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Research Article

Strong Convergence for Generalized Equilibrium Problems, Fixed Point Problems and Relaxed Cocoercive Variational Inequalities

Chaichana Jaiboon[1,2] and Poom Kumam[1]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, KMUTT, Bangkok 10140, Thailand*
[2] *Department of Mathematics, Faculty of Applied Liberal Arts, Rajamangala University of Technology, Rattanakosin, RMUTR, Bangkok 10100, Thailand*

Correspondence should be addressed to Poom Kumam, poom.kum@kmutt.ac.th

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We introduce a new iterative scheme for finding the common element of the set of solutions of the generalized equilibrium problems, the set of fixed points of an infinite family of nonexpansive mappings, and the set of solutions of the variational inequality problems for a relaxed (u, v)-cocoercive and ξ-Lipschitz continuous mapping in a real Hilbert space. Then, we prove the strong convergence of a common element of the above three sets under some suitable conditions. Our result can be considered as an improvement and refinement of the previously known results.

1. Introduction

Variational inequalities introduced by Stampacchia [1] in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences. It is well known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation has been used to suggest and analyze in variational inequalities. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. It is well known that the convergence of a projection method requires the operator to be strongly monotone and Lipschitz continuous. Gabay [2] has shown that the convergence of a projection method can be proved for cocoercive operators. Note that cocoercivity is a weaker condition than strong monotonicity.

Equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization which has been extended and generalized in many directions using novel and innovative technique; see [3, 4]. Related to the equilibrium problems, we also have the problem of finding the fixed points of the nonexpansive mappings. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems and a set of the fixed points of infinitely (finitely) many nonexpansive mappings; see [5–7] and the references therein. In this paper, we suggest and analyze a new iterative method for finding a common element of a set of the solutions of generalized equilibrium problems and a set of fixed points of an infinite family of nonexpansive mappings and the set solution of the variational inequality problems for a relaxed (u, v)-cocoercive mapping in a real Hilbert space.

Let H be a real Hilbert space and let E be a nonempty closed convex subset of H and P_E is the metric projection of H onto E. Recall that a mapping $f : E \to E$ is contraction on E if there exists a constant $\alpha \in (0,1)$ such that $\|f(x)-f(y)\| \leq \alpha\|x-y\|$ for all $x, y \in E$. A mapping S of E into itself is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in E$. We denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in E : Sx = x\}$. If $E \subset H$ is nonempty, bounded, closed, and convex and S is a nonexpansive mapping of E into itself, then $F(S)$ is nonempty; see, for example, [8]. We recalled some definitions as follows.

Definition 1.1. Let $B : E \to H$ be a mapping. Then one has the following.

(1) B is called *monotone* if $\langle Bx - By, x - y\rangle \geq 0$, for all $x, y \in E$.

(2) B is called *v-strongly monotone* if there exists a positive real number v such that

$$\langle Bx - By, x - y\rangle \geq v\|x - y\|^2, \quad \forall x, y \in E. \tag{1.1}$$

(3) B is called *ξ-Lipschitz continuous* if there exists a positive real number ξ such that

$$\|Bx - By\| \leq \xi\|x - y\|, \quad \forall x, y \in E. \tag{1.2}$$

(4) B is called *η-inverse-strongly monotone*, [9, 10] if there exists a positive real number η such that

$$\langle Bx - By, x - y\rangle \geq \eta\|Bx - By\|^2, \quad \forall x, y \in E. \tag{1.3}$$

If $\eta = 1$, we say that B is firmly nonexpansive. It is obvious that any η-inverse-strongly monotone mapping B is monotone and $(1/\eta)$-Lipschitz continuous.

(5) B is called *relaxed (u, v)-cocoercive* if there exists a positive real number u, v such that

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2, \quad \forall x, y \in E. \tag{1.4}$$

For $u = 0$, B is v-strongly monotone. This class of maps is more general than the class of strongly monotone maps. It is easy to see that we have the following implication: v-strongly monotonicity $\Rightarrow$ relaxed (u, v)-cocoercivity.

(6) A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let B be a monotone mapping of E into H and let $N_E w_1$ be the *normal cone* to E at $w_1 \in E$, that is,

$$N_E w_1 = \{w \in H : \langle \vartheta - w_1, w\rangle \geq 0,\ \forall \vartheta \in E\}. \tag{1.5}$$

Define

$$Tw_1 = \begin{cases} Bw_1 + N_E w_1, & \text{if } w_1 \in E, \\ \emptyset, & \text{if } w_1 \notin E. \end{cases} \tag{1.6}$$

Then T is the maximal monotone and $0 \in Tw_1$ if and only if $w_1 \in \mathrm{VI}(E, B)$; see [11, 12]

In addition, let $D : E \to H$ be a inverse-strongly monotone mapping. Let F be a bifunction of $E \times \mathrm{E}$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The generalized equilibrium problem for $F : E \times E \to \mathbb{R}$ is to find $x \in E$ such that

$$F(x, y) + \langle Dx, y - x\rangle \geq 0, \quad \forall y \in E. \tag{1.7}$$

The set of such $x \in E$ is denoted by $\mathrm{EP}(F, D)$, that is,

$$\mathrm{EP}(F, D) = \{x \in E : F(x, y) + \langle Dx, y - x\rangle \geq 0,\ \forall y \in E\}. \tag{1.8}$$

Special Cases

(I) If $D \equiv 0$ (:the zero mapping), then the problem (1.7) is reduced to the equilibrium problem:

$$\text{Find } x \in E \text{ such that } F(x, y) \geq 0, \quad \forall y \in E. \tag{1.9}$$

The set of solutions of (1.9) is denoted by $\mathrm{EP}(F)$, that is,

$$\mathrm{EP}(F) = \{x \in E : F(x, y) \geq 0,\ \forall y \in E\}. \tag{1.10}$$

(II) If $F \equiv 0$, the problem (1.7) is reduced to the variational inequality problem:

$$\text{Find } x \in E \text{ such that } \langle Dx, y - x\rangle \geq 0, \quad \forall y \in E. \tag{1.11}$$

The set of solutions of (1.11) is denoted by $\mathrm{VI}(E, D)$, that is,

$$\mathrm{VI}(E, D) = \{x \in E : \langle Dx, y - x\rangle \geq 0,\ \forall y \in E\}. \tag{1.12}$$

The generalized equilibrium problem (1.7) is very general in the sense that it includes, as special case, some optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, economics, and others (see, e.g., [4, 13]). Some methods have been proposed to solve the equilibrium problem and the generalized equilibrium problem; see, for instance, [5, 14–28]. Recently, Combettes and Hirstoaga [29] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem. Very recently, Moudafi [24] introduced an itertive method for finding an element of $\mathrm{EP}(F, D) \cap F(S)$, where $D : E \to H$ is an inverse-strongly monotone mapping and then proved a weak convergence theorem.

For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problem for an η-inverse-strongly monotone, Takahashi and Toyoda [30] introduced the following iterative scheme:

$$\begin{gathered} x_0 \in E \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\mathrm{SP}_E(x_n - \tau_n B x_n), \quad \forall n \geq 0, \end{gathered} \tag{1.13}$$

where B is an η-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\tau_n\}$ is a sequence in $(0, 2\eta)$. They showed that if $F(S) \cap \mathrm{VI}(E, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.13) converges weakly to some $z \in F(S) \cap \mathrm{VI}(E, B)$. On the other hand, Shang et al. [31] introduced a new iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a relaxed (u, v)-cocoercive mapping in a real Hilbert space. Let $S : E \to E$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in E$, defined sequences $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \mathrm{SP}_E(I - \tau_n B)x_n, \quad \forall n \geq 1. \tag{1.14}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\tau_n\}$, the sequence $\{x_n\}$ converges strongly to $z \in F(S) \cap \mathrm{VI}(E, B)$, where $z = P_{F(S)\cap \mathrm{VI}(E,B)} f(z)$.

In 2008, S. Takahashi and W. Takahashi [27] introduced the following iterative scheme for finding an element of $F(S) \cap \mathrm{EF}(F, D)$ under some mild conditions. Let E be a nonempty closed convex subset of a real Hilbert space H. Let D be an η-inverse-strongly monotone mapping of E into H and let S be a nonexpansive mapping of E into itself such that $F(S) \cap \mathrm{EP}(F, D) \neq \emptyset$. Suppose $x_1 = \sigma \in E$ and let $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ by sequences generated by

$$\begin{gathered} F(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n \sigma + (1-\alpha_n)u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S y_n, \end{gathered} \tag{1.15}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [0,2\eta]$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.15) converges strongly to a common element of $F(S) \cap \mathrm{EF}(F, D)$.

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [32–35] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences.

A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping in a real Hilbert space H:

$$\min_{x\in E}\left\{\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle\right\}, \tag{1.16}$$

where E is the fixed point set of a nonexpansive mapping S on H and b is a given point in H. Assume that A is a *strongly positive bounded linear operator* on H; that is, there exists a constant $\overline{\gamma}>0$ such that

$$\langle Ax,x\rangle\geq\overline{\gamma}\|x\|^2,\quad \forall x\in H. \tag{1.17}$$

In 2006, Marino and Xu [36] considered the following iterative method:

$$x_{n+1}=\epsilon_n\gamma f(x_n)+(1-\epsilon_nA)Sx_n,\quad \forall n\geq 0. \tag{1.18}$$

They proved that if the sequence $\{\epsilon_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.18) converges strongly to the unique of the variational inequality

$$\langle(A-\gamma f)z,x-z\rangle\geq 0,\quad \forall x\in F(S), \tag{1.19}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)}\left\{\frac{1}{2}\langle Ax,x\rangle-h(x)\right\}, \tag{1.20}$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

In 2008, Qin et al. [26] proposed the following iterative algorithm:

$$\begin{gathered}F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in H,\\ x_{n+1}=\epsilon_n\gamma f(x_n)+(I-\epsilon_nA)\mathrm{SP}_E(I-\tau_nB)u_n,\end{gathered} \tag{1.21}$$

where A is a strongly positive linear bounded operator and B is a relaxed cocoercive mapping of E into H. They prove that if the sequences $\{\epsilon_n\}$, $\{\tau_n\}$, and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle(A-\gamma f)z,x-z\rangle\geq 0,\quad \forall x\in F(S)\cap\mathrm{VI}(E,B)\cap\mathrm{EP}(F), \tag{1.22}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap \mathrm{VI}(E,B)\cap \mathrm{EP}(F)} \left\{ \frac{1}{2}\langle Ax, x\rangle - h(x) \right\}, \tag{1.23}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Furthermore, for finding approximate common fixed points of an infinite family of nonexpansive mappings $\{T_n\}$ under very mild conditions on the parameters, we need the following definition.

Definition 1.2 (see [37]). Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of E into itself and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For each $n \geq 1$, define a mapping W_n of E into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n)I, \\ U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I, \\ &\vdots \\ U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \\ U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I, \\ &\vdots \\ U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\ W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I. \end{aligned} \tag{1.24}$$

Such a mappings W_n is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$. It is obvious that W_n is nonexpansive, and if $x = T_n x$, then $x = U_{n,k} = W_n x$.

