi^k - x^k \right\rangle.
\end{aligned} \tag{3.10}$$

On the other hand, if we denote $\pi_C^k = Pr_C(x^k + (1/\beta\rho_k)(\xi^k + w^k))$, then it follows that

$$\begin{aligned}
d_C^2\left(x^k + \frac{1}{\beta\rho_k}\left(\xi^k + w^k\right)\right) &= \left\| \pi_C^k - x^k - \frac{1}{\beta\rho_k}\left(\xi^k + w^k\right) \right\|^2 \\
&= \left\| \pi_C^k - x^k \right\|^2 - \frac{2}{\beta\rho_k}\left\langle \pi_C^k - x^k, \xi^k + w^k \right\rangle + \frac{1}{\beta^2\rho_k^2}\left\|\xi^k + w^k\right\|^2.
\end{aligned} \tag{3.11}$$

Combine (3.10) and (3.11), we get

$$d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^k\right) \geq d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^{k-1}\right)+\left\|x^k-u^k\right\|^2+\left\|\pi_C^k-x^k\right\|^2 \\ -\frac{2}{\beta\rho_k}\left\langle \pi_C^k-x^k,\xi^k+w^k\right\rangle \\ +\frac{1}{\beta^2\rho_k^2}\left\|w^k\right\|^2+\frac{2}{\beta\rho_k}\left\langle w^k,\overline{x}-x^k+\frac{1}{\beta}\overline{w}^{k-1}\right\rangle, \tag{3.12}$$

which proves (3.2).

On the other hand, from (3.9) we have

$$d_C^2\left(u^k+\frac{1}{\beta\rho_k}w^k\right) \geq \left\|x^k-u^k\right\|^2+\frac{1}{\beta^2\rho_k^2}\left\|\xi^k\right\|^2+\frac{2}{\beta\rho_k}\left\langle \xi^k,x^k-u^k\right\rangle \\ +\frac{2}{\beta\rho_k}\left\langle \xi^k+w^k,u^k-\frac{1}{\beta\rho_k}\xi^k-x^k\right\rangle. \tag{3.13}$$

Then the inequality (3.3) is deduced from this inequality and (3.6). □

The dual algorithm is an iterative method which generates a sequence $(u^k, x^k, \overline{w}^k, w^k)$ based on scheme (3.1). The algorithm is presented in detail as follows:

Algorithm 3.2. One has the following.

Initialization:

Given a tolerance $\varepsilon > 0$, fix an arbitrary point $\overline{x} \in \operatorname{int} C$ and choose $\beta \geq L$, $R = \max\{\|x\| \mid x \in C\}$. Take $\overline{w}^{-1} := 0$ and $k := -1$.

Iterations:

For each $k = 0, 1, 2, \ldots, k_\varepsilon$, execute four steps below.

Step 1. Compute a projection point u^k by taking

$$u^k := Pr_C\left(\overline{x}+\frac{1}{\beta}\overline{w}^{k-1}\right). \tag{3.14}$$

Step 2. Solve the strongly convex programming problem

$$\min\left\{\left\langle F\left(u^k\right),y-u^k\right\rangle+\varphi(y)+\frac{\beta}{2}\left\|y-u^k\right\|^2 \mid y \in C\right\} \tag{3.15}$$

to get the unique solution x^k.

Step 3. Find $w^k \in \mathbb{R}^n$ such that

$$w^k \in -F\left(x^k\right) - \partial\varphi\left(x^k\right). \tag{3.16}$$

Set $\overline{w}^k := \overline{w}^{k-1} + w^k$.

Step 4. Compute

$$r_k := \sum_{i=0}^{k}\left\langle w^i, \overline{x} - x^i\right\rangle + \max\left\{\left\langle \overline{w}^k, y - \overline{x}\right\rangle \mid y \in C_R\right\}. \tag{3.17}$$

If $r_k \le (k+1)\varepsilon$, where $\varepsilon > 0$ is a given tolerance, then stop.

