

Research Article

Existence of Solutions for Weighted $p(r)$ -Laplacian Impulsive Integro-Differential System Periodic-Like Boundary Value Problems

Guizhen Zhi,¹ Liang Zhao,² Guangxia Chen,³
Shujuan Wang,¹ and Qihu Zhang^{1,4}

¹ Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, China

² Academic Administration, Henan Institute of Science and Technology, Xinxiang, Henan 453003, China

³ School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo, Henan 454003, China

⁴ School of Mathematics and Statistics, Huazhong Normal University, Wuhan, Hubei 430079, China

Correspondence should be addressed to Qihu Zhang, zhangqh1999@yahoo.com.cn

Received 14 November 2009; Accepted 28 January 2010

Academic Editor: Marta García-Huidobro

Copyright © 2010 Guizhen Zhi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper investigates the existence of solutions for weighted $p(r)$ -Laplacian impulsive integro-differential system periodic-like boundary value problems via Leray-Schauder's degree. The sufficient conditions for the existence of solutions are given.

1. Introduction

In this paper, we consider the existence of solutions for the weighted $p(r)$ -Laplacian integro-differential system

$$-\left(\omega(r)|u'|^{p(r)-2}u'\right)' + f\left(r, u, (\omega(r))^{1/(p(r)-1)}u', S(u)\right) = 0, \quad r \in (0, T), \quad r \neq r_i, \quad (1.1)$$

where $u : [0, 1] \rightarrow \mathbb{R}^N$, with the following impulsive boundary value conditions:

$$\lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) = A_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (\omega(r))^{1/(p(r)-1)} u'(r) \right), \quad i = 1, \dots, k, \quad (1.2)$$

$$\begin{aligned} & \lim_{r \rightarrow r_i^+} (\omega(r))^{1/(p(r)-1)} u'(r) - \lim_{r \rightarrow r_i^-} (\omega(r))^{1/(p(r)-1)} u'(r) \\ &= B_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (\omega(r))^{1/(p(r)-1)} u'(r) \right), \quad i = 1, \dots, k, \end{aligned} \quad (1.3)$$

$$u(0) = u(T), \quad \lim_{r \rightarrow 0^+} \omega(r) |u'|^{p(r)-2} u'(r) = \lim_{r \rightarrow T^-} \omega(r) |u'|^{p(r)-2} u'(r), \quad (1.4)$$

where $p \in C([0, T], \mathbb{R})$ and $p(r) > 1$, $-(\omega(r)|u'|^{p(r)-2}u')'$ is called the weighted $p(r)$ -Laplacian; $0 < r_1 < r_2 < \dots < r_k < T$; $A_i, B_i \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$; $S(u)(t) = \int_0^T \phi(s, t)u(s)ds$, where $0 \leq \phi(\cdot, \cdot) \in C([0, T] \times [0, T], \mathbb{R})$.

Throughout the paper, $o(1)$ means function which uniformly convergent to 0 (as $n \rightarrow +\infty$); for any $v \in \mathbb{R}^N$, v^j will denote the j th component of v ; the inner product in \mathbb{R}^N will be denoted by $\langle \cdot, \cdot \rangle$, $|\cdot|$ will denote the absolute value and the Euclidean norm on \mathbb{R}^N . Denote $J = [0, T]$, $J' = [0, T] \setminus \{r_0, r_1, \dots, r_k, r_{k+1}\}$, $J_0 = [r_0, r_1]$, $J_i = (r_i, r_{i+1})$, $i = 1, \dots, k$, where $r_0 = 0$, $r_{k+1} = T$. Denote J_i^o the interior of J_i , $i = 0, 1, \dots, k$. Let $\text{PC}(J, \mathbb{R}^N) = \{x : J \rightarrow \mathbb{R}^N \mid x \in C(J_i, \mathbb{R}^N), i = 0, 1, \dots, k, \text{ and } x(r_i^+) \text{ exists for } i = 1, \dots, k\}$; $w \in \text{PC}(J, \mathbb{R})$ satisfies $0 < w(r), \forall r \in J'$, and $(w(r))^{-1/(p(r)-1)} \in L^1(0, T)$; $\text{PC}^1(J, \mathbb{R}^N) = \{x \in \text{PC}(J, \mathbb{R}^N) \mid x' \in C(J_i^o, \mathbb{R}^N), \lim_{r \rightarrow r_i^+} (w(r))^{1/(p(r)-1)} x'(r) \text{ and } \lim_{r \rightarrow r_{i+1}^-} (w(r))^{1/(p(r)-1)} x'(r) \text{ exists for } i = 0, 1, \dots, k\}$. The equivalent $\lim_{r \rightarrow 0^+} (w(r))^{1/(p(r)-1)} u'(r) = \lim_{r \rightarrow T^-} (w(r))^{1/(p(r)-1)} u'(r)$ means $\lim_{r \rightarrow 0^+} (w(r))^{1/(p(r)-1)} u'(r)$ and $\lim_{r \rightarrow T^-} (w(r))^{1/(p(r)-1)} u'(r)$ both exist and equal. For any $x = (x^1, \dots, x^N) \in \text{PC}(J, \mathbb{R}^N)$, denote $|x|_0 = \sup\{|x^i(r)| \mid r \in J'\}$. Obviously, $\text{PC}(J, \mathbb{R}^N)$ is a Banach space with the norm $\|x\|_0 = (\sum_{i=1}^k |x^i|_0^2)^{1/2}$, $\text{PC}^1(J, \mathbb{R}^N)$ is a Banach space with the norm $\|x\|_1 = \|x\|_0 + \|(w(r))^{1/(p(r)-1)} x'\|_0$. In the following, $\text{PC}(J, \mathbb{R}^N)$ and $\text{PC}^1(J, \mathbb{R}^N)$ will be simply denoted by PC and PC^1 , respectively. Let $L^1 = L^1(J, \mathbb{R}^N)$ with the norm $\|x\|_{L^1} = (\sum_{i=1}^N |x^i|_{L^1}^2)^{1/2}$, $\forall x \in L^1$, where $|x^i|_{L^1} = \int_0^T |x^i(r)| dr$. We denote

$$\begin{aligned} \omega(0) |u'|^{p(0)-2} u'(0) &= \lim_{r \rightarrow 0^+} \omega(r) |u'|^{p(r)-2} u'(r), \\ \omega(T) |u'|^{p(T)-2} u'(T) &= \lim_{r \rightarrow T^-} \omega(r) |u'|^{p(r)-2} u'(r). \end{aligned} \quad (1.5)$$

The study of differential equations and variational problems with nonstandard $p(r)$ -growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, image processing, and so forth. [1–3]. Many results have been obtained on this kind of problems, for example [1–20]. If $p(r) \equiv p$ (a constant), (1.1) is the well-known p -Laplacian problem. But if $p(r)$ is a general function, the $-\Delta_{p(r)}$ is more complicated than $-\Delta_p$, since it represents a nonhomogeneity and possesses more nonlinearity, many methods and results for p -Laplacian problems are invalid for $p(x)$ -Laplacian problems; for example, if $\Omega \subset \mathbb{R}^N$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (1/p(x)) |\nabla u|^{p(x)} dx}{\int_{\Omega} (1/p(x)) |u|^{p(x)} dx} \quad (1.6)$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [9]), but the fact that $\lambda_p > 0$ is very important in the study of p -Laplacian problems.

Impulsive differential equations have been studied extensively in the recent years. Such equations arise in many applications such as spacecraft control, impact mechanics, chemical engineering and inspection process in operations research (see [21–23] and the references therein). On the Laplacian impulsive differential equations boundary value problems, there are many papers (see [24–27]). The methods include sub-super-solution method, fixed point theorem, monotone iterative method, and coincidence degree, and so forth. Because of the nonlinearity, results on the existence of solutions for p -Laplacian impulsive differential equation boundary value problems are rare (see [23, 28–30]).

In [29], Tian and Ge have studied nonlinear IBVP

$$\begin{aligned}
 & -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = f(t, x(t)), \quad t \neq t_i, \text{ a.e. } t \in [a, b], \\
 & \lim_{t \rightarrow t_i^+} \rho(t)\Phi_p(x'(t)) - \lim_{t \rightarrow t_i^-} \rho(t)\Phi_p(x'(t)) = I_i(x(t_i)), \quad i = 1, \dots, l, \\
 & \alpha x'(a) - \beta x(a) = \sigma_1, \quad \gamma x'(b) + \sigma x(b) = \sigma_2,
 \end{aligned} \tag{1.7}$$

where $\Phi_p(x) = |x|^{p-2}x$, $p > 1$, $\rho, s \in L^\infty[a, b]$ with $\text{essinf}_{[a,b]} \rho > 0$, and $\text{essinf}_{[a,b]} s > 0$, $0 < \rho(a), \rho(b) < \infty$, $\sigma_1 \leq 0, \sigma_2 \geq 0$, $\alpha, \beta, \gamma, \sigma > 0$, $a = t_0 < t_1 < \dots < t_l < t_{l+1} = b$, $I_i \in C([0, +\infty), [0, \infty))$, $i = 1, \dots, l$, $f \in C([a, b] \times 0, +\infty), [0, \infty))$, $f(\cdot, 0)$ is nontrivial. By using variational methods, the existence of at least two positive solutions was obtained.