On the other hand, Yao et al. [38] introduced and considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points of an infinite family of nonexpansive mappings on E. Starting with an arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ &x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A) W_n u_n, \quad \forall n \geq 1, \end{aligned} \tag{1.25}$$

where $\{\epsilon_n\}$ is a sequence in $(0,1)$. It is proved [38] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.25) converges strongly to $z = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F)}(I - A + \gamma f)z$. Very recently, Qin et al. [6] introduced an iterative scheme for finding a common fixed points of a finite family of nonexpansive mappings, the set of solutions of the variational inequality problem for a relaxed cocoercive mapping, and the set of solutions of the equilibrium problems in a real Hilbert space. Starting with an arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H,$$
$$x_{n+1} = \epsilon_n \gamma f(W_n x_n) + (I - \epsilon_n A) W_n P_E (I - \tau_n B) u_n, \quad \forall n \geq 1, \tag{1.26}$$

where B is a relaxed (u, v)-cocoercive mapping and A is a strongly positive linear bounded operator. They proved that under certain appropriate conditions imposed on $\{\epsilon_n\}, \{\tau_n\}$, and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.26) converge strongly to some point $z \in \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \cap \mathrm{VI}(E, B)$, which is a unique solution of the variation inequality:

$$\langle (A - \gamma f) z, x - z\rangle \geq 0, \quad \forall x \in \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \cap \mathrm{VI}(E, B) \tag{1.27}$$

and is also the optimality for some minimization problems.

In this paper, motivated by iterative schemes considered in (1.15), (1.25), and (1.26) we will introduce a new iterative process (3.4) below for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of the generalized equilibrium problem, and the set of solutions of variational inequality problem for a relaxed (u, v)-cocoercive mapping in a real Hilbert space. The results obtained in this paper improve and extend the recent ones announced by Yao et al. [38], S. Takahashi and W. Takahashi [27], and Qin et al. [6] and many others.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let E be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. Recall that the (nearest point) projection P_E from H to E assigns each $x \in H$ the unique point in $P_E x \in E$ satisfying the property

$$\|x - P_E x\| = \min_{y \in E} \|x - y\|. \tag{2.1}$$

The following characterizes the projection P_E.

We need some facts tools in a real Hilbert space H which are listed as follows.

Lemma 2.1. *For any $x \in H$, $z \in E$,*

$$z = P_E x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in E. \tag{2.2}$$

It is well known that P_E is a firmly nonexpansive mapping of H onto E and satisfies

$$\|P_E x - P_E y\|^2 \leq \langle P_E x - P_E y, x - y\rangle, \quad \forall x, y \in H. \tag{2.3}$$

Moreover, $P_E x$ *is characterized by the following properties:* $P_E x \in E$ *and for all* $x \in H, y \in E$,

$$\langle x - P_E x, y - P_E x \rangle \leq 0. \tag{2.4}$$

Lemma 2.2 (see [39]). *Let* H *be a Hilbert space, let* E *be a nonempty closed convex subset of* H*, and let* B *be a mapping of* E *into* H*. Let* $p \in E$*. Then for* $\lambda > 0$,

$$p \in \mathrm{VI}(E, B) \Longleftrightarrow p = P_E(p - \lambda B p), \tag{2.5}$$

where P_E *is the metric projection of* H *onto* E.

It is clear from Lemma 2.2 that variational inequality and fixed point problem are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Lemma 2.3 (see [40]). *Each Hilbert space* H *satisfies Opials condition; that is, for any sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$*, the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.6}$$

holds for each $y \in H$ *with* $y \neq x$.

Lemma 2.4 (see [36]). *Assume that* A *is a strongly positive linear bounded operator on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$*. Then* $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

For solving the equilibrium problem for a bifunction $F : E \times E \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in E$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in E$;

(A3) $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$, for all $x, y, z \in E$;

(A4) for each $x \in E, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [4].

Lemma 2.5 (see [4]). *Let* E *be a nonempty closed convex subset of* H *and let* F *be a bifunction of* $E \times E$ *into* $\mathbb{R}$ *satisfying (A1)–(A4). Let* $r > 0$ *and* $x \in H$*. Then, there exists* $z \in E$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in E. \tag{2.7}$$

The following lemma was also given in [5].

Lemma 2.6 (see [5]). *Assume that $F : E \times E \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to E$ as follows:*

$$T_r(x) = \left\{ z \in E : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in E \right\}, \tag{2.8}$$

for all $z \in H$. Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.9}$$

(3) $F(T_r) = \mathrm{EP}(F)$;

(4) $\mathrm{EP}(F)$ *is closed and convex.*

Remark 2.7. Replacing x with $x - rDx \in H$ in (2.7), then there exists $z \in E$, such that

$$F(z, y) + \langle Dx, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in E. \tag{2.10}$$

Lemma 2.8 (see [41]). *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, for every $x \in E$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.8, one can define a mapping W of E into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \tag{2.11}$$

for every $x \in E$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$. Throughout this paper, we will assume that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, we have the following results.

Lemma 2.9 (see [41]). *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of E into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.10 (see [7]). *If $\{x_n\}$ is a bounded sequence in E, then $\lim_{n\to\infty}\|Wx_n - W_n x_n\| = 0$.*

Lemma 2.11 (see [42]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.12. *Let H be a real Hilbert space. Then the following inequality holds:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$

for all $x, y \in H$.

Lemma 2.13 (see [43]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - l_n)a_n + \sigma_n, \quad \forall n \geq 0, \tag{2.12}$$

where $\{l_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$,

(2) $\limsup_{n\to\infty}(\sigma_n/l_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we prove a strong convergence theorem of a new iterative method (3.4) for an infinite family of nonexpansive mappings and relaxed (u,v)-cocoercive mappings in a real Hilbert space.

We first prove the following lemmas.

Lemma 3.1. *Let H be a real Hilbert space, let E be a nonempty closed convex subset of H, and let $D : E \to H$ be η-inverse-strongly monotone. It $0 \leq r_n \leq 2\eta$, then $I - r_nD$ is a nonexpansive mapping in H.*

Proof. For all $x, y \in E$ and $0 \leq r_n \leq 2\eta$, we have

$$\begin{aligned}
\|(I - r_nD)x - (I - r_nD)y\|^2 &= \|(x - y) - r_n(Dx - Dy)\|^2 \\
&= \|x - y\|^2 - 2r_n\langle x - y, Dx - Dy\rangle + r_n^2\|Dx - Dy\|^2 \\
&\leq \|x - y\|^2 - 2r_n\eta\|Dx - Dy\| + r_n^2\|Dx - Dy\|^2 \\
&= \|x - y\|^2 + r_n(r_n - 2\eta)\|Dx - Dy\|^2 \\
&\leq \|x - y\|^2.
\end{aligned} \tag{3.1}$$

So, $I - r_nD$ is a nonexpansive mapping of E into H. □

Lemma 3.2. *Let H be a real Hilbert space, let E be a nonempty closed convex subset of H, and let $B : E \to H$ be a relaxed (u,v)-cocoercive and ξ-Lipschitz continuous. If $0 \leq \tau_n \leq 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$, then $I - \tau_nB$ is a nonexpansive mapping in H.*

Proof. For any $x, y \in E$ and $\tau_n \leq 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$.

Putting $r = 1 + 2\tau_nu\xi^2 - 2\tau_nv + \tau_n^2\xi^2$, we obtain

$$\begin{aligned}
\tau_n \leq \frac{2(v - u\xi^2)}{\xi^2} &\Longleftrightarrow \tau_n\xi^2 + 2u\xi^2 - 2v \leq 0 \\
&\Longleftrightarrow \tau_n^2\xi^2 + 2\tau_nu\xi^2 - 2\tau_nv \leq 0 \\
&\Longleftrightarrow 1 + \tau_n^2\xi^2 + 2\tau_nu\xi^2 - 2\tau_nv \leq 1,
\end{aligned} \tag{3.2}$$

that is, $r \leq 1$. It follows that

$$\begin{aligned}\|(I-\tau_n B)x-(I-\tau_n B)y\|^2 &= \|(x-y)-\tau_n(Bx-By)\|^2 \\ &= \|x-y\|^2-2\tau_n\langle x-y,Bx-By\rangle+\tau_n^2\|Bx-By\|^2 \\ &\leq \|x-y\|^2-2\tau_n\left\{-u\|Bx-By\|^2+v\|x-y\|^2\right\}+\tau_n^2\|Bx-By\|^2 \\ &\leq \|x-y\|^2+2\tau_n u\xi^2\|x-y\|^2-2\tau_n v\|x-y\|^2+\tau_n^2\xi^2\|x-y\|^2 \\ &= \left(1+2\tau_n u\xi^2-2\tau_n v+\tau_n^2\xi^2\right)\|x-y\|^2 \\ &= r\|x-y\|^2 \\ &\leq \|x-y\|^2,\end{aligned} \tag{3.3}$$

for all $x, y \in E$. Thus $\|(I-\tau_n B)x-(I-\tau_n B)y\| \leq \|x-y\|$.

So, $I-\tau_n B$ is a nonexpansive mapping of E into H. □

Now, we prove the following main theorem.

Theorem 3.3. *Let E be a nonempty closed convex subset of a real Hilbert space H, and let $F : E\times E \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let*

(1) *$\{T_n\}$ be an infinite family of nonexpansive mappings of E into E;*

(2) *D be an η-inverse strongly monotone mappings of E into H;*

(3) *B be relaxed (u,v)-cocoercive and ξ-Lipschitz continuous mappings of E into H.*

Assume that $\Theta := \bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)\neq\emptyset$. Let $f : E \to E$ be a contraction mapping with $0<\alpha<1$ and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma}>0$ and $0<\gamma<\overline{\gamma}/\alpha$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by

$$\begin{gathered} x_1\in E \text{ chosen arbitrary}, \\ F(u_n,y)+\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in E, \\ y_n=\varphi_n u_n+(1-\varphi_n)W_nP_E(u_n-\delta_n Bu_n), \\ k_n=\alpha_n x_n+(1-\alpha_n)W_nP_E(y_n-\lambda_n By_n), \\ x_{n+1}=\epsilon_n\gamma f(W_nx_n)+\beta_n x_n+((1-\beta_n)I-\epsilon_n A)W_nP_E(k_n-\tau_n Bk_n),\quad \forall n\geq 1,\end{gathered} \tag{3.4}$$

where $\{W_n\}$ is the sequence generated by (1.24) and $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\varphi_n\}$, and $\{\beta_n\}$ are sequences in $(0,1)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\epsilon_n=0$, $\sum_{n=1}^{\infty}\epsilon_n=\infty$,

(C2) $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\varphi_n=0$, (C3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C4) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(C5) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty}|\delta_{n+1} - \delta_n| = \lim_{n\to\infty}|\tau_{n+1} - \tau_n| = 0$,

(C6) $\{\tau_n\}$, $\{\lambda_n\}$, $\{\delta_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$,

(C7) $\{r_n\} \subset [c,d]$ *for some* c, d *with* $0 < c < d < 2\eta$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$, *where* $z = P_\Theta(I - A + \gamma f)(z)$, *which solves the variational inequality*

$$\langle (A - \gamma f)z, x - z\rangle \ge 0, \quad \forall x \in \Theta, \tag{3.5}$$

which is the optimality condition fot the minimization problem

$$\min_{x\in\Theta}\left\{\frac{1}{2}\langle Ax, x\rangle - h(x)\right\}, \tag{3.6}$$

where h *is a potential function for* γf *(i.e.,* $h'(x) = \gamma f(x)$ *for* $x \in H$*).*

Proof. Since $\lim_{n\to\infty}\epsilon_n = 0$ by the condition (C1) and $\limsup_{n\to\infty}\beta_n < 1$, we may assume, without loss of generality, that $\epsilon_n \le (1 - \beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on H, then

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}. \tag{3.7}$$

Observe that

$$\begin{aligned}\langle ((1 - \beta_n)I - \epsilon_n A)x, x\rangle &= 1 - \beta_n - \epsilon_n\langle Ax, x\rangle \\ &\ge 1 - \beta_n - \epsilon_n\|A\| \\ &\ge 0,\end{aligned} \tag{3.8}$$

that is to say $(1 - \beta_n)I - \epsilon_n A$ is positive. It follows that

$$\begin{aligned}\|(1 - \beta_n)I - \epsilon_n A\| &= \sup\{|\langle ((1 - \beta_n)I - \epsilon_n A)x, x\rangle| : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \epsilon_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\le 1 - \beta_n - \epsilon_n\overline{\gamma}.\end{aligned} \tag{3.9}$$

We will divide the proof of Theorem 3.3 into six steps.