Otherwise, increase k by 1 and go back to Step 1.

Output:

Compute the final output $\overline{x}^k$ as:

$$\overline{x}_k := \frac{1}{k+1}\sum_{i=0}^{k} x^i. \tag{3.18}$$

Now, we prove the convergence of Algorithm 3.2 and estimate its complexity.

Theorem 3.3. *Suppose that assumptions (A_1)–(A_3) are satisfied and F is L-Lipschitz continuous on C. Then, one has*

$$g_R\left(\overline{x}^k\right) \le \frac{\beta R^2}{2(k+1)}, \tag{3.19}$$

where $\overline{x}^k$ is the final output defined by the sequence $(u^k, x^k, \overline{w}^k, w^k)_{k\ge 0}$ in Algorithm 3.2. As a consequence, the sequence $\{g_R(\overline{x}^k)\}$ converges to 0 and the number of iterations to reach an ε-solution is $k_\varepsilon := [\beta R^2/2\varepsilon]$, where $[x]$ denotes the largest integer such that $[x] \le x$.

Proof. From $\xi^k = \eta^k + F(u^k)$, where $\eta_k \in \partial\varphi(x^k)$ and $\pi_C^k \in C$, we get

$$\begin{aligned}\left\langle \xi^k + w^k, \pi_C^k - x^k\right\rangle &= \left\langle F\left(x^k\right) - F\left(u^k\right), x^k - \pi_C^k\right\rangle \\ &\le \frac{L}{2}\left(\left\|x^k - u^k\right\|^2 + \left\|x^k - \pi_C^k\right\|^2\right).\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.2), we obtain

$$d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^k\right)\geq d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^{k-1}\right)+\left(1-\frac{L}{\beta\rho_k}\right)\left(\left\|x^k-u^k\right\|^2+\left\|\pi_C^k-x^k\right\|^2\right)$$
$$+\frac{1}{\beta^2\rho_k^2}\left\|w^k\right\|^2+\frac{2}{\beta\rho_k}\left\langle w^k,\overline{x}-x^k+\frac{1}{\beta}\overline{w}^{k-1}\right\rangle. \tag{3.21}$$

Using this inequality with $\rho_i=1$ for all $i\geq 0$ and $\beta\geq L$, we obtain

$$\begin{aligned}d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^k\right)&\geq d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^{k-1}\right)+\left(1-\frac{L}{\beta}\right)\left(\left\|x^k-u^k\right\|^2+\left\|\pi_C^k-x^k\right\|^2\right)\\&\quad+\frac{1}{\beta^2}\left\|w^k\right\|^2+\frac{2}{\beta}\left\langle w^k,\overline{x}-x^k+\frac{1}{\beta}\overline{w}^{k-1}\right\rangle\\&\geq d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^{k-1}\right)+\frac{1}{\beta^2}\left\|w^k\right\|^2+\frac{2}{\beta}\left\langle w^k,\overline{x}-x^k+\frac{1}{\beta}\overline{w}^{k-1}\right\rangle.\end{aligned} \tag{3.22}$$

If we choose $\lambda_i=1$ for all $i\geq 0$ in (2.21), then we have

$$\overline{w}^k=\sum_{i=0}^{k}w^i,\qquad \overline{\lambda}_k=k+1,\qquad \overline{x}^k=\frac{1}{k+1}\sum_{i=0}^{k}x^i. \tag{3.23}$$

Hence, from Lemma 2.5(ii), we have

$$(k+1)g_R\left(\overline{x}^k\right)\leq\sum_{i=0}^{k}\langle w^i,\overline{x}-x^i\rangle+\frac{1}{2\beta}\left\|\overline{w}^k\right\|^2-\frac{\beta}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^k\right)+\frac{\beta R^2}{2}. \tag{3.24}$$