On the existence of solutions for $p(r)$ -Laplacian impulsive differential equation boundary value problems, we refer to [31, 32]. If $w(0) = w(T) \neq 0$ and $p(0) = p(T)$, then (1.4) is the periodic boundary value condition, so we call condition (1.4) the periodic-like boundary value condition. In [31], the present author deals with the existence of solutions of (1.1) with (1.2), (1.4) and the following impulsive boundary value condition:

$$\begin{aligned}
 & \lim_{r \rightarrow r_i^+} w(r) |u'|^{p(r)-2} u'(r) - \lim_{r \rightarrow r_i^-} w(r) |u'|^{p(r)-2} u'(r) \\
 & = D_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)} u'(r) \right), \quad i = 1, \dots, k,
 \end{aligned} \tag{1.8}$$

the method in [31] is the coincidence degree.

In this paper, when $p(r)$ is a general function, we investigate the existence of solutions for the weighted $p(r)$ -Laplacian impulsive integro-differential system periodic-like boundary value problems via Leray-Schauder's degree, problems with the impulsive condition (1.8) has been discussed also. The homotopy transformation of this paper is different from [31], our main results partly generalized the results of [23, 28–31].

Let $N \geq 1$, the function $f : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is assumed to be Caratheodory, by this we mean:

- (i) for almost every $t \in J$ the function $f(t, \cdot, \cdot, \cdot)$ is continuous;
- (ii) for each $(x, y, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ the function $f(\cdot, x, y, z)$ is measurable on J ;
- (iii) for each $R > 0$ there is a $\alpha_R \in L^1(J, \mathbb{R})$ such that, for almost every $t \in J$ and every $(x, y, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R, |y| \leq R, |z| \leq R$, one has

$$|f(t, x, y, z)| \leq \alpha_R(t). \quad (1.9)$$

We say a function $u : J \rightarrow \mathbb{R}^N$ is a solution of (1.1) if $u \in PC^1$ with $w(r)|u'|^{p(r)-2}u'(r)$ absolutely continuous on $J_i^o, i = 0, 1, \dots, k$, which satisfies (1.1) a.e. on J .

This paper is divided into four sections; in the second section, we present some preliminary. In the third section, we give the existence of solutions for system (1.1), (1.2), (1.3) (or (1.8)) and (1.4). Finally, in the fourth section, we give several examples.

2. Preliminary

For any $(r, y) \in (J \times \mathbb{R}^N)$, denote $\varphi(r, y) = |y|^{p(r)-2}y$. Obviously, φ has the following properties.

Lemma 2.1. *φ is a continuous function and satisfies the following.*

- (i) For any $r \in [0, T]$, $\varphi(r, \cdot)$ is strictly monotone, that is,

$$\langle \varphi(r, y_1) - \varphi(r, y_2), y_1 - y_2 \rangle > 0, \quad \text{for any } y_1, y_2 \in \mathbb{R}^N, y_1 \neq y_2, \quad (2.1)$$

- (ii) There exists a function $\alpha : [0, +\infty) \rightarrow [0, +\infty)$, $\alpha(s) \rightarrow +\infty$ as $s \rightarrow +\infty$, such that

$$\langle \varphi(r, y), y \rangle \geq \alpha(|y|)|y|, \quad \forall y \in \mathbb{R}^N. \quad (2.2)$$

It is well known that $\varphi(r, \cdot)$ is a homeomorphism from \mathbb{R}^N to \mathbb{R}^N for any fixed $r \in J$. Denote

$$\varphi^{-1}(r, y) = |y|^{(2-p(r))/(p(r)-1)}y, \quad \text{for } y \in \mathbb{R}^N \setminus \{0\}, \varphi^{-1}(r, 0) = 0. \quad (2.3)$$

It is clear that $\varphi^{-1}(r, \cdot)$ is continuous and send bounded sets into bounded sets. Let us now consider the following simple impulsive problem:

$$\begin{aligned} (\omega(r)\varphi(r, u'(r)))' &= f(r), \quad r \in (0, T), r \neq r_i, \\ \lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) &= a_i, \quad i = 1, \dots, k, \\ \lim_{r \rightarrow r_i^+} \omega(r)|u'|^{p(r)-2}u'(r) - \lim_{r \rightarrow r_i^-} \omega(r)|u'|^{p(r)-2}u'(r) &= b_i, \quad i = 1, \dots, k, \\ u(0) = u(T), \quad \lim_{r \rightarrow 0^+} \omega(r)|u'|^{p(r)-2}u'(r) &= \lim_{r \rightarrow T^-} \omega(r)|u'|^{p(r)-2}u'(r), \end{aligned} \quad (2.4)$$

where $a_i, b_i \in \mathbb{R}^N$; $f \in L^1$ and satisfies

$$\int_0^T f(r) dr + \sum_{i=1}^k b_i = 0. \tag{2.5}$$

If u is a solution of (2.4), by integrating (2.4) from 0 to r , we find that

$$w(r)\varphi(r, u'(r)) = w(0)\varphi(0, u'(0)) + \sum_{r_i < r} b_i + \int_0^r f(t) dt. \tag{2.6}$$

Denote $\rho = w(0)\varphi(0, u'(0))$. Define the operator $F : L^1 \rightarrow \text{PC}$ as

$$F(f)(r) = \int_0^r f(t) dt, \quad \forall r \in J, \forall f \in L^1. \tag{2.7}$$

By solving for u' in (2.6) and integrating, we find

$$u(r) = u(0) + \sum_{r_i < r} a_i + F \left\{ \varphi^{-1} \left[r, (w(r))^{-1} \left(\rho + \sum_{r_i < r} b_i + F(f)(r) \right) \right] \right\} (r), \quad \forall r \in J. \tag{2.8}$$

The boundary value conditions imply that

$$\sum_{i=1}^k a_i + \int_0^T \varphi^{-1} \left\{ r, (w(r))^{-1} \left[\rho + \sum_{r_i < r} b_i + F(f)(r) \right] \right\} dr = 0. \tag{2.9}$$

Denote $a = (a_1, \dots, a_k) \in \mathbb{R}^{kN}$, $b = (b_1, \dots, b_k) \in \mathbb{R}^{kN}$. It is easy to see that ρ is dependent on a, b and f . Denote $W = \mathbb{R}^{2kN} \times \text{PC}$ with the norm

$$\|\varpi\|_W = \sum_{i=1}^k |a_i| + \sum_{i=1}^k |b_i| + \|h\|_0, \quad \forall \varpi = (a, b, h) \in W, \tag{2.10}$$

then W is a Banach space.

For fixed $\varpi \in W$, we denote

$$\Lambda_{\varpi}(\rho) = \sum_{i=1}^k a_i + \int_0^T \varphi^{-1} \left\{ r, (w(r))^{-1} \left[\rho + \sum_{r_i < r} b_i + h(r) \right] \right\} dr, \quad \text{where } \varpi = (a, b, h). \tag{2.11}$$

Lemma 2.2. *The function $\Lambda_{\varpi}(\cdot)$ has the following properties.*

(i) *For any fixed $\varpi \in W$, the equation*

$$\Lambda_{\varpi}(\rho) = 0 \quad (2.12)$$

has a unique solution $\tilde{\rho}(\varpi) \in \mathbb{R}^N$.

(ii) *The function $\tilde{\rho} : W \rightarrow \mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover $|\tilde{\rho}(\varpi)| \leq 3N[(((E+1)/E) \sum_{i=1}^k |a_i|)^{p^{\#}-1} + \sum_{i=1}^k |b_i| + \|h\|_0]$, where $\varpi = (a, b, h) \in W$, $E = \int_0^T (\omega(r))^{-1/(p(r)-1)} dr$, the notation $M^{p^{\#}-1}$ means $M^{p^{\#}-1} = \begin{cases} M^{p^{\#}-1}, & M > 1 \\ M^{p^{\#}-1}, & M \leq 1 \end{cases}$.*

Proof. (i) From Lemma 2.1, it is immediate that

$$\langle \Lambda_{\varpi}(y_1) - \Lambda_{\varpi}(y_2), y_1 - y_2 \rangle > 0, \quad \text{for } y_1 \neq y_2, \quad (2.13)$$

and hence, if (2.12) has a solution, then it is unique.