Step 1. We prove that there exists $z \in E$ such that $z = P_{\bigcap_{n=1}^{\infty}F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)}(I - A + \gamma f)z$.

Let $\Im = P_{\bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)}$. Note that f is a contraction mapping of E into itself with coefficient $\alpha \in (0,1)$. Then, we have

$$\begin{aligned}\|\Im(I-A+\gamma f)(x)-\Im(I-A+\gamma f)(y)\| &\leq \|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\| \\ &\leq \|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\| \\ &\leq (1-\overline{\gamma})\|x-y\|+\gamma\alpha\|x-y\| \\ &= (1-(\overline{\gamma}-\alpha\gamma))\|x-y\|, \quad \forall x,y\in H.\end{aligned} \tag{3.10}$$

Therefore, $\Im(I-A+\gamma f)$ is a contraction mapping of E into itself. Therefore by the Banach Contraction Mapping Principle guarantee that $\Im(I-A+\gamma f)$ has a unique fixed point, say $z\in E$. That is, $z=\Im(I-A+\gamma f)(z)=P_{\bigcap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)}(I-A+\gamma f)(z)$.

Step 2. We prove that $\{x_n\}$ is bounded.

Since

$$F(u_n,y)+\langle Dx_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \quad \forall y\in E, \tag{3.11}$$

we obtain

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-(I-r_nD)x_n\rangle \geq 0, \quad \forall y\in E. \tag{3.12}$$

From Lemma 2.6, we have $u_n=T_{r_n}(x_n-r_nDx_n)$, for all $n\in\mathbb{N}$.

For any $p\in\Theta:=\bigcap_{n=1}^{\infty}F(T_n)\cap \mathrm{EP}(F,D)\cap \mathrm{VI}(E,B)$, it follows from $p\in \mathrm{EP}(F,D)$ that

$$F(p,y)+\langle y-p,Dp\rangle \geq 0, \quad \forall y\in E. \tag{3.13}$$

So, we have

$$F(p,y)+\frac{1}{r_n}\langle y-p,p-(p-r_nDp)\rangle \geq 0, \quad \forall y\in E. \tag{3.14}$$

By Lemma 2.6 again, we have $p=T_{r_n}(p-r_nDp)$, for all $n\in\mathbb{N}$. If follows that

$$\begin{aligned}\|u_n-p\| &= \|T_{r_n}(x_n-r_nDx_n)-T_{r_n}(p-r_nDp)\| \\ &\leq \|(x_n-r_nDx_n)-(p-r_nDp)\| \\ &= \|(I-r_nD)x_n-(I-r_nD)p\| \leq \|x_n-p\|.\end{aligned} \tag{3.15}$$

If we applied Lemma 3.2, we get $I-\lambda_n B$ and $I-\delta_n B$ are nonexpansive. Since $p \in \mathrm{VI}(E,B)$ and W_n is a nonexpansive, we have $p = W_n P_E(p-\lambda_n Bp) = W_n P_E(p-\delta_n Bp)$, and we have

$$\begin{aligned}\|y_n - p\| &\leq \varphi_n\|u_n - p\| + (1-\varphi_n)\|W_nP_E(u_n-\delta_nBu_n) - W_nP_E(p-\delta_nBp)\| \\ &\leq \varphi_n\|u_n - p\| + (1-\varphi_n)\|(u_n-\delta_nBu_n) - (p-\delta_nBp)\| \\ &= \varphi_n\|u_n - p\| + (1-\varphi_n)\|(I-\delta_nB)u_n - (I-\delta_nB)p\| \\ &\leq \varphi_n\|u_n - p\| + (1-\varphi_n)\|u_n - p\| \\ &\leq \|u_n - p\| \leq \|x_n - p\|.\end{aligned} \tag{3.16}$$

It follows that

$$\begin{aligned}\|k_n - p\| &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|W_nP_E(y_n-\lambda_nBy_n) - W_nP_E(p-\lambda_nBp)\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|(y_n-\lambda_nBy_n) - (p-\lambda_nBp)\| \\ &= \alpha_n\|x_n - p\| + (1-\alpha_n)\|(I-\lambda_nB)y_n - (I-\lambda_nB)p\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| = \|x_n - p\|,\end{aligned} \tag{3.17}$$

which yields that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\epsilon_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \epsilon_nA)(W_nP_E(k_n - \tau_nBk_n) - p)\| \\ &\leq (1-\beta_n-\epsilon_n\overline{\gamma})\|P_E(I-\tau_nB)k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\ &\leq (1-\beta_n-\epsilon_n\overline{\gamma})\|k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\ &\leq (1-\beta_n-\epsilon_n\overline{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\ &\leq (1-\epsilon_n\overline{\gamma})\|x_n - p\| + \epsilon_n\gamma\|f(x_n) - f(p)\| + \epsilon_n\|\gamma f(p) - Ap\| \\ &\leq (1-\epsilon_n\overline{\gamma})\|x_n - p\| + \epsilon_n\gamma\alpha\|x_n - p\| + \epsilon_n\|\gamma f(p) - Ap\| \\ &= (1-(\overline{\gamma}-\alpha\gamma)\epsilon_n)\|x_n - p\| + (\overline{\gamma}-\alpha\gamma)\epsilon_n\frac{\|\gamma f(p) - Ap\|}{\overline{\gamma}-\alpha\gamma}.\end{aligned} \tag{3.18}$$

This in turn implies that

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma}-\alpha\gamma}\right\}, \quad n \in \mathbb{N}. \tag{3.19}$$

Therefore, $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{k_n\}$, $\{y_n\}$, $\{Bu_n\}$, $\{Bk_n\}$, $\{By_n\}$, $\{W_n u_n\}$, $\{W_n k_n\}$, $\{W_n y_n\}$, and $\{f(W_n x_n)\}$ are all bounded.

Step 3. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|W_n\theta_n - x_n\| = 0$.

From Lemma 2.6, we have $u_n = T_{r_n}(x_n - r_n D x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1} D x_{n+1})$. Let $\varpi_n = x_n - r_n D x_n$, we get $u_n = T_{r_n}\varpi_n$, $u_{n+1} = T_{r_{n+1}}\varpi_{n+1}$, and so

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - \varpi_n\rangle \geq 0, \quad \forall y \in E, \tag{3.20}$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - \varpi_{n+1}\rangle \geq 0, \quad \forall y \in E. \tag{3.21}$$

Putting $y = u_{n+1}$ in (3.20) and $y = u_n$ in (3.21), we have

$$\begin{aligned} &F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - \varpi_n\rangle \geq 0, \\ &F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - \varpi_{n+1}\rangle \geq 0. \end{aligned} \tag{3.22}$$

So, from the monotonicity of F, we get

$$\left\langle u_{n+1} - u_n, \frac{u_n - \varpi_n}{r_n} - \frac{u_{n+1} - \varpi_{n+1}}{r_{n+1}}\right\rangle \geq 0, \tag{3.23}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - \varpi_n - \frac{r_n}{r_{n+1}}(u_{n+1} - \varpi_{n+1})\right\rangle \geq 0. \tag{3.24}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, \varpi_{n+1} - \varpi_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - \varpi_{n+1})\right\rangle \\ &\leq \|u_{n+1} - u_n\|\left\{\|\varpi_{n+1} - \varpi_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - \varpi_{n+1}\|\right\}, \end{aligned} \tag{3.25}$$

and hence

$$\begin{aligned}
\|u_{n+1} - u_n\| &\le \|\varpi_{n+1} - \varpi_n\| + \frac{1}{c}|r_{n+1} - r_n|\|u_{n+1} - \varpi_{n+1}\| \\
&= \|x_{n+1} - r_{n+1}Dx_{n+1} - (x_n - r_nDx_n)\| + \frac{1}{c}|r_{n+1} - r_n|\|u_{n+1} - \varpi_{n+1}\| \\
&\le \|x_{n+1} - r_{n+1}Dx_{n+1} - (x_n - r_{n+1}Dx_n)\| + |r_{n+1} - r_n|\|Dx_n\| \\
&\quad + \frac{1}{c}|r_{n+1} - r_n|\|u_{n+1} - \varpi_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Dx_n\| + \frac{1}{c}|r_{n+1} - r_n|\|u_{n+1} - \varpi_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + M_1|r_{n+1} - r_n|,
\end{aligned} \tag{3.26}$$

where $M_1 = \sup\{\|Dx_n\| + (1/c)\|u_{n+1} - \varpi_{n+1}\| : n \in \mathbb{N}\}$.

Put $\theta_n = P_E(k_n - \tau_nBk_n)$, $\phi_n = P_E(y_n - \lambda_nBy_n)$, and $\varphi_n = P_E(u_n - \delta_nBu_n)$. Since $I - \tau_nB$, $I - \lambda_nB$, and $I - \delta_nB$ are nonexpansive, then we have the following some estimates:

$$\begin{aligned}
\|\varphi_{n+1} - \varphi_n\| &\le \|P_E(u_{n+1} - \delta_{n+1}Bu_{n+1}) - P_E(u_n - \delta_nBu_n)\| \\
&\le \|(u_{n+1} - \delta_{n+1}Bu_{n+1}) - (u_n - \delta_nBu_n)\| \\
&= \|(u_{n+1} - \delta_{n+1}Bu_{n+1}) - (u_n - \delta_{n+1}Bu_n) + (\delta_n - \delta_{n+1})Bu_n\| \\
&\le \|(u_{n+1} - \delta_{n+1}Bu_{n+1}) - (u_n - \delta_{n+1}Bu_n)\| + |\delta_n - \delta_{n+1}|\|Bu_n\| \\
&= \|(I - \delta_{n+1}B)u_{n+1} - (I - \delta_{n+1}B)u_n\| + |\delta_n - \delta_{n+1}|\|Bu_n\| \\
&\le \|u_{n+1} - u_n\| + |\delta_n - \delta_{n+1}|\|Bu_n\|.
\end{aligned} \tag{3.27}$$

Similarly, we can prove that

$$\|\phi_{n+1} - \phi_n\| \le \|y_{n+1} - y_n\| + |\lambda_n - \lambda_{n+1}|\|By_n\|, \tag{3.28}$$

$$\|\theta_{n+1} - \theta_n\| \le \|k_{n+1} - k_n\| + |\tau_n - \tau_{n+1}|\|Bk_n\|. \tag{3.29}$$

Since T_i and $U_{n,i}$ are nonexpansive, we deduce that, for each $n \le 1$,

$$\begin{aligned}
\|W_{n+1}\varphi_n - W_n\varphi_n\| &= \|\mu_1T_1U_{n+1,2}\varphi_n - \mu_1T_1U_{n,2}\varphi_n\| \\
&\le \mu_1\|U_{n+1,2}\varphi_n - U_{n,2}\varphi_n\| \\
&= \mu_1\|\mu_2T_2U_{n+1,3}\varphi_n - \mu_2T_2U_{n,3}\varphi_n\| \\
&\le \mu_1\mu_2\|U_{n+1,3}\varphi_n - U_{n,3}\varphi_n\|
\end{aligned}$$

$$\begin{aligned}&\vdots\\&\le \prod_{i=1}^{n}\mu_i \left\|U_{n+1,n+1}\psi_n - U_{n,n+1}\psi_n\right\|\\&\le M_2\prod_{i=1}^{n}\mu_i,\end{aligned} \tag{3.30}$$

where $M_2 \ge 0$ is a constant such that $\|U_{n+1,n+1}\psi_n - U_{n,n+1}\psi_n\| \le M_2$ for all $n \ge 0$.