Using inequality (3.22) and $\|\overline{w}^k\|^2=\|w^k+\overline{w}^{k-1}\|^2$, it implies that

$$\begin{aligned}a_k&:=\sum_{i=0}^{k}\left\langle w^i,\overline{x}-x^i\right\rangle+\frac{1}{2\beta}\left\|\overline{w}^k\right\|^2-\frac{\beta}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^k\right)+\frac{\beta R^2}{2}\\&=\sum_{i=0}^{k-1}\left\langle w^i,\overline{x}-x^i\right\rangle+\left\langle w^k,\overline{x}-x^k\right\rangle+\frac{1}{2\beta}\left\|\overline{w}^k\right\|^2-\frac{\beta}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^k\right)+\frac{\beta R^2}{2}\\&\leq\sum_{i=0}^{k-1}\left\langle w^i,\overline{x}-x^i\right\rangle+\left\langle w^k,\overline{x}-x^k\right\rangle+\frac{1}{2\beta}\left\|\overline{w}^k\right\|^2+\frac{\beta R^2}{2}\\&\quad-\frac{\beta}{2}\left(d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^{k-1}\right)+\frac{1}{\beta^2}\left\|w^k\right\|^2+\frac{2}{\beta}\left\langle w^k,\overline{x}-x^k+\frac{1}{\beta}\overline{w}^{k-1}\right\rangle\right)\\&=\sum_{i=0}^{k-1}\left\langle w^i,\overline{x}-x^i\right\rangle+\frac{1}{2\beta}\left\|\overline{w}^{k-1}\right\|^2-\frac{\beta}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}\overline{w}^{k-1}\right)+\frac{\beta R^2}{2}\\&=a_{k-1}.\end{aligned} \tag{3.25}$$

Note that $a_{-1} = \beta R^2/2$. It follows from the inequalities (3.24) and (3.25) that

$$(k+1)g_R\left(\overline{x}^k\right) \le \frac{\beta R^2}{2}, \tag{3.26}$$

which implies that $g_R(\overline{x}^k) \le \beta R^2/2(k+1)$. The termination criterion at Step 4, $r_k \le (k+1)\epsilon$, using inequality (2.26) we obtain $g_R(\overline{x}^k) \le \epsilon$ and the number of iterations to reach an ϵ-solution is $k_\varepsilon := [\beta R^2/2\varepsilon]$. □

If there is no the guarantee for the Lipschitz condition, but the sequences (w^k) and (ξ^k) are uniformly bounded, we suppose that

$$M = \sup_k \left\| F\left(x^k\right) - F\left(u^k\right)\right\| = \sup_k \left\| w^k + \xi^k \right\|, \tag{3.27}$$

then the algorithm can be modified to ensure that it still converges. The variant of Algorithm 3.2 is presented as Algorithm 3.4 below.

Algorithm 3.4. One has the following.

Initialization:

Fix an arbitrary point $\overline{x} \in \operatorname{int} C$ and set $R = \max\{\|x\| \mid x \in C\}$. Take $\overline{w}^{-1} := 0$ and $k := -1$. Choose $\beta_k = M/R$ for all $k \ge 0$.

Iterations:

For each $k = 0, 1, 2, \ldots$ execute the following steps.

Step 1. Compute the projection point u_k by taking

$$u^k := Pr_C\left(\overline{x} + \frac{1}{\beta_k}\overline{w}^{k-1}\right). \tag{3.28}$$

Step 2. Solve the strong convex programming problem

$$\min\left\{ \left\langle F\left(u^k\right), y - u^k \right\rangle + \varphi(y) + \frac{\beta_k}{2}\left\| y - u^k\right\|^2 \mid y \in C\right\} \tag{3.29}$$

to get the unique solution x^k.

Step 3. Find $w^k \in \mathbb{R}^n$ such that

$$w^k \in -F\left(x^k\right) - \partial\varphi\left(x^k\right). \tag{3.30}$$

Set $\overline{w}^k := \overline{w}^{k-1} + w^k$.