Let $R_0 = 3N[(((E+1)/E) \sum_{i=1}^k |a_i|)^{p^{\#}-1} + \sum_{i=1}^k |b_i| + \|h\|_0]$. Since $(\omega(r))^{-1/(p(r)-1)} \in L^1(0, T)$ and $h \in \text{PC}$, if $|\rho| > R_0$, it is easy to see that there exists an $j_0 \in \{1, \dots, N\}$ such that the absolute value of the j_0 th component ρ^{j_0} of ρ is bigger than $3[(((E+1)/E) \sum_{i=1}^k |a_i|)^{p^{\#}-1} + \sum_{i=1}^k |b_i| + \|h\|_0]$. Thus the j_0 th component of $\rho + \sum_{r_i < r} b_i + h(r)$ keeps the same sign of ρ^{j_0} on J , namely,

$$\left(\rho^{j_0} + \sum_{r_i < r} b_i^{j_0} + h^{j_0}(r) \right) \rho^{j_0} > 2 \left[\left(\frac{E+1}{E} \sum_{i=1}^k |a_i| \right)^{p^{\#}-1} + \sum_{i=1}^k |b_i| + \|h\|_0 \right]^2, \quad \text{for any } r \in J, \quad (2.14)$$

then it is easy to see that the j_0 th component of $\Lambda_{\varpi}(\rho)$ keep the same sign of ρ^{j_0} , thus

$$\sum_{i=1}^k a_i + \int_0^T \varphi^{-1} \left\{ r, (\omega(r))^{-1} \left[\rho + \sum_{r_i < r} b_i + h(r) \right] \right\} dr \neq 0. \quad (2.15)$$

Let us consider the equation

$$\lambda \Lambda_{\varpi}(\rho) + (1 - \lambda)\rho = 0, \quad \lambda \in [0, 1]. \quad (2.16)$$

According to the former discussion, all the solutions of (2.16) belong to $b(R_0 + 1) = \{x \in \mathbb{R}^N \mid |x| < R_0 + 1\}$. So, we have

$$d_B[\Lambda_{\varpi}(\rho), b(R_0 + 1), 0] = d_B[I, b(R_0 + 1), 0] \neq 0. \quad (2.17)$$

It means the existence of solutions of $\Lambda_{\varpi}(\rho) = 0$.

In this way, we define a function $\tilde{\rho}(\varpi) : W \rightarrow \mathbb{R}^N$, which satisfies

$$\sum_{i=1}^k a_i + \int_0^T \varphi^{-1} \left\{ r, (w(r))^{-1} \left[\tilde{\rho}(\varpi) + \sum_{r_i < r} b_i + h(r) \right] \right\} dr = 0. \tag{2.18}$$

(ii) By the proof of (i), we also obtain $\tilde{\rho}$ the sends bounded set to bounded set, and

$$|\tilde{\rho}(\varpi)| \leq 3N \left[\left(\frac{E+1}{E} \sum_{i=1}^k |a_i| \right)^{p^*-1} + \left\| \sum_{r_i < r} b_i \right\|_0 + \|h\|_0 \right], \quad \forall \varpi = (a, b, h) \in W. \tag{2.19}$$

Finally to show that the continuity of $\tilde{\rho}$. Let $\{\varpi_n = (a_n, b_n, h_n)\}$ is a convergent sequence in W and $\varpi_n \rightarrow \varpi = (a, b, h)$, as $n \rightarrow +\infty$. Since $\{\tilde{\rho}(\varpi_n)\}$ is bounded, it contains a convergent subsequence $\{\tilde{\rho}(\varpi_{n_j})\}$. Set $\tilde{\rho}(\varpi_{n_j}) \rightarrow \rho_0$ as $j \rightarrow +\infty$. Obviously,

$$\sum_{i=1}^k a_{i,n_j} + \int_0^T \varphi^{-1} \left\{ r, (w(r))^{-1} \left[\tilde{\rho}(\varpi_{n_j}) + \sum_{r_i < r} b_{i,n_j} + h_{n_j}(r) \right] \right\} dr = 0, \tag{2.20}$$

where $a_{n_j} = (a_{1,n_j}, \dots, a_{k,n_j})$, $b_{n_j} = (b_{1,n_j}, \dots, b_{k,n_j})$. Letting $j \rightarrow +\infty$, we have

$$\sum_{i=1}^k a_i + \int_0^T \varphi^{-1} \left\{ r, (w(r))^{-1} \left[\rho_0 + \sum_{r_i < r} b_i + h(r) \right] \right\} dr = 0, \tag{2.21}$$

from (i) we get $\rho_0 = \tilde{\rho}(\varpi)$, it means that $\tilde{\rho}$ is continuous.

This completes the proof. □

Now, we define $\rho : \mathbb{R}^N \times \mathbb{R}^N \times L^1 \rightarrow \mathbb{R}^N$ be defined by

$$\rho(a, b, u) = \tilde{\rho}(a, b, F(u)). \tag{2.22}$$

It is clear that ρ is a continuous function which send bounded sets of $\mathbb{R}^N \times \mathbb{R}^N \times L^1$ into bounded sets of \mathbb{R}^N , and hence it is a compact continuous mapping.

Now, we continue with our argument previous to Lemma 2.2.

Denote

$$\begin{aligned} A_i &= A_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)} u'(r) \right), \\ D_i &= D_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)} u'(r) \right). \end{aligned} \tag{2.23}$$

From the definition of B_i and D_i , we have

$$D_i = \varphi \left(r_i, \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)} u'(r) + B_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)} u'(r) \right) \right) - \varphi \left(r_i, \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)} u'(r) \right). \quad (2.24)$$

Let us define

$$P : PC^1 \longrightarrow PC^1, \quad u \longmapsto u(0); \quad Q : L^1 \longrightarrow L^1, \quad h \longmapsto \frac{1}{T} \int_0^T h(r) dr; \quad (2.25)$$

$$\Theta_b : L^1 \longrightarrow L^1, \quad h \longmapsto (I - Q)h - \frac{1}{T} \sum_{i=1}^k b_i;$$

and $K_{(a,b)} : L^1 \rightarrow PC^1$ as

$$K_{(a,b)}(h)(r) = F \left\{ \varphi^{-1} \left[r, (w(r))^{-1} \left(\rho(a, b, h) + \sum_{r_i < r} b_i + F(h) \right) \right] \right\} (r), \quad \forall r \in J. \quad (2.26)$$

Lemma 2.3. *The operator $(K_{(a,b)} \circ \Theta_b)(\cdot)$ is continuous and send equi-integrable sets of L^1 into relatively compact sets of PC^1 .*

Proof. It is easy to check that $(K_{(a,b)} \circ \Theta_b)(h)(\cdot) \in PC^1$. Since $(w(r))^{-1/(p(r)-1)} \in L^1$ and

$$(K_{(a,b)} \circ \Theta_b)(h)'(t) = \varphi^{-1} \left\{ t, (w(t))^{-1} \left[\rho(\Theta_b(h)) + \sum_{r_i < t} b_i + F(\Theta_b(h)) \right] \right\}, \quad \forall t \in J, \quad (2.27)$$

it is easy to check that $(K_{(a,b)} \circ \Theta_b)(\cdot)$ is a continuous operator from L^1 to PC^1 .

Let now U be an equi-integrable set in L^1 , then there exist $\beta \in L^1$, such that, for any $u \in L^1$

$$|u(t)| \leq \beta(t) \quad \text{a.e. in } J. \quad (2.28)$$

We want to show that $\overline{(K_{(a,b)} \circ \Theta_b)(U)} \subset PC^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $(K_{(a,b)} \circ \Theta_b)(U)$, then there exists a sequence $\{h_n\} \in U$ such that $u_n = (K_{(a,b)} \circ \Theta_b)(h_n)$. For any $t_1, t_2 \in J$, we have that

$$|F(\Theta_b(h_n))(t_1) - F(\Theta_b(h_n))(t_2)| \leq \left| \int_{t_1}^{t_2} \beta(t) dt \right| + |t_1 - t_2| \frac{1}{T} \left(\int_0^T \beta(t) dt + \left| \sum_{i=1}^k b_i \right| \right). \quad (2.29)$$

Hence the sequence $\{F(\Theta_b(h_n))\}$ is uniformly bounded and equicontinuous. By Ascoli-Arzela theorem, there exists a subsequence of $\{F(\Theta_b(h_n))\}$ (which we rename the same) is convergent in PC . According to the bounded continuity of the operator ρ , we can choose a subsequence of $\{\rho(\Theta_b(h_n)) + F(\Theta_b(h_n))\}$ (which we still denote by $\{\rho(\Theta_b(h_n)) + F(\Theta_b(h_n))\}$) is convergent in PC , then

$$w(t)\varphi(t, (K \circ \Theta_b)(h_n)'(t)) = \rho(\Theta_b(h_n)) + \sum_{r_i < t} b_i + F(\Theta_b(h_n)) \tag{2.30}$$

is convergent according to the norm of PC . Since

$$(K_{(a,b)} \circ \Theta_b)(h_n)(t) = F \left\{ \varphi^{-1} \left[t, (w(r))^{-1} (\rho(\Theta_b(h_n)) + \sum_{r_i < t} b_i + F(\Theta_b(h_n))) \right] \right\} (t), \quad \forall t \in J, \tag{2.31}$$

according to the continuous of φ^{-1} and the integrability of $(w(t))^{-1/(p(t)-1)}$ in L^1 , we can see that $(K_{(a,b)} \circ \Theta_b)(h_n)$ is convergent in PC . Thus $\{u_n\}$ convergent in PC^1 . This completes the proof. \square

We denote $N_f(u) : PC^1 \times J \rightarrow L^1$ the Nemytski operator associated to f defined by

$$N_f(u)(r) = f \left(r, u(r), (w(r))^{1/(p(r)-1)} u'(r), S(u) \right), \quad \text{a.e. on } J. \tag{2.32}$$

Denote $A = (A_1, \dots, A_k)$, $D = (D_1, \dots, D_k)$ and

$$\Theta_f(u) = (I - Q)N_f(u) - \frac{1}{T} \sum_{i=1}^k D_i, \tag{2.33}$$

$$\rho(u) = \rho(A, D, \Theta_f)(u),$$

$$K(u)(r) = F \left\{ \varphi^{-1} \left[r, (w(r))^{-1} (\rho(u) + \sum_{r_i < r} D_i + F(\Theta_f(u))) \right] \right\} (r), \quad \forall r \in J. \tag{2.34}$$

Lemma 2.4. u is a solution of (1.1), (1.2), (1.8) and (1.4), if and only if u is a solution of the following abstract equation:

$$u = Pu + \sum_{r_i < r} A_i + \frac{1}{T} \sum_{i=1}^k D_i + QN_f(u) + K(u). \tag{2.35}$$

Proof. If u is a solution of (1.1), (1.2), (1.8) and (1.4), it is clear that u is a solution of (2.35).