Similarly, we can obtain that there exist nonnegative numbers M_3, M_4 such that

$$\left\|U_{n+1,n+1}\varphi_n - U_{n,n+1}\varphi_n\right\| \le M_3, \qquad \|U_{n+1,n+1}\theta_n - U_{n,n+1}\theta_n\| \le M_4, \tag{3.31}$$

and so are

$$\left\|W_{n+1}\phi_n - W_n\phi_n\right\| \le M_3\prod_{i=1}^{n}\mu_i, \qquad \|W_{n+1}\theta_n - W_n\theta_n\| \le M_4\prod_{i=1}^{n}\mu_i. \tag{3.32}$$

Observing that

$$\begin{aligned}y_n &= \varphi_n u_n + (1-\varphi_n)W_n\psi_n,\\ y_{n+1} &= \varphi_{n+1}u_{n+1} + (1-\varphi_{n+1})W_{n+1}\psi_{n+1},\end{aligned} \tag{3.33}$$

we obtain

$$y_n - y_{n+1} = \varphi_n(u_n - u_{n+1}) + (1-\varphi_n)\left(W_n\psi_n - W_{n+1}\psi_{n+1}\right) + \left(W_{n+1}\psi_{n+1} - u_{n+1}\right)\left(\varphi_{n+1} - \varphi_n\right), \tag{3.34}$$

which yields that

$$\begin{aligned}\left\|y_n - y_{n+1}\right\| &\le \varphi_n\|u_n - u_{n+1}\| + (1-\varphi_n)\left\|W_{n+1}\psi_{n+1} - W_n\psi_n\right\| + \left|\varphi_{n+1}-\varphi_n\right|\left\|W_{n+1}\psi_{n+1} - u_{n+1}\right\|\\
&\le \varphi_n\|u_n - u_{n+1}\| + (1-\varphi_n)\left\{\left\|W_{n+1}\psi_{n+1} - W_{n+1}\psi_n\right\| + \left\|W_{n+1}\psi_n - W_n\psi_n\right\|\right\}\\
&\quad + \left|\varphi_{n+1}-\varphi_n\right|\left\|W_{n+1}\psi_{n+1} - u_{n+1}\right\|\\
&\le \varphi_n\|u_n - u_{n+1}\| + (1-\varphi_n)\left\{\left\|\psi_{n+1} - \psi_n\right\| + \left\|W_{n+1}\psi_n - W_n\psi_n\right\|\right\}\\
&\quad + \left|\varphi_{n+1}-\varphi_n\right|\left\|W_{n+1}\psi_{n+1} - u_{n+1}\right\|\\
&\le \varphi_n\|u_n - u_{n+1}\| + (1-\varphi_n)\left\|\psi_{n+1} - \psi_n\right\| + \left\|W_{n+1}\psi_n - W_n\psi_n\right\|\\
&\quad + \left|\varphi_{n+1}-\varphi_n\right|\left\|W_{n+1}\psi_{n+1} - u_{n+1}\right\|.\end{aligned} \tag{3.35}$$

Substitution of (3.27) and (3.30) into (3.35) yields that

$$\begin{aligned}\|y_n - y_{n+1}\| &\le \varphi_n\|u_n - u_{n+1}\| + (1-\varphi_n)\{\|u_{n+1} - u_n\| + |\delta_n - \delta_{n+1}|\|Bu_n\|\} \\ &\quad + M_2\prod_{i=1}^{n}\mu_i + |\varphi_{n+1} - \varphi_n|\|W_{n+1}\varphi_{n+1} - u_{n+1}\| \\ &= \|u_n - u_{n+1}\| + (1-\varphi_n)|\delta_n - \delta_{n+1}|\|Bu_n\| \\ &\quad + M_2\prod_{i=1}^{n}\mu_i + \|W_{n+1}\varphi_{n+1} - u_{n+1}\||\varphi_{n+1} - \varphi_n| \\ &\le \|u_n - u_{n+1}\| + M_5(|\delta_n - \delta_{n+1}| + |\varphi_{n+1} - \varphi_n|) + M_2\prod_{i=1}^{n}\mu_i,\end{aligned} \tag{3.36}$$

where M_5 is an appropriate constant such that $M_5 = \max\{\sup_{n\ge1}\|Bu_n\|, \sup_{n\ge1}\|W_n\varphi_n - u_n\|\}$.

Observing that

$$\begin{aligned}k_n &= \alpha_n x_n + (1-\alpha_n)W_n\phi_n, \\ k_{n+1} &= \alpha_{n+1}x_{n+1} + (1-\alpha_{n+1})W_n\phi_{n+1},\end{aligned} \tag{3.37}$$

we obtain

$$k_n - k_{n+1} = \alpha_n(x_n - x_{n+1}) + (1-\alpha_n)(W_n\phi_n - W_{n+1}\phi_{n+1}) + (W_{n+1}\phi_{n+1} - x_{n+1})(\alpha_{n+1} - \alpha_n), \tag{3.38}$$

which yields that

$$\begin{aligned}\|k_n - k_{n+1}\| &\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|W_n\phi_n - W_{n+1}\phi_{n+1}\| + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - x_{n+1}\| \\ &\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\{\|W_{n+1}\phi_{n+1} - W_{n+1}\phi_n\| + \|W_{n+1}\phi_n - W_n\phi_n\|\} \\ &\quad + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - x_{n+1}\| \\ &\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|\phi_{n+1} - \phi_n\| + \|W_{n+1}\phi_n - W_n\phi_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|W_{n+1}\phi_{n+1} - x_{n+1}\|.\end{aligned} \tag{3.39}$$

Substitution of (3.28) and (3.32) into (3.39) yields that

$$\begin{aligned}
\|k_n - k_{n+1}\| &\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\left\{\left\|y_{n+1} - y_n\right\| + |\lambda_n - \lambda_{n+1}|\left\|By_n\right\|\right\} \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + |\alpha_{n+1} - \alpha_n|\left\|W_{n+1}\phi_{n+1} - x_{n+1}\right\| \\
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\left\|y_{n+1} - y_n\right\| + (1-\alpha_n)|\lambda_n - \lambda_{n+1}|\left\|By_n\right\| \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + |\alpha_{n+1} - \alpha_n|\left\|W_{n+1}\phi_{n+1} - x_{n+1}\right\| \\
&\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\left\|y_{n+1} - y_n\right\| + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|),
\end{aligned} \tag{3.40}$$

where M_6 is an appropriate constant such that $M_6 = \max\{\sup_{n\ge1}\|By_n\|, \sup_{n\ge1}\|W_n\phi_n - x_n\|\}$.

Substituting (3.26) and (3.36) into (3.40), we obtain

$$\begin{aligned}
\|k_n - k_{n+1}\| &\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\left\{\|u_n - u_{n+1}\| + M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) + M_2\prod_{i=1}^{n}\mu_i\right\} \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|u_n - u_{n+1}\| + (1-\alpha_n)M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) \\
&\quad + (1-\alpha_n)M_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\le \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\{\|x_{n+1} - x_n\| + M_1|r_{n+1} - r_n|\} \\
&\quad + (1-\alpha_n)M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) + (1-\alpha_n)M_2\prod_{i=1}^{n}\mu_i \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&= \alpha_n\|x_n - x_{n+1}\| + (1-\alpha_n)\|x_{n+1} - x_n\| + (1-\alpha_n)M_1|r_{n+1} - r_n| \\
&\quad + (1-\alpha_n)M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) + (1-\alpha_n)M_2\prod_{i=1}^{n}\mu_i \\
&\quad + M_3\prod_{i=1}^{n}\mu_i + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\le \|x_n - x_{n+1}\| + M_1|r_{n+1} - r_n| + M_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + M_5\left(|\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right|\right) + M_6(|\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|) \\
&\le \|x_n - x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + K_1\left(|r_{n+1} - r_n| + |\delta_n - \delta_{n+1}| + \left|\varphi_{n+1} - \varphi_n\right| + |\lambda_n - \lambda_{n+1}| + |\alpha_{n+1} - \alpha_n|\right),
\end{aligned} \tag{3.41}$$

where K_1 is an appropriate constant such that $K_1 = \max\{M_1, M_5, M_6\}$.

Substituting (3.41) into (3.29), we obtain

$$\begin{aligned}
\|\theta_{n+1}-\theta_n\| &\le \|k_{n+1}-k_n\| + |\tau_n-\tau_{n+1}|\|Bk_n\| \\
&\le \|x_n-x_{n+1}\| + M_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + K_1\left(|r_{n+1}-r_n| + |\delta_n-\delta_{n+1}| + \left|\varphi_{n+1}-\varphi_n\right| + |\lambda_n-\lambda_{n+1}| + |\alpha_{n+1}-\alpha_n|\right) \\
&\quad + |\tau_n-\tau_{n+1}|\|Bk_n\| \\
&\le \|x_n-x_{n+1}\| + \mathrm{M}_2\prod_{i=1}^{n}\mu_i + M_3\prod_{i=1}^{n}\mu_i \\
&\quad + K_2\left(|r_{n+1}-r_n| + |\delta_n-\delta_{n+1}| + \left|\varphi_{n+1}-\varphi_n\right| + |\lambda_n-\lambda_{n+1}| + |\alpha_{n+1}-\alpha_n| + |\tau_n-\tau_{n+1}|\right),
\end{aligned} \tag{3.42}$$

where K_2 is an appropriate constant such that $K_2 = \max\{\sup_{n\ge1}\|Bk_n\|, K_1\}$.