Step 4. Compute

$$r_k := \sum_{i=0}^{k} \langle w^i, \overline{x} - x^i \rangle + \max\left\{ \langle \overline{w}^k, y - \overline{x} \rangle \mid y \in C_R \right\}. \tag{3.31}$$

If $r_k \leq (k+1)\varepsilon$, where $\varepsilon > 0$ is a given tolerance, then stop.

Otherwise, increase k by 1, update $\beta_k := (M/R)\sqrt{k+1}$ and go back to Step 1.

Output:

Compute the final output $\overline{x}^k$ as

$$\overline{x}_k := \frac{1}{k+1} \sum_{i=0}^{k} x^i. \tag{3.32}$$

The next theorem shows the convergence of Algorithm 3.4.

Theorem 3.5. *Let assumptions* (A_1)*–*(A_3) *be satisfied and the sequence* $(u^k, x^k, \overline{w}^k, w^k)$ *be generated by Algorithm 3.4. Suppose that the sequences* $(F(x^k))$ *and* $(F(u^k))$ *are uniformly bounded by* (3.27). *Then, we have*

$$g_R\left(\overline{x}^k\right) \leq \frac{MR}{\sqrt{k+1}}. \tag{3.33}$$

As a consequence, the sequence $\{g_R(\overline{x}^k)\}$ *converges to 0 and the number of iterations to reach an* ε*-solution is* $k_\varepsilon := [M^2R^2/\varepsilon^2]$.

Proof. If we choose $\lambda_k = 1$ for all $k \geq 0$ in (2.21), then we have $\overline{\lambda}_k = k+1$. Since $\overline{w}^{-1} = 0$, it follows from Step 3 of Algorithm 3.4 that

$$\overline{w}^k = \sum_{i=0}^{k} w^k. \tag{3.34}$$

From (3.34) and Lemma 2.5(ii), for all $\beta_k \geq 1$ we have

$$(k+1) g_R\left(\overline{x}^k\right) \leq \sum_{i=0}^{k} \left\langle w^i, \overline{x} - x^i \right\rangle + \frac{1}{2\beta_k} \left\| \overline{w}^k \right\|^2 - \frac{\beta_k}{2} d_C^2\left(\overline{x} + \frac{1}{\beta_k} \overline{w}^k \right) + \frac{\beta_k R^2}{2}. \tag{3.35}$$

We define $b_k := \sum_{i=0}^{k}\langle w^i, \overline{x}-x^i\rangle + (1/2\beta_k)\|\overline{w}^k\|^2 - (\beta_k/2)d_C^2(\overline{x}+(1/\beta)_k\overline{w}^k)$. Then, we have

$$\begin{aligned} b_k - b_{k-1} = {} & \left\langle w^k, \overline{x}-x^k\right\rangle + \frac{1}{2\beta_k}\left\|\overline{w}^k\right\|^2 - \frac{\beta_k}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}_k\overline{w}^k\right) - \frac{1}{2\beta_{k-1}}\left\|\overline{w}^{k-1}\right\|^2 \\ & + \frac{\beta_{k-1}}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}_{k-1}\overline{w}^{k-1}\right). \end{aligned} \tag{3.36}$$

We consider, for all $y \in \mathbb{R}^n$

$$\begin{aligned} q(\beta) &:= \frac{1}{2\beta}\|y\|^2 - \frac{\beta}{2}d_C^2\left(x+\frac{1}{\beta}w\right) \\ &= \frac{1}{2\beta}\|y\|^2 - \frac{\beta}{2}\min_{v\in C}\left\|v-x-\frac{1}{\beta}w\right\|^2. \end{aligned} \tag{3.37}$$

Then derivative of q is given by

$$q'(\beta) = -\left\|Pr_C\left(x+\frac{1}{\beta}y\right)-x\right\|^2 \leq 0. \tag{3.38}$$