Conversely, if u is a solution of (2.35), then (1.2) is satisfied and

$$\frac{1}{T} \sum_{i=1}^k D_i + QN_f(u) = 0. \tag{2.36}$$

Thus $\Theta_f(u) = N_f(u)$.

By the condition of the mapping ρ , we have

$$\sum_{i=1}^k A_i + F \left\{ \varphi^{-1} \left[r, w^{-1}(r) (\rho(u) + \sum_{r_i < r} D_i + F(\Theta_f(u))) \right] \right\} (T) = 0, \quad (2.37)$$

then $u(0) = u(T)$. From (2.35) and (2.36), we also have

$$w(r)\varphi(r, u') = \rho(u) + \sum_{r_i < r} D_i + F(\Theta_f(u))(r), \quad r \in (0, T), \quad r \neq r_i, \quad (2.38)$$

$$(w(r)\varphi(r, u'))' = N_f(u)(r), \quad r \in (0, T), \quad r \neq r_i. \quad (2.39)$$

From (2.38), we get that (1.8) is satisfied. Since $\sum_{i=1}^k D_i + F(\Theta_f(u))(T) = 0$, we have

$$w(0)\varphi(0, u'(0)) = w(T)\varphi(T, u'(T)). \quad (2.40)$$

Hence u is a solutions of (1.1), (1.2), (1.8) and (1.4). This completes the proof. \square

3. Main Results and Proofs

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions for (1.1) with impulsive periodic-like boundary value conditions (1.2), (1.8) (or (1.3)) and (1.4).

Theorem 3.1. *Assume that Ω is an open bounded set in PC^1 such that the following conditions hold.*

(1⁰) *For each $\lambda \in (0, 1)$ the problem*

$$\begin{aligned} (w(r)|u'|^{p(r)-2}u')' &= \lambda f(r, u, (w(r))^{1/(p(r)-1)}u', S(u)), \quad r \in (0, T), \quad r \neq r_i, \\ \lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) &= \lambda A_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)}u'(r) \right), \quad i = 1, \dots, k, \\ \lim_{r \rightarrow r_i^+} w(r)|u'|^{p(r)-2}u'(r) - \lim_{r \rightarrow r_i^-} w(r)|u'|^{p(r)-2}u'(r) & \\ &= \lambda D_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)}u'(r) \right), \quad i = 1, \dots, k, u(0) \\ u(0) = u(T), \quad \lim_{r \rightarrow 0^+} w(r)|u'|^{p(r)-2}u'(r) &= \lim_{r \rightarrow T^-} w(r)|u'|^{p(r)-2}u'(r), \end{aligned} \quad (3.1)$$

has no solution on $\partial\Omega$.

(2⁰) The equation

$$\omega(l) := \frac{1}{T} \int_0^T f(r, l, 0, S(l)) dr + \frac{1}{T} \sum_{i=1}^k D_i(l, 0) = 0, \tag{3.2}$$

has no solution on $\partial\Omega \cap \mathbb{R}^N$.

(3⁰) The Brouwer degree $d_B[\omega, \Omega \cap \mathbb{R}^N, 0] \neq 0$.

Then (1.1) with (1.2), (1.8) and (1.4) has a solution on $\bar{\Omega}$.

Proof. Let us consider the following impulsive equation:

$$\begin{aligned} \left(w(r) |u'|^{p(r)-2} u' \right)' &= \lambda N_f(u) + (1 - \lambda) \left[QN_f(u) + \frac{1}{T} \sum_{i=1}^k D_i \right], \quad r \in (0, T), \quad r \neq r_i, \\ \lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) &= \lambda A_i, \quad i = 1, \dots, k, \\ \lim_{r \rightarrow r_i^+} w(r) |u'|^{p(r)-2} u'(r) - \lim_{r \rightarrow r_i^-} w(r) |u'|^{p(r)-2} u'(r) &= \lambda D_i, \quad i = 1, \dots, k, \\ u(0) = u(T), \quad \lim_{r \rightarrow 0^+} w(r) |u'|^{p(r)-2} u'(r) &= \lim_{r \rightarrow T^-} w(r) |u'|^{p(r)-2} u'(r), \end{aligned} \tag{3.3}$$

where $A = (A_1, \dots, A_k)$ and $D = (D_1, \dots, D_k)$ are defined in (2.23).

For any $\lambda \in (0, 1]$, if u is a solution to (3.1) or u is a solution to (3.3), we have necessarily

$$QN_f(u) + \frac{1}{T} \sum_{i=1}^k D_i = 0. \tag{3.4}$$

It means that (3.1) and (3.3) has the same solutions for $\lambda \in (0, 1]$.

We denote $N(\cdot, \cdot) : PC^1 \times [0, 1] \rightarrow L^1$ defined by

$$N(u, \lambda) = \lambda N_f(u) + (1 - \lambda) \left[QN_f(u) + \frac{1}{T} \sum_{i=1}^k D_i \right], \tag{3.5}$$

where $N_f(u)$ is defined by (2.32). Denote

$$\Theta_\lambda : L^1 \longrightarrow L^1, \quad u \longmapsto (I - Q)N(u, \lambda) - \frac{\lambda}{T} \sum_{i=1}^k D_i \tag{3.6}$$

$$\rho_\lambda(u) = \rho(\lambda A, \lambda B, \Theta_\lambda),$$

$$K_\lambda(u)(t) = F \left\{ \varphi^{-1} \left[r, \left(w(r) \right)^{-1} \left(\rho_\lambda(u) + \lambda \sum_{r_i < r} D_i + F(\Theta_\lambda(u)) \right) \right] \right\} (t), \quad \forall t \in 0, T]. \tag{3.7}$$

Let

$$\begin{aligned}\Phi_f(u, \lambda) &= Pu + \lambda \sum_{r_i < r} A_i + \lambda \frac{1}{T} \sum_{i=1}^k D_i + QN(u, \lambda) + K_\lambda(u) \\ &= Pu + \lambda \sum_{r_i < r} A_i + \frac{1}{T} \sum_{i=1}^k D_i + QN_f(u) + K_\lambda(u),\end{aligned}\tag{3.8}$$

the fixed point of $\Phi_f(u, 1)$ is a solution for (1.1) with (1.2), (1.8) and (1.4). Also problem (3.3) can be written in the equivalent form

$$u = \Phi_f(u, \lambda).\tag{3.9}$$

Since f is Caratheodory, it is easy to see that $N(\cdot, \cdot)$ is continuous and sends bounded sets into equi-integrable sets. According to Lemma 2.3 we can conclude that Φ_f is continuous and compact for any $\lambda \in [0, 1]$. We assume that for $\lambda = 1$, (3.9) does not have a solution on $\partial\Omega$, otherwise we complete the proof. Now from hypothesis (1⁰) it follows that (3.9) has no solutions for $(u, \lambda) \in \partial\Omega \times (0, 1]$. For $\lambda = 0$, (3.3) is equivalent to the following usual problem:

$$\begin{aligned}-\left(w(r)|u'|^{p(r)-2}u'\right)' &= QN_f(u) + \frac{1}{T} \sum_{i=1}^k D_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)} u'(r) \right), \quad r \in (0, T), \\ u(0) &= u(T), \quad \lim_{r \rightarrow 0^+} w(r)|u'|^{p(r)-2}u'(r) = \lim_{r \rightarrow T^-} w(r)|u'|^{p(r)-2}u'(r).\end{aligned}\tag{3.10}$$

If u is a solution to this problem, we must have

$$\int_0^T f\left(r, u(r), (w(r))^{1/(p(r)-1)}u'(r), S(u)\right)dr + \sum_{i=1}^k D_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} (w(r))^{1/(p(r)-1)}u'(r) \right) = 0.\tag{3.11}$$