Define

$$x_{n+1} = (1-\beta_n)z_n + \beta_n x_n, \quad n \ge 1. \tag{3.43}$$

Observe that from the definition z_n, we obtain

$$\begin{aligned}
z_{n+1}-z_n &= \frac{\epsilon_{n+1}\gamma f(W_{n+1}x_{n+1}) + ((1-\beta_{n+1})I-\epsilon_{n+1}A)W_{n+1}\theta_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\epsilon_n\gamma f(W_nx_n) + ((1-\beta_n)I-\epsilon_nA)W_n\theta_n}{1-\beta_n} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\gamma f(W_{n+1}x_{n+1}) - \frac{\epsilon_n}{1-\beta_n}\gamma f(W_nx_n) + W_{n+1}\theta_{n+1} - W_n\theta_n \\
&\quad + \frac{\epsilon_n}{1-\beta_n}AW_n\theta_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}AW_{n+1}\theta_{n+1} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\gamma f(W_{n+1}x_{n+1}) - AW_{n+1}\theta_{n+1}\right) + \frac{\epsilon_n}{1-\beta_n}\left(AW_n\theta_n - \gamma f(W_nx_n)\right) \\
&\quad + W_{n+1}\theta_{n+1} - W_{n+1}\theta_n + W_{n+1}\theta_n - W_n\theta_n.
\end{aligned} \tag{3.44}$$

It follows from (3.32), (3.42), and (3.44) that

$$\begin{aligned}
&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(W_{n+1}x_{n+1})\right\|+\|AW_{n+1}\theta_{n+1}\|\right)+\frac{\epsilon_n}{1-\beta_n}\left(\|AW_n\theta_n\|+\left\|\gamma f(W_nx_n)\right\|\right) \\
&\qquad+\|W_{n+1}\theta_{n+1}-W_{n+1}\theta_n\|+\|W_{n+1}\theta_n-W_n\theta_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(W_{n+1}x_{n+1})\right\|+\|AW_{n+1}\theta_{n+1}\|\right)+\frac{\epsilon_n}{1-\beta_n}\left(\|AW_n\theta_n\|+\left\|\gamma f(W_nx_n)\right\|\right) \\
&\qquad+\|\theta_{n+1}-\theta_n\|+\|W_{n+1}\theta_n-W_n\theta_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(W_{n+1}x_{n+1})\right\|+\|AW_{n+1}\theta_{n+1}\|\right)+\frac{\epsilon_n}{1-\beta_n}\left(\|AW_n\theta_n\|+\left\|\gamma f(W_nx_n)\right\|\right) \\
&\qquad+M_2\prod_{i=1}^{n}\mu_i+M_3\prod_{i=1}^{n}\mu_i+M_4\prod_{i=1}^{n}\mu_i \\
&\qquad+K_2\left(|r_{n+1}-r_n|+|\delta_n-\delta_{n+1}|+\left|\varphi_{n+1}-\varphi_n\right|+|\lambda_n-\lambda_{n+1}|+|\alpha_{n+1}-\alpha_n|+|\tau_n-\tau_{n+1}|\right) \\
&\quad\le \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(W_{n+1}x_{n+1})\right\|+\|AW_{n+1}\theta_{n+1}\|\right)+\frac{\epsilon_n}{1-\beta_n}\left(\|AW_n\theta_n\|+\left\|\gamma f(W_nx_n)\right\|\right) \\
&\qquad+3K\prod_{i=1}^{n}\mu_i \\
&\qquad+K_2\left(|r_{n+1}-r_n|+|\delta_n-\delta_{n+1}|+\left|\varphi_{n+1}-\varphi_n\right|+|\lambda_n-\lambda_{n+1}|+|\alpha_{n+1}-\alpha_n|+|\tau_n-\tau_{n+1}|\right),
\end{aligned} \tag{3.45}$$

where K is an appropriate constant such that $K=\max\{M_2,M_3,M_4\}$.

It follows from conditions (C1), (C2), (C3), (C4), (C5), and $0<\mu_i\le b<1$, for all $i\ge 1$

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.46}$$

Hence, by Lemma 2.11, we obtain

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.47}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{3.48}$$

Applying (3.48) and conditions in Theorem 3.3 to (3.26), (3.41), and (3.42), we obtain that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = \lim_{n\to\infty}\|k_{n+1}-k_n\| = \lim_{n\to\infty}\|\theta_{n+1}-\theta_n\| = 0. \tag{3.49}$$

From (3.49), (C2), (C5), and $0 < \mu_i \le b < 1$, for all $i \ge 1$, we also have

$$\lim_{n\to\infty}\left\|y_{n+1}-y_n\right\| = 0. \tag{3.50}$$

Since $x_{n+1} = \epsilon_n\gamma f(W_nx_n) + \beta_nx_n + ((1-\beta_n)I - \epsilon_nA)W_n\theta_n$, we have

$$\begin{aligned}\|x_n - W_n\theta_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n\theta_n\| \\ &= \|x_n - x_{n+1}\| + \left\|\epsilon_n\gamma f(W_nx_n) + \beta_nx_n + \left((1-\beta_n)I - \epsilon_nA\right)W_n\theta_n - W_n\theta_n\right\| \\ &= \|x_n - x_{n+1}\| + \left\|\epsilon_n\left(\gamma f(W_nx_n) - AW_n\theta_n\right) + \beta_n(x_n - W_n\theta_n)\right\| \\ &\le \|x_n - x_{n+1}\| + \epsilon_n\left(\left\|\gamma f(W_nx_n)\right\| + \|AW_n\theta_n\|\right) + \beta_n\|x_n - W_n\theta_n\|,\end{aligned} \tag{3.51}$$

that is,

$$\|x_n - W_n\theta_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}\left(\left\|\gamma f(W_nx_n)\right\| + \|AW_n\theta_n\|\right). \tag{3.52}$$

By (C1), (C3), and (3.48) it follows that

$$\lim_{n\to\infty}\|W_n\theta_n - x_n\| = 0. \tag{3.53}$$

Step 4. We claim that the following statements hold:

(i) $\lim_{n\to\infty}\|u_n - \theta_n\| = 0$;

(ii) $\lim_{n\to\infty}\|x_n - u_n\| = 0$;

(iii) $\lim_{n\to\infty}\|W_n\theta_n - \theta_n\| = 0$.

Since B is relaxed (u,v)-cocoercive and ξ-Lipschitz continuous mappings, by the assumptions imposed on $\{\tau_n\}$ for any $p \in \Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F,D) \cap \mathrm{VI}(E,B)$, we have

$$
\begin{aligned}
\|W_n\theta_n - p\|^2 &\le \|P_E(k_n - \tau_n Bk_n) - P_E(p - \tau_n Bp)\|^2 \\
&\le \|(k_n - \tau_n Bk_n) - (p - \tau_n Bp)\|^2 \\
&= \|(k_n - p) - \tau_n(Bk_n - Bp)\|^2 \\
&\le \|k_n - p\|^2 - 2\tau_n\langle k_n - p, Bk_n - Bp\rangle + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - 2\tau_n\langle k_n - p, Bk_n - Bp\rangle + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - 2\tau_n\left\{-u\|Bk_n - Bp\|^2 + v\|k_n - p\|^2\right\} + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 + 2\tau_n u\|Bk_n - Bp\|^2 - 2\tau_n v\|k_n - p\|^2 + \tau_n^2\|Bk_n - Bp\|^2 \\
&\le \|x_n - p\|^2 + 2\tau_n u\|Bk_n - Bp\|^2 - \frac{2\tau_n v}{\xi^2}\|Bk_n - Bp\|^2 + \tau_n^2\|Bk_n - Bp\|^2 \\
&= \|x_n - p\|^2 + \left(2\tau_n u + \tau_n^2 - \frac{2\tau_n v}{\xi^2}\right)\|Bk_n - Bp\|^2.
\end{aligned}
\tag{3.54}
$$

Similarly, we have

$$
\begin{aligned}
\|W_n\phi_n - p\|^2 &\le \|x_n - p\|^2 + \left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2, \\
\|W_n\psi_n - p\|^2 &\le \|x_n - p\|^2 + \left(2\delta_n u + \delta_n^2 - \frac{2\delta_n v}{\xi^2}\right)\|Bu_n - Bp\|^2.
\end{aligned}
\tag{3.55}
$$

Observe that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p) + \beta_n(x_n - p) + \epsilon_n(\gamma f(W_n x_n) - Ap)\|^2 \\
&= \|((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p) + \beta_n(x_n - p)\|^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Ap\|^2 \\
&\quad + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(W_n x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(W_n x_n) - Ap\rangle \\
&\le ((1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n - p\| + \beta_n\|x_n - p\|)^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Ap\|^2 \\
&\quad + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(W_n x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(W_n x_n) - Ap\rangle \\
&\le ((1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n - p\| + \beta_n\|x_n - p\|)^2 + c_n \\
&\le (1-\beta_n-\epsilon_n\overline{\gamma})^2\|W_n\theta_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + 2(1-\beta_n-\epsilon_n\overline{\gamma})\beta_n\|W_n\theta_n - p\|\|x_n - p\| + c_n \\
&\le (1-\beta_n-\epsilon_n\overline{\gamma})^2\|W_n\theta_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\epsilon_n\overline{\gamma})\beta_n\left(\|W_n\theta_n - p\|^2 + \|x_n - p\|^2\right) + c_n
\end{aligned}
$$

$$\begin{aligned}
&= \left[(1-\epsilon_n\overline{\gamma})^2 - 2(1-\epsilon_n\overline{\gamma})\beta_n + \beta_n^2\right]\left\|W_n\theta_n - p\right\|^2 + \beta_n^2\left\|x_n - p\right\|^2 \\
&\quad + \left((1-\epsilon_n\overline{\gamma})\beta_n - \beta_n^2\right)\left(\left\|W_n\theta_n - p\right\|^2 + \left\|x_n - p\right\|^2\right) + c_n \\
&= \left[(1-\epsilon_n\overline{\gamma})^2 - (1-\epsilon_n\overline{\gamma})\beta_n\right]\left\|W_n\theta_n - p\right\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n - p\right\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\|W_n\theta_n - p\right\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n - p\right\|^2 + c_n,
\end{aligned} \tag{3.56}$$

where

$$\begin{aligned}
c_n &= \epsilon_n^2\left\|\gamma f(x_n) - Ap\right\|^2 + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(W_n\theta_n - p), \gamma f(x_n) - Ap\rangle.
\end{aligned} \tag{3.57}$$

It follows from condition (C1) that

$$\lim_{n\to\infty} c_n = 0. \tag{3.58}$$

Substituting (3.54) into (3.56), and using condition (C6), we have

$$\begin{aligned}
\left\|x_{n+1} - p\right\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\left\|x_n - p\right\|^2 + \left(2\tau_n u + \tau_n^2 - \frac{2\tau_n v}{\xi^2}\right)\left\|Bk_n - Bp\right\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\left\|x_n - p\right\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\left\|x_n - p\right\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma}) \\
&\quad \times\left(2\tau_n u + \tau_n^2 - \frac{2\tau_n v}{\xi^2}\right)\left\|Bk_n - Bp\right\|^2 + c_n \\
&\le \left\|x_n - p\right\|^2 + \left(2\tau_n u + \tau_n^2 - \frac{2\tau_n v}{\xi^2}\right)\left\|Bk_n - Bp\right\|^2 + c_n.
\end{aligned} \tag{3.59}$$

It follows that

$$\begin{aligned}
\left(\frac{2av}{\xi^2} - 2bu - b^2\right)\left\|Bk_n - Bp\right\|^2 &\le \left(\frac{2\tau_n v}{\xi^2} - 2\tau_n u - \tau_n^2\right)\left\|Bk_n - Bp\right\|^2 \\
&\le \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 + c_n \\
&= \left(\left\|x_n - p\right\| - \left\|x_{n+1} - p\right\|\right)\left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right) + c_n \\
&\le \left\|x_n - x_{n+1}\right\|\left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right) + c_n.
\end{aligned} \tag{3.60}$$