Thus q is nonincreasing. Combining this with (3.36) and $0 < \beta_{k-1} < \beta_k$, we have

$$\begin{aligned} b_k - b_{k-1} \leq {} & \langle w^k, \overline{x}-x^k\rangle + \frac{1}{2\beta_k}\left\|\overline{w}^k\right\|^2 - \frac{\beta_k}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}_k\overline{w}^k\right) - \frac{1}{2\beta_k}\left\|\overline{w}^{k-1}\right\|^2 \\ & + \frac{\beta_k}{2}d_C^2\left(\overline{x}+\frac{1}{\beta}_k\overline{w}^{k-1}\right). \end{aligned} \tag{3.39}$$

From Lemma 3.1, $\beta=\beta_k$ and $\rho_k = 1$, we have

$$\begin{aligned} d_C^2\left(\overline{x}+\frac{1}{\beta_k}\overline{w}^k\right) - d_C^2\left(\overline{x}+\frac{1}{\beta_k}\overline{w}^{k-1}\right) \geq {} & \frac{2}{\beta_k}\langle w^k, \overline{x}-x^k\rangle + \frac{1}{\beta_k^2}\left\|\overline{w}^k\right\|^2 - \frac{1}{\beta_k^2}\left\|\overline{w}^{k-1}\right\|^2 \\ & - \frac{1}{\beta_k^2}\left\|\xi^k + w^k\right\|^2. \end{aligned} \tag{3.40}$$

Combining (3.39) and this inequality, we have

$$b_k - b_{k-1} \leq \frac{\left\|\xi^k + w^k\right\|^2}{2\beta_k} = \frac{\left\|F(x^k)-F(u^k)\right\|^2}{2\beta_k} \leq \frac{MR}{2\sqrt{k+1}}. \tag{3.41}$$

By induction on k, it follows from (3.41) and $\beta_0 := (M_x+M_u)/R$ that

$$b_k \leq \frac{MR}{2}\sum_{i=0}^{k}\frac{1}{\sqrt{i+1}} \leq \frac{MR}{2}\sqrt{k+1} \equiv \frac{\beta_k R^2}{2}. \tag{3.42}$$

From (3.35) and (3.42), we obtain

$$(k+1)g_R\left(\overline{x}^k\right) \le \beta_k R^2 = MR\sqrt{k+1}, \tag{3.43}$$

which implies that $g_R(x^k) \le MR/\sqrt{k+1}$. The remainder of the theorem is trivially follows from (3.33). □

4. Illustrative Example and Numerical Results

In this section, we illustrate the proposed algorithms on a class of generalized variational inequalities (GVI), where C is a polyhedral convex set given by

$$C := \{x \in \mathbb{R}^n \mid Ax \le b\}, \tag{4.1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. The cost function $F : C \to \mathbb{R}$ is defined by

$$F(x) = D(x) - Mx + q, \tag{4.2}$$

where $D : C \to \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix and $q \in \mathbb{R}^n$. The function φ is defined by

$$\varphi(x) := \sum_{i=1}^{n}\left(x_i^2 + |x_i - i|\right). \tag{4.3}$$

Then φ is subdifferentiable, but it is not differentiable on $\mathbb{R}^n$.

For this class of problem (GVI) we have the following results.