When $\lambda = 0$, the problem is a usual differential equation. Hence

$$w(r)|u'|^{p(r)-2}u' \equiv c,\tag{3.12}$$

where $c \in \mathbb{R}^N$ is a constant. Since $u(0) = u(T)$, there exist $t_0^i \in (0, T)$, such that $(u^i)'(t_0^i) = 0$, hence $(u^i)' \equiv 0$, it holds $u \equiv l$, a constant. Thus, by (3.11) we have

$$\int_0^T f(r, l, 0, S(l))dr + \sum_{i=1}^k D_i(l, 0) = 0,\tag{3.13}$$

which together with hypothesis (2⁰), implies that $u = l \notin \partial\Omega$. Thus, we have proved that (3.9) has no solution (u, λ) on $\partial\Omega \times [0, 1]$, then we get that the Leray-Schauder's degree $d_{LS}[I - \Phi_f(\cdot, \lambda), \Omega, 0]$ is well defined for $\lambda \in [0, 1]$, and from the homotopy invariant property of that

degree, we have

$$d_{LS}[I - \Phi_f(\cdot, 1), \Omega, 0] = d_{LS}[I - \Phi_f(\cdot, 0), \Omega, 0]. \tag{3.14}$$

Now it is clear that the problem

$$u = \Phi_f(u, 1) \tag{3.15}$$

is equivalent to problem (1.1) with (1.2), (1.8) and (1.4), and (3.14) tell us that problem (3.15) will have a solution if we can show that

$$d_{LS}[I - \Phi_f(\cdot, 0), \Omega, 0] \neq 0. \tag{3.16}$$

Since $K_0(\cdot) \equiv 0$, we have

$$\Phi_f(u, 0) = Pu + \frac{1}{T} \sum_{i=1}^k D_i + QN_f(u) + K_0(u) = Pu + \frac{1}{T} \sum_{i=1}^k D_i + QN_f(u), \tag{3.17}$$

and then

$$u - \Phi_f(u, 0) = u - Pu - \frac{1}{T} \sum_{i=1}^k D_i - QN_f(u) = -\frac{1}{T} \sum_{i=1}^k D_i - QN_f(u), \quad \text{on } \overline{\Omega}. \tag{3.18}$$

By the properties of the Leray-Schauder degree, we have

$$d_{LS}[I - \Phi_f(\cdot, 0), \Omega, 0] = (-1)^N d_B[\omega, \Omega \cap \mathbb{R}^N, 0], \tag{3.19}$$

where the function ω is defined in (3.2) and d_B denotes the Brouwer degree. Since by hypothesis (3⁰), this last degree is different from zero. This completes the proof. \square

Our next theorem is a consequence of Theorem 3.1. As an application of Theorem 3.1, let us consider the system

$$\begin{aligned} (w(r)|u'|^{p(r)-2}u')' &= g(r, u, (w(r))^{1/(p(r)-1)}u', S(u)) \\ &+ e(r, u(r), (w(r))^{1/(p(r)-1)}u'(r), S(u)), \quad r \in J', \end{aligned} \tag{3.20}$$

with (1.2), (1.8) and (1.4), where $e : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is Caratheodory, $g = (g^1, \dots, g^N) : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous, and for any fixed $y_0 \in \mathbb{R}^N$, $g^i(r, y_0, 0, S(y_0)) \neq 0$ if $y_0^i \neq 0, \forall r \in J, i = 1, \dots, N$. Denote

$$z^- = \min_{r \in J} z(r), \quad z^+ = \max_{r \in J} z(r), \quad \text{for } z \in C(J, \mathbb{R}). \tag{3.21}$$

Theorem 3.2. Assume that the following conditions hold:

$$(1^0) \quad g(r, kx, ky, kz) = k^{q(r)-1}g(r, x, y, z) \text{ for all } k > 0 \text{ and all } (r, x, y, z) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N, \\ \text{where } q(r) \in C(J, \mathbb{R}), \text{ and } 1 < q^- \leq q^+ < p^-;$$

$$(2^0) \quad \lim_{|u|+|v| \rightarrow +\infty} (e(r, u, v, S(u)) / (|u| + |v|)^{q(r)-1}) = 0, \text{ for } r \in J \text{ uniformly};$$

$$(3^0) \quad \sum_{i=1}^k |A_i(u, v)| \leq C(1 + |u| + |v|)^\theta, \forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ where } 0 < \theta < (p^- - 1) / (p^+ - 1);$$

$$(4^0) \quad \sum_{i=1}^k |D_i(u, v)| \leq C(1 + |u| + |v|)^{\beta-1}, \forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ where } 1 \leq \beta < q^+;$$

$$(5^0) \quad \text{For large enough } R_0 > 0, \text{ the equation}$$

$$\omega_g(l) := \frac{1}{T} \int_0^T g(r, l, 0, S(l)) dr + \frac{1}{T} \sum_{i=1}^k D_i(l, 0) = 0, \quad (3.22)$$

has no solution on $\partial B(R_0) \cap \mathbb{R}^N$, where $B(R_0) = \{u \in PC^1 \mid \|u\|_1 < R_0\}$;

$$(6^0) \quad d_B[\omega_g, b(R_0), 0] \neq 0 \text{ for large enough } R_0 > 0, \text{ where } b(R_0) = \{x \in \mathbb{R}^N \mid |x| < R_0\}.$$

Then problem (3.20) with (1.2), (1.8) and (1.4) has at least one solution.

Proof. For any $u \in PC^1$, $\lambda \in [0, 1]$, we denote

$$N_{f_\lambda}(u) = g\left(r, u, (\omega(r))^{1/(p(r)-1)} u', S(u)\right) + \lambda e\left(r, u(r), (\omega(r))^{1/(p(r)-1)} u'(r), S(u)\right). \quad (3.23)$$

At first, we consider the following problem:

$$\begin{aligned} (\omega(r)|u'|^{p(r)-2}u')' &= N_{f_\lambda}(u), \quad r \in (0, T), \quad r \neq r_i, \\ \lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) &= A_i, \quad i = 1, \dots, k, \\ \lim_{r \rightarrow r_i^+} \omega(r)|u'|^{p(r)-2}u'(r) - \lim_{r \rightarrow r_i^-} \omega(r)|u'|^{p(r)-2}u'(r) &= D_i, \quad i = 1, \dots, k, \\ u(0) = u(T), \quad \lim_{r \rightarrow 0^+} \omega(r)|u'|^{p(r)-2}u'(r) &= \lim_{r \rightarrow T^-} \omega(r)|u'|^{p(r)-2}u'(r), \end{aligned} \quad (3.24)$$

where A_i and D_i are defined in (2.23).

According to the proof of Theorem 3.1, we know that (3.24) has the same solutions of

$$u = \Phi_f(u, \lambda) = Pu + \sum_{r_i < r} A_i + \frac{1}{T} \sum_{i=1}^k D_i + QN_{f_\lambda}(u) + K(\Theta_{f_\lambda}(u)), \quad (3.25)$$

where Θ_{f_λ} is defined in (2.33).

We claim that all the solutions of (3.24) are bounded for each $\lambda \in [0, 1]$. In fact, if it is false, we can find a sequence (u_n, λ_n) of solutions for (3.24) such that $\|u_n\|_1 > 1$ and $\|u_n\|_1 \rightarrow +\infty$ when $n \rightarrow +\infty$. Since (u_n, λ_n) are solutions of (3.24), we have

$$w(r)\varphi(r, u'_n(r)) = \rho(u_n) + \sum_{r_i < r} D_i + F(N_{f_{\lambda_n}}(u_n))(r), \tag{3.26}$$

$$u_n(r) = u_n(0) + \sum_{r_i < r} A_i + F\left\{\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho(u_n) + \sum_{r_i < r} D_i + F(N_{f_{\lambda_n}}(u_n))(r)\right)\right]\right\}(r). \tag{3.27}$$

Since $u_n(0) = u_n(T)$, we have

$$\sum_{i=1}^k A_i + F\left\{\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho(u_n) + \sum_{r_i < r} D_i + F(N_{f_{\lambda_n}}(u_n))(r)\right)\right]\right\}(T) = 0. \tag{3.28}$$

From Lemma 2.2, we have

$$|\rho(u_n)| \leq 3NC \left(1 + \left(\frac{E+1}{E}\right)^{(p^+-1)} \|u_n\|_1^{\theta(p^+-1)} + \|u_n\|_1^{q^+-1}\right). \tag{3.29}$$

From (3⁰), (4⁰), (3.26) and (3.29), we can see that

$$\left\| (w(r))^{1/(p(r)-1)} u'_n \right\|_0 \leq o(1) \|u_n\|_1. \tag{3.30}$$

From (3.30), we have

$$\lim_{n \rightarrow +\infty} \frac{\|u_n\|_0}{\|u_n\|_1} = 1. \tag{3.31}$$

Denote $\delta_n = (|u_n^1|_0 / \|u_n\|_0, |u_n^2|_0 / \|u_n\|_0, \dots, |u_n^N|_0 / \|u_n\|_0)$, then $\delta_n \in \mathbb{R}^N$ and $|\delta_n| = 1$ ($n = 1, 2, \dots$). Since $\{\delta_n\}$ possesses a convergent subsequence (which still denoted by δ_n), there exists a vector $\delta_0 = (\delta_0^1, \delta_0^2, \dots, \delta_0^N) \in \mathbb{R}^N$ such that

$$|\delta_0| = 1, \quad \lim_{n \rightarrow +\infty} \delta_n = \delta_0. \tag{3.32}$$