Since $c_n \to 0$ as $n \to \infty$ and (3.48), we obtain

$$\lim_{n\to\infty} \|Bk_n - Bp\| = 0. \tag{3.61}$$

Note that

$$\begin{aligned}
\|k_n - p\|^2 &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|W_n\phi_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{ \|x_n - p\|^2 + \left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2 \right\} \\
&\le \|x_n - p\|^2 + (1-\alpha_n)\left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2,
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
\|y_n - p\|^2 &\le \varphi_n \|u_n - p\| + (1-\varphi_n)\|W_n\psi_n - p\|^2 \\
&\le \varphi_n \|x_n - p\|^2 + (1-\varphi_n)\left\{ \|x_n - p\|^2 + \left(2\delta_n u + \delta_n^2 - \frac{2\delta_n v}{\xi^2}\right)\|Bu_n - Bp\|^2 \right\} \\
&\le \|x_n - p\|^2 + (1-\varphi_n)\left(2\delta_n u + \delta_n^2 - \frac{2\delta_n v}{\xi^2}\right)\|Bu_n - Bp\|^2.
\end{aligned} \tag{3.63}$$

Using (3.56) again, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n.
\end{aligned} \tag{3.64}$$

Substituting (3.62) into (3.64) and using condition (C2) and (C6), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n - p\|^2 + (1-\alpha_n)\left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})^2\|x_n - p\|^2 + c_n \\
&\le \|x_n - p\|^2 + (1-\alpha_n)\left(2\lambda_n u + \lambda_n^2 - \frac{2\lambda_n v}{\xi^2}\right)\|By_n - Bp\|^2 + c_n.
\end{aligned} \tag{3.65}$$

It follows that

$$
\begin{aligned}
(1-\alpha_n)\left(\frac{2av}{\xi^2}-2bu-b^2\right)\|By_n-Bp\|^2 &\le (1-\alpha_n)\left(\frac{2\tau_n v}{\xi^2}-2\tau_n u-\tau_n^2\right)\|By_n-Bp\|^2 \\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n \\
&\le \|x_n-x_{n+1}\|\left(\|x_n-p\|+\|x_{n+1}-p\|\right)+c_n.
\end{aligned}
\tag{3.66}
$$

Since $c_n \to 0$ as $n \to \infty$ and (3.48), we obtain

$$
\lim_{n\to\infty}\|By_n-Bp\|=0. \tag{3.67}
$$

In a similar way, we can prove

$$
\lim_{n\to\infty}\|Bu_n-Bp\|=0. \tag{3.68}
$$

By (2.3), we also have

$$
\begin{aligned}
\|\theta_n-p\|^2 &= \|P_E(k_n-\tau_n Bk_n)-P_E(p-\tau_n Bp)\|^2 \\
&= \|P_E(I-\tau_n B)k_n-P_E(I-\tau_n B)p\|^2 \\
&\le \langle (I-\tau_n B)k_n-(I-\tau_n B)p,\theta_n-p\rangle \\
&= \frac{1}{2}\Big\{\|(I-\tau_n B)k_n-(I-\tau_n B)p\|^2+\|\theta_n-p\|^2 \\
&\qquad -\|(I-\tau_n B)k_n-(I-\tau_n B)p-(\theta_n-p)\|^2\Big\} \\
&\le \frac{1}{2}\|k_n-p\|^2+\|\theta_n-p\|^2-\|(k_n-\theta_n)-\tau_n(Bk_n-Bp)\|^2 \\
&\le \frac{1}{2}\Big\{\|x_n-p\|^2+\|\theta_n-p\|^2-\|k_n-\theta_n\|^2-\tau_n^2\|Bk_n-Bp\|^2+2\tau_n\langle k_n-\theta_n,Bk_n-Bp\rangle\Big\},
\end{aligned}
\tag{3.69}
$$

which yields that

$$
\|\theta_n-p\|^2 \le \|x_n-p\|^2-\|k_n-\theta_n\|^2+2\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|. \tag{3.70}
$$

Substituting (3.70) into (3.56), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n-p\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-p\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n-p\|^2-\|k_n-\theta_n\|^2+2\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|\right\}\\
&\quad+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&=(1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-\theta_n\|^2\\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|+c_n\\
&\le \|x_n-p\|^2-(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-\theta_n\|^2\\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|+c_n.
\end{aligned}\tag{3.71}$$

It follows that

$$\begin{aligned}
(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-\theta_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|+c_n\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\quad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n-\theta_n\|\|Bk_n-Bp\|+c_n.
\end{aligned}\tag{3.72}$$

Applying $\|x_{n+1}-x_n\|\to 0$, $\|Bk_n-Bp\|\to 0$ and $c_n\to 0$ as $n\to\infty$ to the last inequality, we have

$$\lim_{n\to\infty}\|k_n-\theta_n\|=0.\tag{3.73}$$

On the other hand, we have

$$\begin{aligned}
\|W_n\theta_n-p\|^2 &\le \|P_E(k_n-\tau_nBk_n)-P_E(p-\tau_nBp)\|^2\\
&=\|P_E(I-\tau_nB)k_n-P_E(I-\tau_nB)p\|^2\\
&\le \langle (I-\tau_nB)k_n-(I-\tau_nB)p, W_n\theta_n-p\rangle\\
&=\frac{1}{2}\Big\{\|(I-\tau_nB)k_n-(I-\tau_nB)p\|^2+\|W_n\theta_n-p\|^2\\
&\qquad-\|(I-\tau_nB)k_n-(I-\tau_nB)p-(W_n\theta_n-p)\|^2\Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\|k_n - p\|^2 + \|W_n\theta_n - p\|^2 - \|(k_n - W_n\theta_n) - \tau_n(Bk_n - Bp)\|^2 \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|W_n\theta_n - p\|^2 - \|k_n - W_n\theta_n\|^2 \\
&\qquad -\tau_n^2\|Bk_n - Bp\|^2 + 2\tau_n\langle k_n - W_n\theta_n, Bk_n - Bp\rangle\Big\},
\end{aligned} \tag{3.74}$$

which yields that

$$\|W_n\theta_n - p\|^2 \leq \|x_n - p\|^2 - \|k_n - W_n\theta_n\|^2 + 2\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\|. \tag{3.75}$$

Similarly, we can prove

$$\|W_n\phi_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - W_n\phi_n\|^2 + 2\lambda_n\|y_n - W_n\phi_n\|\|By_n - Bp\|, \tag{3.76}$$

$$\|W_n\psi_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - W_n\psi_n\|^2 + 2\delta_n\|u_n - W_n\psi_n\|\|Bu_n - Bp\|. \tag{3.77}$$

Substituting (3.75) into (3.56), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma}) \\
&\quad \times \Big\{\|x_n - p\|^2 - \|k_n - W_n\theta_n\|^2 + 2\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\|\Big\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n - p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n - W_n\theta_n\|^2 \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\| + c_n \\
&\leq \|x_n - p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n - W_n\theta_n\|^2 \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\| + c_n,
\end{aligned} \tag{3.78}$$

which yields that

$$\begin{aligned}
&(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n - W_n\theta_n\|^2 \\
&\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\| + c_n \\
&\quad \leq \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\qquad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\tau_n\|k_n - W_n\theta_n\|\|Bk_n - Bp\| + c_n.
\end{aligned} \tag{3.79}$$

Applying (3.48) and (3.61) to the last inequality, we have

$$\lim_{n\to\infty}\|k_n - W_n\theta_n\| = 0. \tag{3.80}$$

Using (3.64) again, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|\alpha_n(x_n-p)+(1-\alpha_n)(W_n\phi_n-p)\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|W_n\phi_n-p\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|x_n-p\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|W_n\phi_n-p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|x_n-p\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n) \\
&\quad \times\left\{\|x_n-p\|^2 - \|y_n-W_n\phi_n\|^2 + 2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\|\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|x_n-p\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|x_n-p\|^2 \\
&\quad - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\| \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-p\|^2 \\
&\quad - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\| \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\| + c_n \\
&\le \|x_n-p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\|y_n-W_n\phi_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)2\lambda_n\|y_n-W_n\phi_n\|\|By_n-Bp\| + c_n,
\end{aligned} \tag{3.81}$$

which implies that

$$\begin{aligned}
&(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\left\|y_n-W_n\phi_n\right\|^2 \\
&\quad\le \left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2 \\
&\qquad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\lambda_n\left\|y_n-W_n\phi_n\right\|\left\|By_n-Bp\right\|+c_n \\
&\quad\le \|x_n-x_{n+1}\|\left(\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right) \\
&\qquad+2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)\lambda_n\left\|y_n-W_n\phi_n\right\|\left\|By_n-Bp\right\|+c_n.
\end{aligned} \tag{3.82}$$

From (3.48) and (3.67), we obtain

$$\lim_{n\to\infty}\left\|y_n-W_n\phi_n\right\|=0. \tag{3.83}$$

By using the same argument, we can prove that

$$\lim_{n\to\infty}\left\|u_n-W_n\varphi_n\right\|=0. \tag{3.84}$$

Note that

$$\begin{aligned}
k_n-W_n\phi_n&=\alpha_n(x_n-W_n\phi_n),\\
y_n-W_n\varphi_n&=\varphi_n(u_n-W_n\varphi_n).
\end{aligned} \tag{3.85}$$

Since $\alpha_n \to 0$ and $\varphi_n \to 0$ as $n \to \infty$, respectively, we also have

$$\lim_{n\to\infty}\left\|k_n-W_n\phi_n\right\|=\lim_{n\to\infty}\left\|y_n-W_n\varphi_n\right\|=0. \tag{3.86}$$

On the other hand, we observe

$$\|u_n-\theta_n\|\le\left\|u_n-W_n\varphi_n\right\|+\left\|W_n\varphi_n-y_n\right\|+\left\|y_n-W_n\phi_n\right\|+\left\|W_n\phi_n-k_n\right\|+\|k_n-\theta_n\|. \tag{3.87}$$

Applying (3.73), (3.83), (3.84), and (3.86), we have

$$\lim_{n\to\infty}\|u_n-\theta_n\|=0. \tag{3.88}$$

On the other hand, we have

$$
\begin{aligned}
\|k_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|W_n\phi_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|\phi_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\Big\{\varphi_n\|u_n - p\|^2 + (1-\varphi_n)\|W_n\psi_n - p\|\Big\} \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\Big\{\varphi_n\|u_n - p\|^2 + (1-\varphi_n)\|\psi_n - p\|\Big\} \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\Big\{\varphi_n\|u_n - p\|^2 + (1-\varphi_n)\|u_n - p\|\Big\} \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|T_{r_n}(I - r_nD)x_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|(I - r_nD)x_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 - r_n(2\eta - r_n)\|Dx_n - Dp\|^2\Big\} \\
&= \|x_n - p\|^2 - (1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\|^2.
\end{aligned}
\tag{3.89}
$$

Substituting (3.89) into (3.64) and using conditions (C2) and (C7), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n - p\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\Big\{\|x_n - p\|^2 - (1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\|^2\Big\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})(1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})^2\|x_n - p\|^2 + c_n \\
&\le \|x_n - p\|^2 - (1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\|.
\end{aligned}
\tag{3.90}
$$

This implies that

$$
(1-\alpha_n)r_n(2\eta - r_n)\|Dx_n - Dp\| \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + c_n. \tag{3.91}
$$

In view of the restrictions (C2) and (C7), we obtain that

$$
\lim_{n\to\infty}\|Dx_n - Dp\| = 0. \tag{3.92}
$$

Let $p \in \Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F, D) \cap \mathrm{VI}(E, B)$. Since $u_n = T_{r_n}(x_n - r_n D x_n)$ and T_{r_n} is firmly nonexpansive (Lemma 2.6), then we obtain