Lemma 4.1. *Let $D : C \to \mathbb{R}^n$. Then*

(i) *if D is τ-strongly monotone on C, then F is monotone on C whenever $\tau = \|M\|$.*

(ii) *if D is τ-strongly monotone on C, then F is $(\tau - \|M\|)$-strongly monotone on C whenever $\tau > \|M\|$.*

(iii) *if D is L-Lipschitz on C, then F is $(L + \|M\|)$-Lipschitz on C.*

Proof. Since D is τ-strongly monotone on C, that is

$$\begin{gathered}\langle D(x) - D(y), x - y\rangle \ge \tau\|x - y\|^2, \quad \forall x, y \in C,\\ \langle M(x - y), x - y\rangle \le \|M\| \ \ \|x - y\|^2, \quad \forall x, y \in C,\end{gathered} \tag{4.4}$$

we have

$$\langle F(x) - F(y), x - y\rangle = \langle D(x) - D(y), x - y\rangle - \langle M(x - y), x - y\rangle$$
$$\geq (\tau - \|M\|)\|x - y\|^2, \quad \forall x, y \in C. \tag{4.5}$$

Then (i) and (ii) easily follow.

Using the Lipschitz condition, it is not difficult to obtain (iii). □

To illustrate our algorithms, we consider the following data.

$$n = 10, \quad D(x) := \tau x, \quad q = (1, -1, 2, -3, 1, -4, 5, 6, -2, 7)^T,$$
$$C := \left\{ x \in \mathbb{R}^{10} \mid \sum_{i=1}^{10} x_i \geq -2, -1 \leq x_i \leq 1 \right\},$$
$$M = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.5 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.5 \end{bmatrix}, \tag{4.6}$$
$$\overline{x} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \in \operatorname{int} C, \quad \epsilon = 10^{-6}, \quad R = \sqrt{10},$$

with $\tau = \|M\| = 2.2071$, $L = \tau + \|M\| = 4.4142$, $\beta = L/2 = 2.2071$. From Lemma 4.1, we have F is monotone on C. The subproblems in Algorithm 3.2 can be solved efficiently, for example, by using MATLAB Optimization Toolbox R2008a. We obtain the approximate solution

$$x^{10} = (0.0510, 0.6234, -0.2779, 1.0000, 0.0449, 1.0000, -1.0000, 1.0000, 0.7927, -1.0000)^T. \tag{4.7}$$

Now we use Algorithm 3.4 on the same variational inequalities except that

$$F(x) := \tau x + D(x) - Mx + q, \tag{4.8}$$

where the n components of the $D(x)$ are defined by: $D_j(x) = d_j \arctan(x_j)$, with d_j randomly chosen in $(0, 1)$ and the n components of q are randomly chosen in $(-1, 3)$. The function D is given by Bnouhachem [19]. Under these assumptions, it can be proved that F is continuous and monotone on C.

Table 1: Numerical results: Algorithm 3.4 with $n = 10$.

P	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	x_9^k	x_{10}^k
1	−0.278	0.001	−0.006	−0.377	0.272	−0.007	−0.462	−0.227	0.395	−0.364
2	−0.054	0.133	−0.245	−0.435	−0.348	0.080	0.493	−0.223	−0.146	0.307
3	−0.417	0.320	−0.027	−0.270	0.463	−0.375	−0.381	0.255	−0.087	−0.403
4	0.197	0.161	0.434	−0.090	0.505	−0.001	0.451	−0.358	−0.320	0.278
5	0.291	0.071	−0.383	−0.290	0.453	−0.035	−0.393	−0.536	0.238	0.166
6	−0.021	0.246	0.211	−0.036	0.044	−0.241	0.466	−0.186	0.486	−0.072
7	−0.429	0.220	0.134	0.321	−0.312	0.364	−0.278	0.551	0.421	−0.118
8	−0.349	−0.448	0.365	−0.467	−0.137	0.387	0.217	−0.049	−0.443	−0.453
9	−0.115	0.562	−0.371	−0.536	−0.198	−0.248	−0.233	0.124	−0.149	0.319
10	0.071	0.134	−0.268	−0.340	0.307	0.010	0.052	−0.168	−0.206	−0.244

With $\overline{x} = (0,0,0,0,0,0,0,0,0,0) \in \operatorname{int} C$ and the tolerance $\epsilon = 10^{-6}$, we obtained the computational results (see, the Table 1).

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