With out loss of generality, we assume that $\delta_0^1 > 0$. Since $u_n \in PC^1$, there exist $\eta_n^i \in J$ such that

$$\left| u_n^i(\eta_n^i) \right| \geq \left(1 - \frac{1}{n}\right) |u_n^i|_0, \quad i = 1, 2, \dots, N, \quad n = 1, 2, \dots \tag{3.33}$$

Obviously

$$0 \leq \left| u_n^1(r) - u_n^1(\eta_n^1) \right| = \left| \int_{\eta_n^1}^r (u_n^1)'(t) dt + \sum_{\eta_n^1 < r_i < r} A_i^1 \right| \leq o(1) \|u_n\|_1 \int_0^T (w(t))^{-1/(p(t)-1)} dt + \sum_{i=1}^k \left| A_i^1 \right|. \quad (3.34)$$

Since $\|u_n\|_1 \rightarrow +\infty$ (as $n \rightarrow +\infty$) and $\delta_0^1 > 0$, according to (3.31), (3.33) and (3⁰), we have

$$\lim_{n \rightarrow +\infty} \frac{1}{|u_n^1(\eta_n^1)|} \left\{ o(1) \|u_n\|_1 \int_0^T (w(t))^{-1/(p(t)-1)} dt + \sum_{i=1}^k \left| A_i^1 \right| \right\} = 0. \quad (3.35)$$

From (3.31), (3.34) and (3.35) we have

$$\lim_{n \rightarrow +\infty} \frac{u_n^1(r)}{u_n^1(\eta_n^1)} = 1, \quad \text{for } r \in J \text{ uniformly.} \quad (3.36)$$

So we get

$$\lim_{n \rightarrow +\infty} \frac{u_n(r)}{\|u_n\|_1} = \delta_*, \quad \lim_{n \rightarrow +\infty} \frac{(w(r))^{1/(p(r)-1)} u_n'(r)}{\|u_n\|_1} = 0, \quad \text{for } r \in J \text{ uniformly,} \quad (3.37)$$

where $\delta_* \in \mathbb{R}^N$ with $|\delta_*^i| = \delta_0^i$, $i = 1, \dots, N$.

From (1.4), we have

$$\begin{aligned} \sum_{i=1}^k D_i + \int_0^T \left\{ g(r, u_n, (w(r))^{1/(p(r)-1)} u_n', S(u_n)) + \right. \\ \left. + e(r, u_n(r), (w(r))^{1/(p(r)-1)} u_n'(r), S(u_n)) \right\} dr = 0. \end{aligned} \quad (3.38)$$

Since $g^1(t, \delta_0, 0, S(\delta_0)) \neq 0$, according to (3.37), (4⁰) and the continuity of g , we have

$$\sum_{i=1}^k D_i + \int_0^T \|u_n\|_1^{q(t)-1} \{ g[t, \delta_0 + o(1), o(1), S(\delta_0)] + o(1) \} dt \neq 0, \quad (3.39)$$

it is a contradiction to (3.38). This implies that there is a large enough $R_0 > 0$ such that all the solutions of (3.24) belong to $B(R_0)$, then we have

$$d_{LS}[I - \Phi_f(\cdot, 1), B(R_0), 0] = d_{LS}[I - \Phi_f(\cdot, 0), B(R_0), 0]. \quad (3.40)$$

If we prove that $d_{LS}[I - \Phi_f(\cdot, 0), B(R_0), 0] \neq 0$, then we obtain the existence of solutions (3.20) with (1.2), (1.8) and (1.4).

Now we consider the following equation:

$$\begin{aligned} (\omega(r)|u'|^{p(r)-2}u')' &= \lambda N_g(u) + (1-\lambda) \left[QN_g(u) + \frac{1}{T} \sum_{i=1}^k D_i \right], \quad r \in J', \\ \lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) &= \lambda A_i, \quad i = 1, \dots, k, \\ \lim_{r \rightarrow r_i^+} \omega(r)|u'|^{p(r)-2}u'(r) - \lim_{r \rightarrow r_i^-} \omega(r)|u'|^{p(r)-2}u'(r) &= D_i, \quad i = 1, \dots, k, \\ u(0) = u(T), \quad \lim_{r \rightarrow 0^+} \omega(r)|u'|^{p(r)-2}u'(r) &= \lim_{r \rightarrow T^-} \omega(r)|u'|^{p(r)-2}u'(r), \end{aligned} \tag{3.41}$$

where $N_g(u) = g(r, u, (\omega(r))^{1/(p(r)-1)}u', S(u))$, A_i and D_i are defined in (2.23).

Similar to the above discussions, for any $\lambda \in (0, 1]$, all the solutions of (3.41) are uniformly bounded.

If u is a solution of the following usual equation with (1.4):

$$(\omega(r)|u'|^{p(r)-2}u')' = QN_g(u) + \frac{1}{T} \sum_{i=1}^k D_i, \quad r \in (0, T), \tag{3.42}$$

then we have

$$QN_g(u) + \frac{1}{T} \sum_{i=1}^k D_i = 0, \quad \omega(r)|u'|^{p(r)-2}u' \equiv c. \tag{3.43}$$

Since $u(0) = u(T)$, we have $\omega(r)|u'|^{p(r)-2}u' \equiv 0$, it means that u is a solution of

$$\omega_g(l) = \frac{1}{T} \int_{T_1}^{T_2} g(r, l, 0, S(l))dr + \frac{1}{T} \sum_{i=1}^k D_i(l, 0) = 0, \tag{3.44}$$

according to hypothesis (5⁰), (3.41) has no solution on $[0, 1] \times \partial B(R_0)$, from Theorem 3.1 we obtain that (3.20) with (1.2), (1.8) and (1.4) has at least one solution. This completes the proof. □

Corollary 3.3. *If $e : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is Caratheodory, conditions (2⁰), (3⁰) and (4⁰) of Theorem 3.2 are satisfied, $g(r, u, v, S(u)) = \psi(r)(|u|^{q(r)-2}u + |v|^{q(r)-2}v + |S(u)|^{q(r)-2}S(u))$, where $\psi(\cdot), q(\cdot) \in C(J, \mathbb{R})$ are positive functions, and satisfies $1 < q^- \leq q^+ < p^-$; then (3.20) with (1.2), (1.8) and (1.4) has at least one solution.*

Proof. Denote

$$H(l, \lambda) = \frac{1}{T} \int_0^T g(r, l, 0, S(l))dr + \frac{1}{T} \sum_{i=1}^k \lambda D_i(l, 0). \tag{3.45}$$

From condition (4⁰), we have

$$|D_i(l, 0)| \leq C(1 + |l|)^{\beta-1}, \quad 1 \leq \beta < q^+. \quad (3.46)$$

Since ϕ is nonnegative, from the above inequality, we can see that all the solutions of $H(l, \lambda) = 0$ are uniformly bounded for $\lambda \in [0, 1]$. Thus the Leray-Schauder degree $d_B[H(l, \lambda), b(R_0), 0]$ is well defined for $\lambda \in [0, 1]$, and we have

$$\begin{aligned} d_B[\omega_g, b(R_0), 0] &= d_B[H(l, 1), b(R_0), 0] = d_B[H(l, 0), b(R_0), 0]. \\ H(l, 0) &= \frac{1}{T} \int_0^T g(r, l, 0, S(l)) dr = \frac{1}{T} \int_0^T \psi(r) (|l|^{q(r)-2} l + S(l)) dr, \end{aligned} \quad (3.47)$$

then it is easy to see that $H(l, 0) = 0$ has only one solution in \mathbb{R}^N and

$$d_B[\omega_g, b(R_0), 0] = d_B[I, b(R_0), 0] \neq 0. \quad (3.48)$$

According to Theorem 3.2 we get that (3.20) with (1.2), (1.8) and (1.4) has at least a solution. This completes the proof. \square

Let us consider

$$-\left(\omega(r) |u'|^{p(r)-2} u'\right)' + f\left(r, u, (\omega(r))^{1/(p(r)-1)} u', S(u), \delta\right) = 0, \quad r \in (0, 1), \quad r \neq r_i, \quad (3.49)$$

where δ is a parameter, and

$$\begin{aligned} &f\left(r, u, (\omega(r))^{1/(p(r)-1)} u', S(u), \delta\right) \\ &= g\left(r, u, (\omega(r))^{1/(p(r)-1)} u', S(u)\right) + \delta h\left(r, u, (\omega(r))^{1/(p(r)-1)} u', S(u)\right), \end{aligned} \quad (3.50)$$

where $g, h : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ are Caratheodory.

We have the following theorem.

Theorem 3.4. *One assumes that conditions of (1⁰) and (3⁰)–(6⁰) of Theorem 3.2 are satisfied, then problem (3.49) with (1.2), (1.8) and (1.4) has at least a solution when the parameter δ is small enough.*

Proof. Denote

$$\begin{aligned}
 & f_{\lambda\delta}\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) \\
 &= g\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) + \lambda\delta h\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right).
 \end{aligned}
 \tag{3.51}$$

Let us consider the existence of solutions of the following

$$-\left(w(r)|u'|^{p(r)-2}u'\right)' + f_{\lambda\delta}\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) = 0, \quad r \in (0, 1), \quad r \neq r_i,
 \tag{3.52}$$

with (1.2), (1.8) and (1.4).