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n D x_n) - T_{r_n}(p - r_n D p)\|^2 \\
&\leq \langle T_{r_n}(x_n - r_n D x_n) - T_{r_n}(p - r_n D p), u_n - p \rangle \\
&= \langle x_n - r_n D x_n - (p - r_n D p), u_n - p \rangle \\
&= \frac{1}{2}\Big\{ \|(x_n - r_n D x_n) - (p - r_n D p)\|^2 + \|u_n - p\|^2 \\
&\qquad - \|(x_n - r_n D x_n) - (p - r_n D p) - (u_n - p)\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(D x_n - D p)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, D x_n - D p\rangle \\
&\qquad - r_n^2\|D x_n - D p\|^2 \Big\}.
\end{aligned} \tag{3.93}$$

So, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|D x_n - D p\|. \tag{3.94}$$

Therefore, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|W_n\theta_n - p\|^2 + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|\theta_n - p\|^2 + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|(\theta_n - u_n) + (u_n - p)\|^2 + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\Big\{\|\theta_n - u_n\|^2 + \|u_n - p\|^2 + 2\langle\theta_n - u_n, u_n - p\rangle\Big\} \\
&\quad + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|\theta_n - u_n\|^2 + (1 - \epsilon_n\overline{\gamma})(1 - \epsilon_n\overline{\gamma} - \beta_n)\|u_n - p\|^2 \\
&\quad + 2(1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|\theta_n - u_n\|\|u_n - p\| + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\leq (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|\theta_n - u_n\|^2 \\
&\quad + (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\Big\{\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|D x_n - D p\|\Big\} \\
&\quad + 2(1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|\theta_n - u_n\|\|u_n - p\| + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + c_n
\end{aligned}$$

$$
\begin{aligned}
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-p\|^2 \\
&\quad - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2 + c_n \\
&= (1-\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n \\
&= \left(1-2\epsilon_n\overline{\gamma} + (\epsilon_n\overline{\gamma})^2\right)\|x_n-p\|^2 - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n \\
&\le \|x_n-p\|^2 + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 \\
&\quad - (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\quad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n.
\end{aligned}
\tag{3.95}
$$

It follows that

$$
\begin{aligned}
&(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-u_n\|^2 \\
&\quad \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 \\
&\qquad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\qquad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n \\
&\quad \le \|x_n-x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + (\epsilon_n\overline{\gamma})^2\|x_n-p\|^2 \\
&\qquad + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2r_n\|x_n-u_n\|\|Dx_n-Dp\| \\
&\qquad + 2(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-u_n\|\|u_n-p\| + c_n.
\end{aligned}
\tag{3.96}
$$

Using $\epsilon_n \to 0$, $c_n \to 0$ as $n \to \infty$, (3.48), (3.88), and (3.92), we obtain

$$
\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.97}
$$

Since $\liminf_{n\to\infty} r_n > 0$, we obtain

$$\lim_{n\to\infty}\left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.98}$$

Note that

$$\|x_n - \theta_n\| \le \|x_n - u_n\| + \|u_n - \theta_n\|, \tag{3.99}$$

and thus from (3.88) and (3.97), we have

$$\lim_{n\to\infty}\|x_n - \theta_n\| = 0. \tag{3.100}$$

Observe that

$$\|W_n\theta_n - \theta_n\| \le \|W_n\theta_n - x_n\| + \|x_n - \theta_n\|. \tag{3.101}$$

Applying (3.53) and (3.100), we obtain

$$\lim_{n\to\infty}\|W_n\theta_n - \theta_n\| = 0. \tag{3.102}$$

Let W be the mapping defined by (2.11). Since $\{\theta_n\}$ is bounded, applying Lemma 2.10 and (3.102), we have

$$\|W\theta_n - \theta_n\| \le \|W\theta_n - W_n\theta_n\| + \|W_n\theta_n - \theta_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.103}$$

Step 5. We claim that $\limsup_{n\to\infty}\langle (A-\gamma f)z, z - x_n\rangle \le 0$, where z is the unique solution of the variational inequality $\langle (A-\gamma f)z, x - z\rangle \ge 0$, for all $x \in \Theta$.

Since $z = P_\Theta(I - A + \gamma f)(z)$ is a unique solution of the variational inequality (3.5), to show this inequality, we choose a subsequence $\{\theta_{n_i}\}$ of $\{\theta_n\}$ such that

$$\lim_{i\to\infty}\langle (A-\gamma f)z, z - \theta_{n_i}\rangle = \limsup_{n\to\infty}\langle (A-\gamma f)z, z - \theta_n\rangle. \tag{3.104}$$

Since $\{\theta_{n_i}\}$ is bounded, there exists a subsequence $\{\theta_{n_{i_j}}\}$ of $\{\theta_{n_i}\}$ which converges weakly to $w \in E$. Without loss of generality, we can assume that $\theta_{n_i} \rightharpoonup w$. From $\|W\theta_n - \theta_n\| \to 0$, we obtain $W\theta_{n_i} \rightharpoonup w$. Next, We show that $w \in \Theta$, where $\Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F, D) \cap \mathrm{VI}(E, B)$.

(a) First, we prove $w \in \mathrm{EP}(F, D)$.

Since $u_n = T_{r_n}(x_n - r_n D x_n)$, we know that

$$F(u_n, y) + \langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in E. \tag{3.105}$$

From (A2), we also have

$$\langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq -F(u_n, y) \geq F(y, u_n). \tag{3.106}$$

Replacing n by n_i, we have

$$\langle Dx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}). \tag{3.107}$$

For any t with $0 < t \leq 1$ and $y \in E$, let $\varphi_t = ty + (1-t)z$. Since $y \in E$ and $z \in E$, we have $\varphi_t \in E$. So, from (3.107) we have

$$\begin{aligned}
\langle \varphi_t - u_{n_i}, D\varphi_t\rangle &\geq \langle \varphi_t - u_{n_i}, D\varphi_t\rangle - \langle Dx_{n_i}, \varphi_t - u_{n_i}\rangle - \left\langle \varphi_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(\varphi_t, u_{n_i}) \\
&\geq \langle \varphi_t - u_{n_i}, D\varphi_t - Du_{n_i}\rangle + \langle \varphi_t - u_{n_i}, Du_{n_i} - Dx_{n_i}\rangle \\
&\quad - \langle \varphi_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(\varphi_t, u_{n_i}).
\end{aligned} \tag{3.108}$$

Since D is Lipschitz continuous, from (3.97), we have $\|Du_{n_i} - Dx_{n_i}\| \to 0$ as $i \to \infty$.

Further, from the monotonicity of D, we get that

$$\langle \varphi_t - u_{n_i}, D\varphi_t - Du_{n_i}\rangle \geq 0. \tag{3.109}$$

It follows from (A4) and (3.108) that

$$\langle \varphi_t - z, D\varphi_t\rangle \geq F(\varphi_t, z). \tag{3.110}$$

From (A1), (A4), and (3.110), we also have

$$\begin{aligned}
0 = F(\varphi_t, \varphi_t) &\leq tF(\varphi_t, y) + (1-t)F(\varphi_t, z) \\
&\leq tF(\varphi_t, y) + (1-t)\langle \varphi_t - z, D\varphi_t\rangle \\
&= tF(\varphi_t, y) + (1-t)t\langle y - z, D\varphi_t\rangle,
\end{aligned} \tag{3.111}$$

and hence

$$F(\varphi_t, y) + (1-t)\langle y - z, D\varphi_t\rangle \geq 0. \tag{3.112}$$

Letting $t \to \infty$ in the above inequality, we have, for each $y \in E$,

$$F(z,y) + \langle y - z, Dz \rangle \geq 0. \tag{3.113}$$

Thus $z \in \mathrm{EP}(F, D)$.

(b) Next, we show that $w \in \bigcap_{n=1}^{\infty} F(T_n)$.

By Lemma 2.9, we have $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$. Assume $w \notin F(W)$. Since $\|x_n - \theta_n\| \to 0$, we know that $\theta_{n_i} \rightharpoonup w$ $(i \to \infty)$ and $w \neq Ww$, and it follows by the Opial's condition (Lemma 2.3) that

$$\begin{aligned} \liminf_{i \to \infty} \|\theta_{n_i} - w\| &< \liminf_{i \to \infty} \|\theta_{n_i} - Ww\| \\ &\leq \liminf_{i \to \infty} (\|\theta_{n_i} - W\theta_{n_i}\| + \|W\theta_{n_i} - Ww\|) \\ &< \liminf_{i \to \infty} \|\theta_{n_i} - w\|, \end{aligned} \tag{3.114}$$

that is a contradiction. Thus, we have $w \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.

(c) Finally, Now we prove that $w \in \mathrm{VI}(E, B)$. Define,

$$Tw_1 = \begin{cases} Bw_1 + N_E w_1, & \text{if } w_1 \in E, \\ \emptyset, & \text{if } w_1 \notin E. \end{cases} \tag{3.115}$$

Since B is relaxed (u, v)-cocoercive and condition (C6), we have

$$\langle Bx - By, x - y \rangle \geq (-u) \|Bx - By\|^2 + v\|x - y\|^2 \geq \left(v - u\xi^2 \right) \|x - y\|^2 \geq 0, \tag{3.116}$$

which yields that B is monotone. Then, T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Bw_1 \in N_E w_1$ and $\theta_n \in E$, we have $\langle w_1 - \theta_n, w_2 - Bw_1 \rangle \geq 0$. On the other hand, from $\theta_n = P_E(k_n - \tau_n Bk_n)$, we have

$$\langle w_1 - \theta_n, \theta_n - (k_n - \tau_n Bk_n) \rangle \geq 0, \tag{3.117}$$

and hence

$$\left\langle w_1 - \theta_n, \frac{(\theta_n - k_n)}{\tau_n} + Bk_n \right\rangle \geq 0. \tag{3.118}$$

Therefore, we have

$$\begin{aligned}
\langle w_1 - \theta_{n_i}, w\rangle &\geq \langle w_1 - \theta_{n_i}, Bw_1\rangle \\
&\geq \langle w_1 - \theta_{n_i}, Bw_1\rangle - \left\langle w_1 - \theta_{n_i}, \frac{(\theta_{n_i} - k_{n_i})}{\tau_{n_i}} + Bk_{n_i}\right\rangle \\
&= \left\langle w_1 - \theta_{n_i}, Bw_1 - Bk_{n_i} - \frac{(\theta_{n_i} - k_{n_i})}{\tau_{n_i}}\right\rangle \\
&= \langle w_1 - \theta_{n_i}, Bv - B\theta_{n_i}\rangle + \langle w_1 - \theta_{n_i}, B\theta_{n_i} - Bk_{n_i}\rangle - \left\langle w_1 - \theta_{n_i}, \frac{(\theta_{n_i} - k_{n_i})}{\tau_{n_i}}\right\rangle \\
&\geq \langle w_1 - \theta_{n_i}, B\theta_{n_i} - Bk_{n_i}\rangle - \left\langle w_1 - \theta_{n_i}, \frac{(\theta_{n_i} - k_{n_i})}{\tau_{n_i}}\right\rangle,
\end{aligned} \tag{3.119}$$

which implies that

$$\langle w_1 - w, w_2\rangle \geq 0. \tag{3.120}$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in \mathrm{VI}(E, B)$. That is, $w \in \Theta$, where $\Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F, D) \cap \mathrm{VI}(E, B)$. Since $z = P_\Theta(I - A + \gamma f)(z)$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\langle (A - \gamma f)z, z - x_n\rangle &= \limsup_{n\to\infty}\langle (A - \gamma f)z, z - \theta_n\rangle \\
&= \lim_{i\to\infty}\langle (A - \gamma f)z, z - \theta_{n_i}\rangle \\
&= \langle (A - \gamma f)z, z - w\rangle \leq 0.
\end{aligned} \tag{3.121}$$

On the other hand, we have

$$\begin{aligned}
\langle (A - \gamma f)z, z - W_n\theta_n\rangle &= \langle (A - \gamma f)z, x_n - W_n\theta_n\rangle + \langle (A - \gamma f)z, z - x_n\rangle \\
&\leq \|(A - \gamma f)z\|\|x_n - W_n\theta_n\| + \langle (A - \gamma f)z, z - x_n\rangle.
\end{aligned} \tag{3.122}$$

From (3.53) and (3.121), we obtain that

$$\limsup_{n\to\infty}\langle \gamma f(z) - Az, W_n\theta_n - z\rangle \leq 0. \tag{3.123}$$

Step 6. Finally, we show that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta(I - A + \gamma f)(z)$.

Indeed, from (3.4) and Lemma 2.4, we obtain

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &= \left\|\epsilon_n\gamma f(W_nx_n)+\beta_nx_n+((1-\beta_n)I-\epsilon_nA)W_n\theta_n-z\right\|^2\\
&= \left\|((1-\beta_n)I-\epsilon_nA)(W_n\theta_n-z)+\beta_n(x_n-z)+\epsilon_n(\gamma f(W_nx_n)-Az)\right\|^2\\
&= \left\|((1-\beta_n)I-\epsilon_nA)(W_n\theta_n-z)+\beta_n(x_n-z)\right\|^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2\\
&\quad+2\beta_n\epsilon_n\langle x_n-z,\gamma f(W_nx_n)-Az\rangle\\
&\quad+2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(W_n\theta_n-z),\gamma f(W_nx_n)-Az\rangle\\
&\le \left((1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n-z\|+\beta_n\|x_n-z\|\right)^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2\\
&\quad+2\beta_n\epsilon_n\gamma\langle x_n-z,f(W_nx_n)-f(z)\rangle+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+2(1-\beta_n)\gamma\epsilon_n\langle W_n\theta_n-z,f(W_nx_n)-f(z)\rangle\\
&\quad+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle,\\
&\le \left((1-\beta_n-\epsilon_n\overline{\gamma})\|W_n\theta_n-z\|+\beta_n\|x_n-z\|\right)^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2\\
&\quad+2\beta_n\epsilon_n\gamma\|x_n-z\|\left\|f(W_nx_n)-f(z)\right\|+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+2(1-\beta_n)\gamma\epsilon_n\|W_n\theta_n-z\|\left\|f(W_nx_n)-f(z)\right\|\\
&\quad+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle,\\
&\le \left((1-\beta_n-\epsilon_n\overline{\gamma})\|\theta_n-z\|+\beta_n\|x_n-z\|\right)^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2\\
&\quad+2\beta_n\epsilon_n\gamma\|x_n-z\|\left\|f(W_nx_n)-f(z)\right\|+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+2(1-\beta_n)\gamma\epsilon_n\|\theta_n-z\|\left\|f(W_nx_n)-f(z)\right\|+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&\quad-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle\\
&\le \left((1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|\right)^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2\\
&\quad+2\beta_n\epsilon_n\gamma\alpha\|x_n-z\|^2+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n-z\|^2\\
&\quad+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle\\
&= \left[(1-\epsilon_n\overline{\gamma})^2+2\beta_n\epsilon_n\gamma\alpha+2(1-\beta_n)\gamma\epsilon_n\alpha\right]\|x_n-z\|^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2\\
&\quad+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&\quad-2\epsilon_n^2\langle A(W_n\theta_n-z),\gamma f(z)-Az\rangle\\
&\le \left[1-2(\overline{\gamma}-\alpha\gamma)\epsilon_n\right]\|x_n-z\|^2+\overline{\gamma}^2\epsilon_n^2\|x_n-z\|^2+\epsilon_n^2\left\|\gamma f(W_nx_n)-Az\right\|^2\\
&\quad+2\beta_n\epsilon_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\epsilon_n\langle W_n\theta_n-z,\gamma f(z)-Az\rangle\\
&\quad+2\epsilon_n^2\|A(W_n\theta_n-z)\|\left\|\gamma f(z)-Az\right\|
\end{aligned}
$$

$$= \left[1 - 2(\overline{\gamma} - \alpha\gamma)\epsilon_n\right]\|x_n - z\|^2$$
$$+ \epsilon_n\left\{\epsilon_n\left(\overline{\gamma}^2\|x_n - z\|^2 + \left\|\gamma f(W_n x_n) - Az\right\|^2 + 2\|A(W_n\theta_n - z)\|\left\|\gamma f(z) - Az\right\|\right)\right.$$
$$\left. + 2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle W_n\theta_n - z, \gamma f(z) - Az\rangle\right\}. \quad (3.124)$$

Since $\{x_n\}$, $\{f(W_n x_n)\}$, and $\{W_n\theta_n\}$ are bounded, we can take a constant $M > 0$ such that

$$\overline{\gamma}^2\|x_n - z\|^2 + \left\|\gamma f(W_n x_n) - Az\right\|^2 + 2\|A(W_n\theta_n - z)\|\left\|\gamma f(z) - Az\right\| \leq M \quad (3.125)$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - z\|^2 \leq (1 - l_n)\|x_n - z\|^2 + \epsilon_n\sigma_n, \quad (3.126)$$

where

$$l_n = 2(\overline{\gamma} - \alpha\gamma)\epsilon_n,$$
$$\sigma_n = \epsilon_n M + 2\beta_n\langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta_n)\langle W_n\theta_n - z, \gamma f(z) - Az\rangle. \quad (3.127)$$

Using (C1), (3.121), and (3.123), we get $l_n \to 0$, $\sum_{n=1}^{\infty} l_n = \infty$ and $\limsup_{n\to\infty}(\sigma_n/l_n) \leq 0$. Applying Lemma 2.13 to (3.126), we conclude that $x_n \to z$ in norm. Finally, noticing $\|u_n - z\| = \|T_{r_n}(x_n - r_n D x_n) - T_{r_n}(z - r_n D z)\| \leq \|x_n - z\|$, we also conclude that $u_n \to z$ in norm. This completes the proof. □

Corollary 3.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $F : E\times E \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4), let $B : E \to H$ be relaxed (u,v)-cocoercive and ξ-Lipschitz continuous mappings, and let $\{T_n\}$ be an infinite family of nonexpansive mappings of E into itself such that $\Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \cap \mathrm{VI}(E,B) \neq \emptyset$. Let f be a contraction mapping of E into itself with $\alpha \in (0,1)$. Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$, and $\{u_n\}$ be sequences generated by*

$$x_1 \in E \text{ chosen arbitrary},$$
$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E,$$
$$y_n = \varphi_n u_n + (1-\varphi_n)W_n P_E(u_n - \delta_n B u_n),$$
$$k_n = \alpha_n x_n + (1-\alpha_n)W_n P_E(y_n - \lambda_n B y_n),$$
$$x_{n+1} = \epsilon_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n P_E(k_n - \tau_n B k_n), \quad \forall n \geq 1, \quad (3.128)$$

where $\{W_n\}$ is the sequence generated by (1.24) *and $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\varphi_n\}$, and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a real sequence in $(0,\infty)$ satisfying the following conditions:*

(C1) $\epsilon_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\to\infty}\epsilon_n = 0$, $\sum_{n=1}^{\infty}\epsilon_n = \infty$, (C4) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\varphi_n = 0$,

(C5) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C6) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty}|\delta_{n+1} - \delta_n| = \lim_{n\to\infty}|\tau_{n+1} - \tau_n| = 0$,

(C7) $\{\tau_n\}, \{\lambda_n\}, \{\delta_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$, *where* $z = P_\Theta f(z)$.

Proof. Put $A = I, \gamma \equiv 1, \gamma_n = 1-\epsilon_n-\beta_n, D = 0$ (:the zero mapping) and $\{\epsilon_n\} = 0$ in Theorem 3.3. Then $y_n = v_n = u_n$, and for any $\eta > 0$, we see that

$$\langle Dx - Dy, x - y\rangle \ge \eta\|Dx - Dy\|^2, \quad \forall x, y \in E. \tag{3.129}$$

Let $\{r_n\}$ be a sequence satisfying the restriction: $c \le r_n \le d$, where $c, d \in (0,\infty)$. Then we can obtain the desired conclusion easily from Theorem 3.3. □

Corollary 3.5. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let* $\{T_n\}$ *be an infinite family of nonexpansive mappings of E into itself and let* $B : E \to H$ *be relaxed* (u, v)*-cocoercive and* ξ*-Lipschitz continuous mappings such that* $\Theta := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(E, B) \ne \emptyset$. *Let* $f : E \to E$ *be a contraction mapping with* $0 < \alpha < 1$ *and let A be a strongly positive linear bounded operator on H with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$. *Let* $\{x_n\}$,$\{y_n\}$, *and* $\{k_n\}$ *be sequences generated by*

$$\begin{aligned}
&x_1 \in E \text{ chosen arbitrary},\\
&y_n = \varphi_n x_n + (1-\varphi_n)W_n P_E(x_n - \delta_n B x_n),\\
&k_n = \alpha_n x_n + (1-\alpha_n)W_n P_E(y_n - \lambda_n B y_n),\\
&x_{n+1} = \epsilon_n\gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)W_n P_E(k_n - \tau_n B k_n), \quad \forall n \ge 1,
\end{aligned} \tag{3.130}$$

where $\{W_n\}$ *is the sequence generated by* (1.24) *and* $\{\epsilon_n\}$, $\{\alpha_n\}$, $\{\varphi_n\}$, *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\epsilon_n = 0, \sum_{n=1}^{\infty}\epsilon_n = \infty$,

(C2) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\varphi_n = 0$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C4) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty}|\delta_{n+1} - \delta_n| = \lim_{n\to\infty}|\tau_{n+1} - \tau_n| = 0$,

(C5) $\{\tau_n\}, \{\lambda_n\}, \{\delta_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v - u\xi^2)/\xi^2$, $v > u\xi^2$.

Then, $\{x_n\}$ *converges strongly to a point* $z \in \Theta$, *where* $z = P_\Theta(I - A + \gamma f)(z)$, *which solves the variational inequality*

$$\langle (A - \gamma f)z, x - z\rangle \ge 0, \quad \forall x \in \Theta, \tag{3.131}$$

which is the optimality condition fot the minimization problem

$$\min_{x\in\Theta}\left\{\frac{1}{2}\langle Ax, x\rangle - h(x)\right\}, \tag{3.132}$$

where h *is a potential function for* γf *(i.e.,* $h'(x) = \gamma f(x)$ *for* $x \in H$*).*

Proof. Put $D = 0$, $F(x, y) = 0$ for all $x, y \in E$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.3. Then, we have $u_n = P_C x_n = x_n$. So, by Theorem 3.3, we can conclude the desired conclusion easily. □

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