Denote

$$\begin{aligned}
 & \rho_\lambda^\#(u, \delta) = \rho(A, D, \Theta_{f_{\lambda\delta}})(u), \\
 & K_\lambda^\#(u, \delta) = F\left\{\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho_\lambda^\#(u, \delta) + \sum_{r_i < r} D_i + F(\Theta_{f_{\lambda\delta}}(u))(r)\right)\right]\right\}, \\
 & \Psi_\delta(u, \lambda) = P(u) + \sum_{r_i < r} A_i + \frac{1}{T} \sum_{i=1}^k D_i + QN_{f_{\lambda\delta}}(u) + K_\lambda^\#(u, \delta),
 \end{aligned}
 \tag{3.53}$$

where $N_{f_{\lambda\delta}}(u)$ is defined in (2.32), $\Theta_{f_{\lambda\delta}}$ is defined in (2.33).

We know that (3.52) with (1.2), (1.8) and (1.4) has the same solution of

$$u = \Psi_\delta(u, \lambda).
 \tag{3.54}$$

Obviously, $f_0 = g$. Thus $\Psi_\delta(u, 0) = \Phi_g(u, 1)$. From the proof of Theorem 3.2, we can see that all the solutions of $u = \Psi_\delta(u, 0)$ are uniformly bounded, then there exists a large enough $R_0 > 0$ such that all the solutions of $u = \Psi_\delta(u, 0)$ belong to $B(R_0) = \{u \in PC^1 \mid \|u\|_1 < R_0\}$. Since $\Psi_\delta(\cdot, 0)$ is compact continuous from PC^1 to PC^1 , we have

$$\inf_{u \in \partial B(R_0)} \|u - \Psi_\delta(u, 0)\|_1 > 0.
 \tag{3.55}$$

Since g, h are Caratheodory, we have

$$\begin{aligned}
 & \|F(N_{f_{\lambda\delta}}(u)) - F(N_{f_0}(u))\|_0 \longrightarrow 0 \quad \text{for } (u, \lambda) \in \overline{B(R_0)} \times [0, 1] \text{ uniformly, as } \delta \longrightarrow 0, \\
 & \left|\rho_\lambda^\#(u, \delta) - \rho_0^\#(u, \delta)\right| \longrightarrow 0 \quad \text{for } (u, \lambda) \in \overline{B(R_0)} \times [0, 1] \text{ uniformly, as } \delta \longrightarrow 0, \\
 & \|K_\lambda^\#(u, \delta) - K_0^\#(u, \delta)\|_1 \longrightarrow 0 \quad \text{for } (u, \lambda) \in \overline{B(R_0)} \times [0, 1] \text{ uniformly, as } \delta \longrightarrow 0.
 \end{aligned}
 \tag{3.56}$$

Thus

$$\|\Psi_\delta(u, \lambda) - \Psi_0(u, \lambda)\|_1 \longrightarrow 0 \text{ for } (u, \lambda) \in \overline{B(R_0)} \times [0, 1] \text{ uniformly, as } \delta \longrightarrow 0.
 \tag{3.57}$$

Obviously, $\Psi_0(u, \lambda) = \Psi_\delta(u, 0) = \Psi_0(u, 0)$. Therefore

$$\|\Psi_\delta(u, \lambda) - \Psi_\delta(u, 0)\|_1 \longrightarrow 0 \quad \text{for } (u, \lambda) \in \overline{B(R_0)} \times [0, 1] \text{ uniformly, as } \delta \longrightarrow 0. \quad (3.58)$$

Thus, when δ is small enough, from (3.55), we can conclude

$$\begin{aligned} & \inf_{(u, \lambda) \in \partial B(R_0) \times [0, 1]} \|u - \Psi_\delta(u, \lambda)\|_1 \\ & \geq \inf_{u \in \partial B(R_0)} \|u - \Psi_\delta(u, 0)\|_1 - \sup_{(u, \lambda) \in \overline{B(R_0)} \times [0, 1]} \|\Psi_\delta(u, 0) - \Psi_\delta(u, \lambda)\|_1 > 0. \end{aligned} \quad (3.59)$$

Thus $u = \Psi_\delta(u, \lambda)$ has no solution on $\partial B(R_0)$ for any $\lambda \in [0, 1]$, when δ is small enough. It means that the Leray-Schauder degree $d_{LS}[I - \Psi_\delta(u, \lambda), B(R_0), 0]$ is well defined for any $\lambda \in [0, 1]$, and

$$d_{LS}[I - \Psi_\delta(u, \lambda), B(R_0), 0] = d_{LS}[I - \Psi_\delta(u, 0), B(R_0), 0]. \quad (3.60)$$

From the proof of Theorem 3.2, we can see that the right-hand side is nonzero, then (3.49) with (1.2), (1.8) and (1.4) has at least a solution, when the parameter δ is small enough. \square

Now, let us consider the existence of solutions of (1.1) with (1.2), (1.3) and (1.4).

Theorem 3.5. *One assumes that conditions of (1⁰)-(3⁰) and (5⁰)-(6⁰) of Theorem 3.2 are satisfied, one also assumes that B satisfy*

$$|B_i(u, v)| \leq C(1 + |u| + |v|)^{\alpha_i}, \quad \forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N, \quad i = 1, \dots, k, \quad (3.61)$$

where

$$\alpha_i < \frac{q^+ - 1}{p(r_i) - 1}, \quad p(r_i) - 1 < q^+ - \alpha_i, \quad i = 1, \dots, k, \quad (3.62)$$

then problem (3.20) with (1.2), (1.3) and (1.4) has at least a solution.

Proof. We only need to prove that condition (4⁰) of Theorem 3.2 is satisfied.

(a) When $|v| \leq M|B_i(u, v)|$, where M is a large enough positive constant, from the definition of D , we can see that

$$|D_i(u, v)| \leq C_1 |B_i(u, v)|^{p(r_i)-1} \leq C_2 (1 + |u| + |v|)^{\alpha_i(p(r_i)-1)}. \quad (3.63)$$

Since $\alpha_i < (q^+ - 1)/(p(r_i) - 1)$, we have

$$\alpha_i(p(r_i) - 1) < q^+ - 1. \quad (3.64)$$

Thus condition (4⁰) of Theorem 3.2 is satisfied.

(b) When $|v| \geq M|B_i(u, v)|$, we can see that

$$|D_i(u, v)| \leq C_3|v|^{p(r_i)-1} \frac{|B_i(u, v)|}{|v|} = C_4|v|^{p(r_i)-2}|B_i(u, v)|. \tag{3.65}$$

(i) If $p(r_i) - 1 \geq 1$, since $p(r_i) - 1 < q^+ - \alpha_i$, we have

$$\begin{aligned} p(r_i) - 2 + \alpha_i &< q^+ - 1, \\ |D_i(u, v)| &\leq C_5|v|^{p(r_i)-2}|B_i(u, v)| \leq C_6(1 + |u| + |v|)^{p(r_i)-2+\alpha_i}. \end{aligned} \tag{3.66}$$

Thus the condition (4^0) of Theorem 3.2 is satisfied.

(ii) If $p(r_i) - 1 < 1$, since $\alpha_i < (q^+ - 1)/(p(r_i) - 1)$, we have

$$\begin{aligned} \alpha_i(p(r_i) - 1) &< q^+ - 1, \\ |D_i(u, v)| &\leq C_7|v|^{p(r_i)-2}|B_i(u, v)| \leq C_8|B_i(u, v)|^{p(r_i)-1} \leq C_9(1 + |u| + |v|)^{\alpha_i(p(r_i)-1)}. \end{aligned} \tag{3.67}$$

Thus the condition (4^0) of Theorem 3.2 is satisfied.

Summarizing the discussion, we can see that condition (4^0) of Theorem 3.2 is satisfied. Thus problem (3.20) with (1.2), (1.3) and (1.4) has at least a solution. \square

Corollary 3.6. *If $e : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is Caratheodory, (3.61), (3.62) and conditions (2^0) and (3^0) of Theorem 3.2 are satisfied, $g(r, u, v, S(u)) = \psi(r)(|u|^{q(r)-2}u + |v|^{q(r)-2}v + |S(u)|^{q(r)-2}S(u))$, where $\psi(r), q(r) \in C(J, \mathbb{R})$ are positive functions, and satisfies $1 < q^- \leq q^+ < p^-$; then (3.20) with (1.2), (1.3) and (1.4) has at least one solution.*

Proof. It is easy to see that from the proof of Corollary 3.3 and Theorem 3.5. We omit it here. \square

4. Examples

Example 4.1. Consider the following problem:

$$\begin{aligned} &-\left(\omega(t)|u'|^{p(t)-2}u'\right)' + |u|^{q(t)-2}u + \omega(t)|u'|^{q(t)-2}u' + S(u)(t) = 0, \quad t \in J', \\ &\lim_{t \rightarrow t_i^+} u(t) - \lim_{t \rightarrow t_i^-} u(t) = \lim_{t \rightarrow t_i^-} |u(t)|^{-1/2}u(t) + \lim_{t \rightarrow t_i^-} \left|(\omega(t))^{1/(p(t)-1)}u'(t)\right|^{-1/2}(\omega(t))^{1/(p(t)-1)}u'(t), \\ &\lim_{t \rightarrow t_i^+} \omega(t)|u'|^{p(t)-2}u'(t) - \lim_{t \rightarrow t_i^-} \omega(t)|u'|^{p(t)-2}u'(t) = \lim_{t \rightarrow t_i^-} |u(t)|^2u(t) + \lim_{t \rightarrow t_i^-} (\omega(t))^{3/(p(t)-1)}|u'(t)|^2u'(t), \\ &u(0) = u(T), \quad \lim_{r \rightarrow 0^+} \omega(r)|u'|^{p(r)-2}u'(r) = \lim_{r \rightarrow T^-} \omega(r)|u'|^{p(r)-2}u'(r), \end{aligned} \tag{S_1}$$

where $p(t) = 6 + \sin t, q(t) = 3 + \cos t, S(u)(t) = \int_0^T (\sin st + 1)u(s)ds$.

Obviously, $|u|^{q(t)-2}u + w(t)|u|^{q(t)-2}u' + S(u)(t)$ is Caratheodory, $q(t) \leq 4 < 5 \leq p(t)$, and the conditions of Theorem 3.2 are satisfied, then (S_1) has a solution.

Example 4.2. Consider the following problem:

$$\begin{aligned} & -\left(w(t)|u|^{p(t)-2}u'\right)' + f\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) \\ & \quad + \delta h\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) = 0, \quad t \in J', \\ & \lim_{t \rightarrow t_i^+} u(t) - \lim_{t \rightarrow t_i^-} u(t) = \lim_{t \rightarrow t_i^-} |u(t)|^{-1/3} u(t) + \lim_{t \rightarrow t_i^-} \left| (w(t))^{1/(p(t)-1)} u'(t) \right|^{-1/3} (w(t))^{1/(p(t)-1)} u'(t), \\ & \lim_{t \rightarrow t_i^+} w(t) |u|^{p(t)-2} u'(t) - \lim_{t \rightarrow t_i^-} w(t) |u|^{p(t)-2} u'(t) = \lim_{t \rightarrow t_i^-} |u(t)|^2 u(t) + \lim_{t \rightarrow t_i^-} (w(t))^{3/(p(t)-1)} |u'(t)|^2 u'(t), \\ & u(0) = u(T), \quad \lim_{r \rightarrow 0^+} w(r) |u|^{p(r)-2} u'(r) = \lim_{r \rightarrow T^-} w(r) |u|^{p(r)-2} u'(r), \end{aligned} \tag{S_2}$$

where

$$\begin{aligned} f\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) &= |u|^{q(t)-2}u + w(t)|u|^{q(t)-2}u' + \delta u e^{|u| + |(w(t))^{1/(p(t)-1)}u'|} + S(u)(t), \\ p(t) &= 7 + \cos 3t, \quad q(t) = 4 + \sin 2t, \quad S(u)(t) = \int_0^T (\cos st + 1)u(s)ds. \end{aligned} \tag{4.1}$$

Obviously, $|u|^{q(t)-2}u + w(t)|u|^{q(t)-2}u' + \delta u e^{|u| + |(w(t))^{1/(p(t)-1)}u'|} + S(u)(t)$ is Caratheodory, $q(t) \leq 5 < 6 \leq p(t)$, and the conditions of Theorem 3.4 are satisfied, then (S_2) has a solution when δ is small enough.

Acknowledgment

This work is partly supported by the National Science Foundation of China (10701066, 10926075 and 10971087), China Postdoctoral Science Foundation funded project (20090460969), the Natural Science Foundation of Henan Education Committee (2008-755-65), and the Natural Science Foundation of Jiangsu Education Committee (08KJD110007).

References

- [1] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM Journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.
- [2] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, vol. 1748 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.
- [3] V. V. Zhikov, "Averaging of functionals of the calculus of variations and elasticity theory," *Mathematics of the USSR-Izvestiya*, vol. 29, pp. 33–66, 1987.
- [4] L. Diening, P. Hästö, and S. Roudenko, "Function spaces of variable smoothness and integrability," *Journal of Functional Analysis*, vol. 256, no. 6, pp. 1731–1768, 2009.

- [5] X.-L. Fan, H.-Q. Wu, and F.-Z. Wang, "Hartman-type results for $p(t)$ -Laplacian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 52, no. 2, pp. 585–594, 2003.
- [6] X.-L. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.
- [7] X.-L. Fan and Q.-H. Zhang, "Existence of solutions for $p(x)$ -Laplacian Dirichlet problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 52, no. 8, pp. 1843–1852, 2003.
- [8] X.-L. Fan, "Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form," *Journal of Differential Equations*, vol. 235, no. 2, pp. 397–417, 2007.
- [9] X.-L. Fan, Q.-H. Zhang, and D. Zhao, "Eigenvalues of $p(x)$ -Laplacian Dirichlet problem," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 306–317, 2005.
- [10] A. El Hamidi, "Existence results to elliptic systems with nonstandard growth conditions," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 1, pp. 30–42, 2004.
- [11] P. Hästö, Y. Mizuta, T. Ohno, and T. Shimomura, "Sobolev inequalities for Orlicz spaces of two variable exponents," to appear in *Glasgow Mathematical Journal*.
- [12] P. Harjulehto, P. Hästö, M. Koskenoja, and S. Varonen, "The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values," *Potential Analysis*, vol. 25, no. 3, pp. 205–222, 2006.
- [13] P. Harjulehto, P. Hästö, and V. Latvala, "Harnack's inequality for $p(\cdot)$ -harmonic functions with unbounded exponent p ," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 1, pp. 345–359, 2009.
- [14] O. Kováčik and J. Rákosník, "On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$," *Czechoslovak Mathematical Journal*, vol. 41(116), no. 4, pp. 592–618, 1991.
- [15] P. Marcellini, "Regularity and existence of solutions of elliptic equations with $(p \cdot q)$ -growth conditions," *Journal of Differential Equations*, vol. 90, no. 1, pp. 1–30, 1991.
- [16] M. Mihăilescu and V. Rădulescu, "Continuous spectrum for a class of nonhomogeneous differential operators," *Manuscripta Mathematica*, vol. 125, no. 2, pp. 157–167, 2008.
- [17] J. Musielak, *Orlicz Spaces and Modular Spaces*, vol. 1034 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.
- [18] S. G. Samko, "Density $C_0^\infty(\mathbb{R}^N)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbb{R}^N)$," *Rossiiskaya Akademiya Nauk. Doklady Akademii Nauk*, vol. 369, no. 4, pp. 451–454, 1999.
- [19] Q. Zhang, "A strong maximum principle for differential equations with nonstandard $p(x)$ -growth conditions," *Journal of Mathematical Analysis and Applications*, vol. 312, no. 1, pp. 24–32, 2005.
- [20] X. Guo, M. Lu, and Q. Zhang, "Infinitely many periodic solutions for variable exponent systems," *Journal of Inequalities and Applications*, vol. 2009, Article ID 714179, 10 pages, 2009.
- [21] D. D. Bañov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK, 1989.
- [22] X. Fu, B. Yan, and Y. Liu, *Introduction of Impulsive Differential Systems*, Science Press, Beijing, China, 2005.
- [23] Y.-H. Lee and X. Liu, "Study of singular boundary value problems for second order impulsive differential equations," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 159–176, 2007.
- [24] J. Jiao, L. Chen, and L. Li, "Asymptotic behavior of solutions of second-order nonlinear impulsive differential equations," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 458–463, 2008.
- [25] L. Liu, L. Hu, and Y. Wu, "Positive solutions of two-point boundary value problems for systems of nonlinear second-order singular and impulsive differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 11, pp. 3774–3789, 2008.
- [26] J. Shen and W. Wang, "Impulsive boundary value problems with nonlinear boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 11, pp. 4055–4062, 2008.
- [27] M. Yao, A. Zhao, and J. Yan, "Periodic boundary value problems of second-order impulsive differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 1, pp. 262–273, 2009.
- [28] A. Cabada and J. Tomeček, "Extremal solutions for nonlinear functional ϕ -Laplacian impulsive equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 3, pp. 827–841, 2007.
- [29] Y. Tian and W. Ge, "Applications of variational methods to boundary-value problem for impulsive differential equations," *Proceedings of the Edinburgh Mathematical Society. Series II*, vol. 51, no. 2, pp. 509–527, 2008.

- [30] X. Zhang and W. Ge, "Impulsive boundary value problems involving the one-dimensional p -Laplacian," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 4, pp. 1692–1701, 2009.
- [31] Q. Zhang, X. Liu, and Z. Qiu, "Existence of solutions for weighted $p(r)$ -Laplacian impulsive system periodic-like boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 3596–3611, 2009.
- [32] Q. Zhang, Z. Qiu, and X. Liu, "Existence of solutions and nonnegative solutions for weighted $p(r)$ -Laplacian impulsive system multi-point boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 3814–3825, 2009.